Optimal control for reconstruction of curves without cusps

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Abstract

We consider the problem of minimizing $\int_0^L \sqrt{1+K(t)^2}\ dt$ for a planar curve having fixed initial and final positions and directions. Here $K(t)$ is the curvature of the curve and the total length L is free. This problem comes from a model of geometry of vision due to Petitot, Citti and Sarti.

We study existence of local and global minimizers for this problem. We prove that, depending on the boundary conditions, only two cases are possible: either there exists a global minimizer that is smooth and without cusps; or there is neither a global nor a local minimizer nor a geodesic.

Our main tool is the construction of the optimal synthesis for the Reed and Shepp car with quadratic cost.

1 Introduction

In this paper we are interested to the following variational problem:

(P) Fix $\xi > 0$ and $(x_{in}, y_{in}, \theta_{in}), (x_{fin}, y_{fin}, \theta_{fin}) \in \mathbf{R}^2 \times S^1$. Assume that $(x_{in}, y_{in}) \neq (x_{fin}, y_{fin})$. On the space of (regular enough) planar curves, parameterized by plane-arclength[1] find the solutions of:

$$\gamma(0) = (x_{in}, y_{in}), \quad \gamma(\ell) = (x_{fin}, y_{fin}), \quad \dot\gamma(0) = (\cos(\theta_{in}), \sin(\theta_{in})), \quad \dot\gamma(\ell) = (\cos(\theta_{fin}), \sin(\theta_{fin})),$$
$$\int_0^\ell \sqrt{\xi^2 + K(s)^2}\ ds \to \min \quad \text{(here } \ell \text{ is free.)}$$

Here $K(s) = \frac{\dot x \ddot y - \dot y \ddot x}{(\dot x^2 + \dot y^2)^{3/2}}$ is the geodesic curvature of the planar curve $\gamma(.) = (x(.), y(.))$. This problem can be formulated as a problem of optimal control, for which the functional spaces where the problem is formulated are also more naturally specified.

($\mathbf{P_{curve}}$) Fix $\xi > 0$ and $(x_{in}, y_{in}, \theta_{in}), (x_{fin}, y_{fin}, \theta_{fin}) \in \mathbf{R}^2 \times S^1$. Assume that $(x_{in}, y_{in}) \neq (x_{fin}, y_{fin})$. In the space of integrable controls $v(.) : [0, \ell] \to \mathbf{R}$, find the solutions of:

$$(\dot x, \dot y, \dot\theta) = (\cos(\theta), \sin(\theta), 0) + v(t)(0, 0, 1),$$
$$(x(0), y(0), \theta(0)) = (x_{in}, y_{in}, \theta_{in}), \quad (x(\ell), y(\ell), \theta(\ell)) = (x_{fin}, y_{fin}, \theta_{fin}),$$
$$\int_0^\ell \sqrt{\xi^2 + v(s)^2}\ ds \to \min \quad \text{(here } \ell \text{ is free)}$$

Remark that we have used here that $v(s) = K(s)$. Since in this problem we are taking $v(.) \in L^1([0, \ell])$, we have that the curve $(x(.).y(.), \theta(.)) : [0, \ell] \to \mathbf{R}^2 \times S^1$ is absolutely continuous and the curve $(x(.).y(.)) : [0, \ell] \to \mathbf{R}^2$ is in $W^{2,1}$.

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[1]Here by plane-arclength we mean the arclength in $\mathbf{R}^2$. Later on, we consider also parameterizations by arclength on $\mathbf{R}^2 \times S^1$ or $\mathbf{R}^2 \times P^1$, that we call sR-arclength.

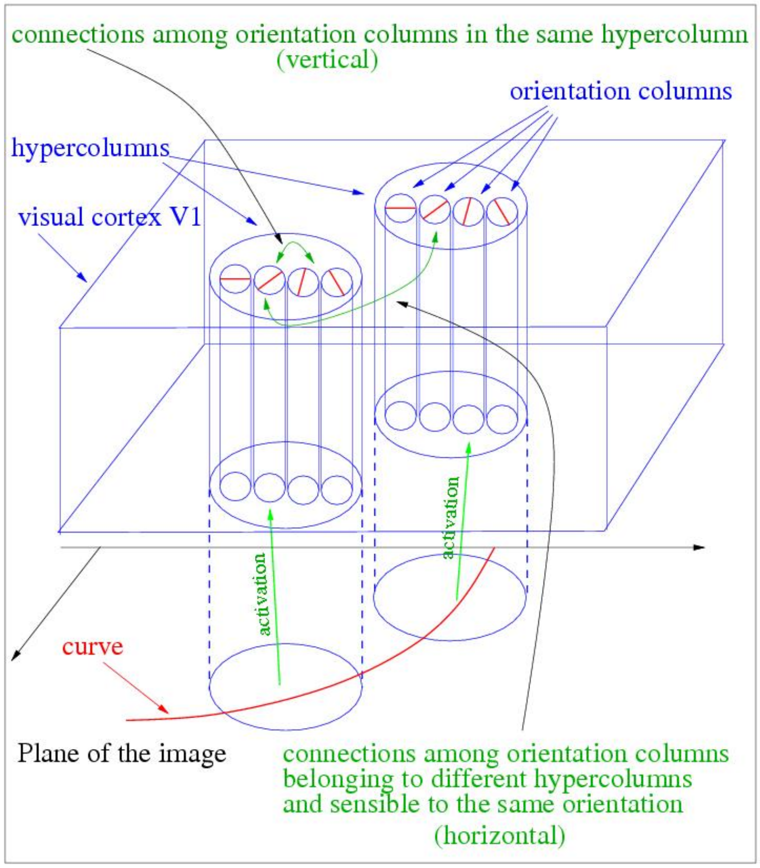

Figure 1: A scheme of the primary visual cortex V1.

1.1 Origin of the problem

This variational problem was studied as a possible model of the mechanism used by the visual cortex V1 to reconstruct curves which are partially hidden or corrupted. This model was initially due to Petitot (see [14, 15] and references therein), then refined by Citti and Sarti [7, 18], and by the authors of the present paper in [4, 8, 9]. It was also studied by Hladky and Pauls in [10].

In a simplified model (see [15, p. 79]), neurons of V1 are grouped into *orientation columns*, each of them being sensitive to visual stimuli at a given point of the retina and for a given direction on it. The retina is modeled by the real plane, i.e. each point is represented by $(x, y) \in \mathbf{R}^2$, while the directions at a given point are modeled by the projective[2] line, i.e. $\theta \in P^1$. Hence, the primary visual cortex V1 is modeled by the so called *projective tangent bundle* $PT\mathbf{R}^2 := \mathbf{R}^2 \times P^1$. From a neurological point of view, orientation columns are in turn grouped into *hypercolumns*, each of them being sensitive to stimuli at a given point (x, y) with any direction. In the same hypercolumn, relative to a point (x, y) of the plane, we also find neurons that are sensitive to other stimuli properties, like colors. In this paper, we focus only on directions and therefore each hypercolumn is represented by a fiber $P^1 = (x, y, .)$ of the bundle $PT\mathbf{R}^2$. Orientation columns are connected between them in two different ways. The first kind is given by *vertical connections*, which connect orientation columns belonging to the same hypercolumn and sensible to similar directions. The second is given by the *horizontal connections*, which connect orientation columns in different (but not too far) hypercolumns and sensible to the same directions. See Figure 1.

In other words, when V1 detects a (regular enough) planar curve $(x(.), y(.)) : [0, T] \to \mathbf{R}^2$ it computes a lift in

[2]In this paper by S^1 we mean $\mathbf{R}/\sim$ where $\theta \sim \theta'$ if $\theta = \theta' + 2n\pi$, $n \in \mathbb{N}$. By P^1 we mean $\mathbf{R}/\approx$ where $\theta \approx \theta'$ if $\theta = \theta' + n\pi$, $n \in \mathbb{N}$.

$PT\mathbf{R}^2$ by adding a new variable $\theta(.) : [0, T] \to P^1$ which satisfies:

$$(\dot{x}, \dot{y}, \dot{\theta}) = u(t)(\cos(\theta), \sin(\theta), 0) + v(t)(0, 0, 1) \tag{1}$$

for some real functions $u(.), v(.)$ defined on $[0, T]$. Here it is natural to take $u(.), v(.) \in L^1([0, T])$. This specifies also which regularity we need for the planar curve to be able to compute its lift: we need a curve in $W^{2,1}$ such that it has integrable curvature. In the following we call such a planar curve a *liftable curve*. Given a liftable curve γ, we note with $\Gamma : [0, T] \to SE(2)$ its lift. Given a curve $\Gamma(.) = (x(.), y(.), \theta(.))$ satisfying (1), we denote with $\gamma(.) := (x(.), y(.))$ the corresponding planar curve (that is, in general, not parametrized by arclength).

Consider now a liftable curve $(x(.), y(.)) : [0, T] \to \mathbf{R}^2$ which is interrupted in an interval $]a, b[\subset [0, T]$. Let us call $(x_{in}, y_{in}) := (x(a), y(a))$ and $(x_{fin}, y_{fin}) := (x(b), y(b))$. Assume moreover that, after computing its lift, the limits $\theta_{in} := \lim_{t \to a^-} \theta(t)$ and $\theta_{fin} := \lim_{t \to b^+} \theta(t)$ are well defined. In the model by Petitot, Citti, Sarti the visual cortex reconstructs it by minimizing the energy necessary to activate orientation columns which are not activated by the curve itself. This gives rise to the variational problem with dynamics (1) and

$$\begin{array}{l} J = \int_a^b \left(\xi^2 u(t)^2 + v(t)^2\right) \, dt \to \min, \\ (x(a), y(a), \theta(a)) = (x_{in}, y_{in}, \theta_{in}), \quad (x(b), y(b), \theta(b)) = (x_{fin}, y_{fin}, \theta_{fin}). \end{array} \tag{2}$$

Here $u(t)^2$ (resp. $v(t)^2$) represents the (infinitesimal) energy necessary to activate horizontal (resp. vertical) connections. The parameter $\xi > 0$ is used to fix the relative weight of the horizontal and vertical connections, which have different natures. The minimum is taken on the set of curves which are solution of (1) for some $u(.), v(.) \in L^1([a, b])$.

Minimization of (2) is equivalent to the minimization of

$$\int_a^b \sqrt{\xi^2 u(t)^2 + v(t)^2} \, dt = \int_a^b \|\dot{\gamma}(t)\| \sqrt{\xi^2 + K(t)^2} \, dt,$$

where we have expressed the cost in function of the planar curve $\gamma(t) = (x(t), y(t))$ and of its curvature. Notice that it is invariant by reparameterization of the curve. Hence, it is equivalent to look for minimizers of (2) in $L^1([a, b])$ or $L^\infty([a, b])$. See [4] for more details about these equivalences. Resuming the previous observations, we call $\mathbf{P_{projective}}$ the following problem

($\mathbf{P_{projective}}$) Fix $\xi > 0$ and $(x_{in}, y_{in}, \theta_{in}), (x_{fin}, y_{fin}, \theta_{fin}) \in \mathbf{R}^2 \times P^1$. Assume that $(x_{in}, y_{in}) \neq (x_{fin}, y_{fin})$. In the space of integrable controls $u(.), v(.) : [0, \ell] \to \mathbf{R}$, find the solutions of:

$$\begin{array}{l} (\dot{x}, \dot{y}, \dot{\theta}) = u(t)(\cos(\theta), \sin(\theta), 0) + v(t)(0, 0, 1), \\ (x(0), y(0), \theta(0)) = (x_{in}, y_{in}, \theta_{in}), \quad (x(\ell), y(\ell), \theta(\ell)) = (x_{fin}, y_{fin}, \theta_{fin}), \\ \mathcal{L} = \int_0^\ell \sqrt{\xi^2 u(s)^2 + v(s)^2} \, ds \to \min \quad (\ell \text{ free}) \end{array}$$

The variational problem $\mathbf{P_{projective}}$ is well posed, and we have proved in [5] that a solution always exists. One of its main interests is the possibility of associating to it a hypoelliptic diffusion equation which can be used to reconstruct images (and not just curves), and for contour completion. This point of view was developed in [5, 7, 8, 9].

However, its main drawback (at least for the problem of reconstruction of curves) is the existence of minimizers with cusps. We say that a liftable curve $(x(.), y(.)) : [0, T] \to \mathbf{R}^2$ has a cusp at $\bar{t} \in]0, T[$ if $(\dot{x}(\bar{t}), \dot{y}(\bar{t})) = (0, 0)$ and $\dot{\theta}(.)$ is defined and different from zero in a neighbourhood of $\bar{t}$. Notice that in a neighborhood of a cusp point, the tangent direction (with no orientation) is well defined. Minimizers with cusps are represented in Figures 3 and 4.

Since the presence of cusps has not been observed in human perception experiments [15, 7], people started to look for a way of requiring that no trajectories with cusps appear as solution of the variational problem. In [7] the authors proposed to require that trajectories are parameterized by plane-arclength i.e. with $\|\dot{\gamma}\| = u = 1$. In this way cusps cannot appear. Notice that assuming $u = 1$, directions must be considered with orientation, since now the direction of $\dot{\gamma}$ is defined in S^1. This constraint gives us the variational problem $\mathbf{P_{curve}}$ on which this paper is focused.

The first question we are interested in for $\mathbf{P_{curve}}$ is:

Q1) Is it true that for every initial and final condition, problem $\mathbf{P_{curve}}$ admits a **global minimum**?

In [4] it was shown that there are initial and final conditions for which the $\mathbf{P_{curve}}$ does not admit a minimizer.

From the modelization point of view, the non-existence of global minimizers is not a problem. It is very natural to believe that the visual cortex looks only for local minimizers, since it is able to make a comparison only with close trajectories. The main purpose of this paper is to study the existence of local minimizers[3] for the problem $\mathbf{P_{curve}}$. More precisely, we answer to the following question:

Q2) Is it true that for every initial and final condition the problem $\mathbf{P_{curve}}$ admits a **local minimum**? If not, what is the set of boundary conditions for which a local minimizer exists?

The last question is interesting, since one could compare the limit boundary conditions for which a mathematical reconstruction occurs with the limit boundary conditions for which a reconstruction in human perception experiments is observed. Indeed, it is well known that in human perception experiments, a person connects two configurations if they are sufficiently close in position and orientation, otherwise he simply does not connect them. See e.g. [11].

The main result of this paper is the following.

Theorem 1 *Fix an initial and a final condition $q_{in} = (x_{in}, y_{in}, \theta_{in})$ and $q_{fin} = (x_{fin}, y_{fin}, \theta_{fin})$ in $\mathbf{R}^2 \times S^1$. The only two following cases are possible:*
1) There exists a solution for $\mathbf{P_{curve}}$ from q_{in} to q_{fin}.
2) The problem $\mathbf{P_{curve}}$ from q_{in} to q_{fin} does not admit neither a global nor a local minimum nor a geodesic.
Both cases occur, depending on the initial conditions.

We prove this result by introducing an auxiliary mechanical problem (called $\mathbf{P_{MEC}}$ in the following) which is formulated as $\mathbf{P_{projective}}$ but in which $\theta \in S^1$. More precisely, we consider the following:

($\mathbf{P_{MEC}}$) Fix $(x_{in}, y_{in}, \theta_{in})(x_{fin}, y_{fin}, \theta_{fin}) \in \mathbf{R}^2 \times S^1$ and $\xi > 0$. In the space of L^∞ controls $u(.), v(.) : [0, \ell] \to \mathbf{R}$, find the solutions of:

$$
\begin{aligned}
&(\dot{x}, \dot{y}, \dot{\theta}) = (\cos(\theta), \sin(\theta), 0) + v(t)(0, 0, 1),\\
&(x(0), y(0), \theta(0)) = (x_{in}, y_{in}, \theta_{in}), \quad (x(\ell), y(\ell), \theta(\ell)) = (x_{fin}, y_{fin}, \theta_{fin}),\\
&\textstyle\int_0^\ell \sqrt{\xi^2 u(s)^2 + v(s)^2}\, ds \to \min \quad \text{(here } \ell \text{ is free)}
\end{aligned}
$$

This problem (which cannot be interpreted as a problem of reconstruction of planar curves, as explained in [5]) has been completely solved in a series of papers by one of the authors (see [13, 16, 17]). He developed a software for finiding numerical solutions to problem $\mathbf{P_{MEC}}$ for arbitrary boudary conditions.

One of the features of $\mathbf{P_{MEC}}$ and $\mathbf{P_{projective}}$ is that they are sub-Riemannian problems, see an introduction to sub-Riemannian geometry in [3]. This helps in computing the minimizers. Indeed, one can use first-order conditions (like the Pontryagin Maximum Principle, see Section 2.3) to find geodesics. One then has to study where geodesics lose optimality.

The second part of answer for **Q2** is given by the computation of the boundary conditions for which a solution for $\mathbf{P_{curve}}$ exists. Due to invariance of the problem under rototranslations on the plane, one can always assume that $q_{in} = (0, 0, 0)$. Under this hypothesis, we have computed numerically the final configurations for which a solution exists, see Figure 2.

It is interesting to observe that for $\mathbf{P_{curve}}$ the Lavrentiev phenomenon occurs, i.e. there exist minimizing curves that are absolutely continuous but not Lipschitz. See Section 4.2 This is a quite exotic phenomenon which is rarely observed. The interest in it comes from the fact that in optimal control, standard first order necessary conditions require the knowledge "a priori" of the existence of Lipschitz minimizers and do not permit in general to detect minimizers which are just absolutely continuous. See [12] for more details.

The structure of the paper is the following. In Section 2 we collect the problems we study in this article (Table 1) and we study the equivalences between minimizers of such problems. We then define precisely the concepts of global and local minimizers, and geodesics. We finally introduce the Pontryagin Maximum Principle. In Section 3 we give results about $\mathbf{P_{MEC}}$, describing the minimizing curves and some of their properties. They are used in the main section, that is Section 4, in which we study minimizers for $\mathbf{P_{curve}}$, proving Theorem 1 and showing the Lavrentiev phenomenon.

[3] See Definition 1 for the precise definition of local and global minimizers, and geodesics.

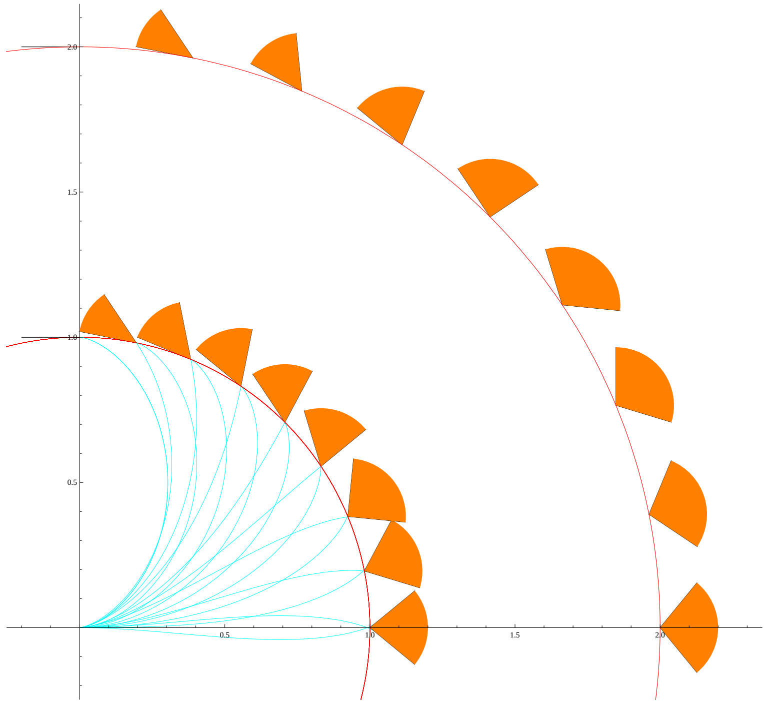

Figure 2: Final configurations for which we have existence of minimizers. We study the cases $x_{fin}^2 + y_{fin}^2 = 1$ or 4, with $y \geq 0$. The case $y \leq 0$ can be recovered by symmetry. In the first case, minimizing curves are also shown.

2 Statement of the problems

In this section, we study minimizers of $\mathbf{P_{curve}}$ and $\mathbf{P_{projective}}$ by introducing the auxiliary mechanical problem $\mathbf{P_{MEC}}$ defined above. These variational problems are recalled in Table 1 for the reader's convenience.

2.1 Equivalence of the problems

We now state precisely the connections between minimizers of such problems.

We first observe that, for any start and end conditions, the problem $\mathbf{P_{MEC}}$ admits a solution. This is a simple result of sub-Riemannian geometry, as we will show in the following. The same result holds for $\mathbf{P_{projective}}$.

Also recall that the definitions of $\mathbf{P_{projective}}$ and $\mathbf{P_{MEC}}$ are very similar, with the only difference that $\theta \in P^1$ or $\theta \in S^1$, respectively. This is based on the fact that $\mathbf{R}^2 \times S^1$ is a double covering of $\mathbf{R}^2 \times P^1$. Moreover, both the dynamics and the infinitesimal cost in $\mathbf{P_{curve}}$ are compatible with the projection $\mathbf{R}^2 \times S^1 \to \mathbf{R}^2 \times P^1$. Thus, the geodesics for $\mathbf{P_{projective}}$ are the projection of the geodesics for $\mathbf{P_{curve}}$. Then, the differences can occur only on the "global" problem. Indeed, given $\mathbf{P_{projective}}$ from $(x_{in}, y_{in}, \theta_{in})$ to $(x_{fin}, y_{fin}, \theta_{fin})$, the minimizer correspond to the shortest between the four minimizing geodesics connecting the following points in $\mathbf{P_{MEC}}$:

- $(x_{in}, y_{in}, \theta_{in})$ and $(x_{fin}, y_{fin}, \theta_{fin})$
- $(x_{in}, y_{in}, \theta_{in} + \pi)$ and $(x_{fin}, y_{fin}, \theta_{fin})$
- $(x_{in}, y_{in}, \theta_{in})$ and $(x_{fin}, y_{fin}, \theta_{fin} + \pi)$
- $(x_{in}, y_{in}, \theta_{in} + \pi)$ and $(x_{fin}, y_{fin}, \theta_{fin} + \pi)$

A detailed description of the geodesics for $\mathbf{P_{MEC}}$ can be found in Section 3.

It is also easy to prove that, given a minimizer Γ of $\mathbf{P_{MEC}}$ without cusps, the corresponding curve γ is a minimizer of $\mathbf{P_{curve}}$. Indeed, take a minimizer of $\mathbf{P_{MEC}}$ such that $\dot{\gamma} = \dot{x}(t)^2 + \dot{y}(t)^2 > 0$ for $t \in [0, T]$. Then, reparametrize

Notation $q = (x, y, \theta)$, $X_1 = (\cos(\theta), \sin(\theta), 0)$, $X_2 = (0, 0, 1)$ here $(x, y) := \gamma \in \mathbf{R}^2$ and $\theta \in S^1$ or P^1 as specified below. Let us call s the plane-arclength parameter and τ the sR-arclength parameter. In all problems written below we have the following: • initial and final conditions $(x_{in}, y_{in}, \theta_{in}), (x_{fin}, y_{fin}, \theta_{fin})$ are fixed in such a way that $(x_{in}, y_{in}) \neq (x_{fin}, y_{fin})$. • the final time T (or length ℓ) is free
Problem $\mathbf{P_{curve}}$: $q \in \mathbf{R}^2 \times S^1$ $\dot{q} = X_1 + vX_2$, $\int_0^\ell \sqrt{\xi^2 + v^2}\, ds = \int_0^\ell \sqrt{\xi^2 + K(s)^2} ds \to \min$
Problem $\mathbf{P_{MEC}}$: $q \in \mathbf{R}^2 \times S^1$ $\dot{q} = uX_1 + vX_2$, $\int_0^T \sqrt{\xi^2 u^2 + v^2}\, d\tau \to \min$
Problem $\mathbf{P_{projective}}$: $q \in \mathbf{R}^2 \times P^1$ $\dot{q} = uX_1 + vX_2$, $\int_0^T \sqrt{\xi^2 u^2 + v^2} d\tau = \int_0^T \|\dot{\gamma}\| \sqrt{\xi^2 + K(\tau)^2} dt \to \min$

Table 1: The different problems we treat in the paper.

the time to have $u = \dot{\gamma} \equiv 1$. This new parametrization of γ satisfies the dynamics for $\mathbf{P_{curve}}$ and the boundary conditions. By contradiction, assume that there exists a curve $\tilde{\gamma}$ satisfying the dynamics for $\mathbf{P_{curve}}$ and the boundary conditions with a cost smaller than the cost of γ. Then the lift of $\tilde{\gamma}$ satisfies the dynamics for $\mathbf{P_{MEC}}$ too and boundary conditions, with a smaller cost, hence Γ is not a minimizer. Contradiction.

Finally, observe that all the problems we treated depend on a parameter $\xi > 0$. It is easy to reduce our study to the case $\xi = 1$. Indeed, consider the problem $\mathbf{P_{MEC}}$ with a fixed $\xi > 0$, that we call $\mathbf{P_{MEC}}(\xi)$. Given a curve Γ with cost $C_\xi(\Gamma)$, apply the dilation $(x, y) \to (\xi x, \xi y)$ to find a curve $\tilde{\Gamma}$. This curve has boundary conditions that are dilations of the previous boundary conditions, and it satisfies the dynamics for $\mathbf{P_{MEC}}$. If one consider now its cost $C_1(\tilde{\Gamma})$ for the problem $\mathbf{P_{MEC}}(1)$, one finds that $C_1(\tilde{\Gamma}) = C_\xi(\Gamma)$. Hence, the problem of minimization for all $\mathbf{P_{MEC}}$ is equivalent to the case $\mathbf{P_{MEC}}(1)$. The same holds for $\mathbf{P_{projective}}$, $\mathbf{P_{curve}}$, with an identical proof. For this reason, we will fix $\xi = 1$ from now on.

2.2 Minimizers, local minimizers, geodesics

Let M be a n dimensional smooth manifold and $f : (q, u) \mapsto f(q, u) \in T_qM$ be a smooth vector field depending on the parameter $u \in \mathbf{R}^m$. Consider the following variational problem (denoted by **VP** for short).

$$\dot{q}(t) = f(q(t), u(t)), \quad q(0) = q_0, \quad q(T) = q_1, \tag{3}$$

$$\int_0^T f^0(q(s), u(s))\, ds \to \min, \quad T \text{ free} \tag{4}$$

$$u(.) \in \bigcup_{T>0} L^1([0,T], \mathbf{R}^m),\ q(.) \in \bigcup_{T>0} AC([0,T], M) \tag{5}$$

Definition 1 *We say that a pair trajectory-control $(q(.), u(.))$ is a* minimizer *if it is a solution of* **VP**.
We say that it is a local minimizer *if there exists an open neighborhood $B_{u(.)}$ of $u(.)$ in $\cup_{T>0} L^1([0, T], \mathbf{R}^n)$ such that all $(\bar{q}(.), \bar{u}(.))$ satisfying (3) with $\bar{u}(.) \in B_{u(.)}$ have a bigger cost.*
We say that it is a geodesic *if for every sufficiently small interval $[t_1, t_2] \subset Dom(q(.))$, the pair $(q(.), u(.))|_{[t_1,t_2]}$ is a minimizer of $\int_{t_1}^T f^0(q(s), u(s))\, ds$ from $q(t_1)$ to $q(t_2)$ with T free.*

In this paper we are interested in studying the two problems $\mathbf{P_{curve}}$ and $\mathbf{P_{MEC}}$, that are two particular cases of **VP**. For $\mathbf{P_{MEC}}$, which is a 3D contact problem (see the definition below), we apply a standard tool of optimal control, namely the Pontryagin maximum Principle (PMP in the following) which is described in the next section. Then we derive properties for $\mathbf{P_{curve}}$ from the solution of $\mathbf{P_{MEC}}$.

2.3 The Pontryagin Maximum Principle on 3D contact manifolds

In the following we recall some classical results from geometric control theory which hold for the 3D contact case. We use later these results for $\mathbf{P_{MEC}}$.

Definition 2 (3D contact problem) *Let M be a 3D manifold and let X_1, X_2 be two smooth vector fields such that dim(Span$\{X_1, X_2, [X_1, X_2]\}(q)$)=3 for every $q \in M$. The variational problem*

$$\dot{q} = u_1 X_1 + u_2 X_2, \quad q(0) = q_0, \quad q(T) = q_1, \quad \int_0^T \sqrt{u_1(t)^2 + u_2(t)^2}dt \to \min$$

is called a 3D-contact problem.

Remark 1 *In the problem above the final time T can be free or fixed since the cost is invariant by time reparameterization. As a consequence the spaces L^1 and AC in (5) can be replaced with L^∞ and Lip, since we can always reparameterize trajectories in such a way that $u_1(t)^2 + u_2(t)^2 = 1$ for every $t \in [0, T]$. If $u_1(t)^2 + u_2(t)^2 = 1$ for a.e. $t \in [0, T]$ we say that the curve is parameterized by sR-arclength. See [4, Section 2.1.1] for more details.*

We now state the PMP to our problem.

Proposition 1 (PMP for 3D contact problems) *In the 3D contact case, a curve parameterized by sR-arclength is a geodesic if and only if it is the projection of a solution of the Hamiltonian system of variables $q \in M,\ p \in T_q^*M$ corresponding to the Hamiltonian*

$$H(q, p) = \tfrac{1}{2}(\langle p, X_1(q)\rangle^2 + \langle p, X_2(q)\rangle^2), \tag{6}$$

lying on the level set $H = 1/2$.

This simple form of the PMP follows from the absence of abnormal extremals in 3D-contact geometry, as a consequence of the condition $dim(Span\{X_1, X_2, [X_1, X_2]\}(q)) = 3$ for every $q \in M$, see [1]. For a general form of the PMP, see [3]. As a consequence, geodesics are always smooth and even analytic if M, X_1, X_2 are analytic. In general, geodesics are not optimal for all times. Instead, minimizers and local minimizers are by definition geodesics.

A 3D contact manifold is said to be "complete" if all geodesics are defined for all times. This is the case for $\mathbf{P_{MEC}}$.

In the following we denote by $(q(t), p(t)) = e^{t\vec{H}}(q_0, p_0)$ the unique solution at time t of the Hamiltonian system

$$\dot{q} = \partial_p H, \quad \dot{p} = -\partial_q H,$$

with initial condition $(q(0), p(0)) = (q_0, p_0)$, that is the unique pair geodesic-covector starting in q_0 with covector p_0. We denote by $\pi : T^*M \to M$ the canonical projection $(q, p) \to q$.

Definition 3 *Let (M, X_1, X_2) be a 3D contact manifold and $q_0 \in M$. Let $\Lambda_{q_0} := \{p_0 \in T^*_{q_0}M \,|\, H(q_0, p_0) = 1/2\}$. We define the* exponential map *starting from q_0:*

$$\mathsf{Exp}_{q_0} : \Lambda_{q_0} \times \mathbf{R}^+ \to M,$$
$$\mathsf{Exp}_{q_0}(p_0, t) = \pi(e^{t\vec{H}}(q_0, p_0)).$$

We now recall the definition of cut and conjugate time.

Definition 4 *Let $q_0 \in M$ and $\gamma(t)$ be a geodesic parameterized by sR-arclength starting from q_0. The* cut time *for γ is*

$$T_{cut}(\gamma) = \sup\{t > 0,\ \gamma|_{[0,t]} \text{ is optimal}\}.$$

The corresponding cut point *is $\gamma(T_{cut}(\gamma))$. The* cut locus *is the set of all cut points.*

Definition 5 *Let $q_0 \in M$ and $q(.)$ be a geodesic parameterized by sR-arclength starting from q_0 with initial covector p_0. The* first conjugate time *of γ is*

$$T_{conj}(q(.)) = \min\{t > 0, |(p_0, t) \text{ critical point of } \mathsf{Exp}_{q_0}\}.$$

The corresponding conjugate point *is $q(T_{conj}(q(.)))$. The* conjugate locus *is the set of all conjugate points.*

It is well known that, for a geodesic $q(.)$, the cut time $t_* = T_{cut}(q(.))$ is either equal to the conjugate time or there exists another geodesic $\widetilde{q}(.)$ such that $q(t_*) = \widetilde{q}(t_*)$ (see for instance [1]). Such a point $q(t_*)$ is called a Maxwell point.

Theorem 2 *Let $q(.)$ be a geodesic starting from q_0 and let T_{cut} and T_{conj} be its cut and conjugate times (possibly $+\infty$). Then*

- $T_{cut} \leq T_{conj}$
- *$q(.)$ is globally optimal from $t = 0$ to T_{cut} and it is not globally optimal from $t = 0$ to $T_{cut} + \varepsilon$, for every $\varepsilon > 0$.*
- *$q(.)$ is locally optimal from $t = 0$ to T_{conj} and it is not locally optimal from $t = 0$ to $T_{conj} + \varepsilon$, for every $\varepsilon > 0$.*

The first two items are direct consequences of the definitions. The third item has been proved in [1] for 3D contact structures.

Remark 2 *In 3D contact geometry (and more in general in sub-Riemannian geometry) the exponential map is never a local diffeomorphism in a neighborhood of a point. As a consequence, spheres are never smooth and both the cut and the conjugate locus from q_0 are adjacent to the point q_0 itself (see [2]).*

3 Structure of the geodesics for the mechanical problem

We recall here results of [13, 16, 17] about the computation of minimizers for the problem $\mathbf{P_{MEC}}$. There, the authors fist apply PMP to the problem $\mathbf{P_{MEC}}$ to compute geodesics, then use symmetries to determine global minimizers, i.e. to evaluate where geodesics lose global optimality.

Theorem 3 *There exist 5 types of geodesics corresponding to the following curves $(x(t), y(t))$:*

1. *$(x(t), y(t)) \equiv (0, 0)$ is stationary,*
2. *$(x(t), y(t)) = (t, 0)$ and $(x(t), y(t)) = (-t, 0)$ are straight lines,*
3. *$(x(t), y(t))$ has infinite number of cusps and no inflection points (Fig. 3-Left). Its expression is*

$$\begin{aligned} x(t) &= \pm(1/k)[\operatorname{cn}\varphi(\operatorname{dn}\varphi - \operatorname{dn}(\varphi + t)) + \operatorname{sn}\varphi(t + \operatorname{E}(\varphi) - \operatorname{E}(\varphi + t))], \\ y(t) &= (1/k)[\operatorname{sn}\varphi(\operatorname{dn}\varphi - \operatorname{dn}(\varphi + t)) - \operatorname{cn}\varphi(t + \operatorname{E}(\varphi) - \operatorname{E}(\varphi + t))]. \end{aligned}$$

4. *$(x(t), y(t))$ has infinite number of cusps and infinite number of inflection points (Fig. 3-Right). Its expression is*

$$\begin{aligned} x(t) &= \pm k[\operatorname{dn}(\varphi/k)(\operatorname{cn}(\varphi/k) - \operatorname{cn}(\varphi + t)/k) + \operatorname{sn}(\varphi/k)(t/k + \operatorname{E}(\varphi/k) - \operatorname{E}((\varphi + t)/k)], \\ y(t) &= \pm[k^2 \operatorname{sn}(\varphi/k)(\operatorname{cn}(\varphi/k) - \operatorname{cn}(\varphi + t)/k) - \operatorname{dn}(\varphi/k)(t/k + \operatorname{E}(\varphi/k) - \operatorname{E}(\varphi + t)/k)]. \end{aligned}$$

5. *$(x(t), y(t))$ has one cusp and no inflection points (Fig. 4). Its expression is*

$$\begin{aligned} x(t) &= \pm[(1/\cosh\varphi)(1/\cosh\varphi - 1/\cosh(\varphi + t)) + \tanh\varphi(t + \tanh\varphi - \tanh(\varphi + t))], \\ y(t) &= \pm[\tanh\varphi(1/\cosh\varphi - 1/\cosh(\varphi + t)) - (1/\cosh\varphi)(t + \tanh\varphi - \tanh(\varphi + t))]. \end{aligned}$$

In the previous formulas, the Jacobian functions cn , sn , dn , E *are used. Variables (φ, k) are action-angle coordinates in the state space of mathematical pendulum that rectify its flow: $\dot{\varphi} = 1$, $\dot{k} = 0$.*

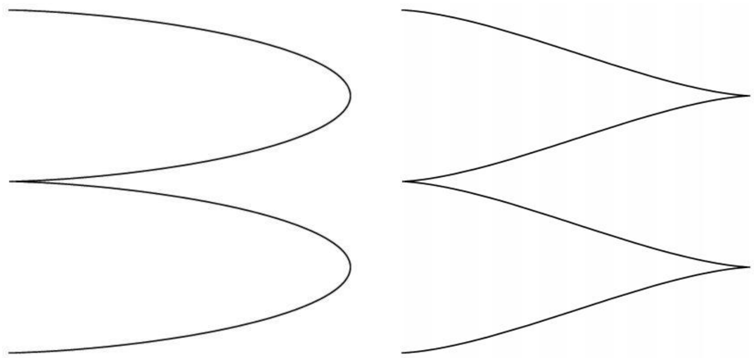

Figure 3: Left: Non-inflectional trajectory. Right: Inflectional trajectory.

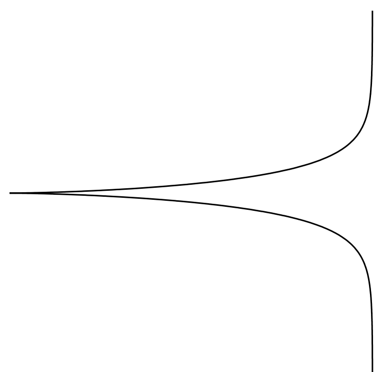

Figure 4: Critical trajectory.

3.1 Symmetries and global optimality

From the analysis developed in [13, 16, 17], it is important to consider two symmetries for the problem.

Definition 6 *Let $\Gamma(s) = (x(s), y(s), \theta(s))$ be a geodesic parameterized by sR-arclength. Consider the following mappings of geodesics:*

$$\varepsilon^i \; : \; \Gamma(s) \mapsto \Gamma^i(s), \qquad s \in [0,t], \qquad i = 2, 5,$$

where

$$\begin{aligned} \theta^2(s) &= \theta(t) - \theta(t-s), \\ x^2(s) &= -\cos\theta(t)(x(t) - x(t-s)) - \sin\theta(t)(y(t) - y(t-s)), \\ y^2(s) &= -\sin\theta(t)(x(t) - x(t-s)) + \cos\theta(t)(y(t) - y(t-s)), \end{aligned}$$

and

$$\begin{aligned} \theta^5(s) &= \theta(t-s) - \theta(t), \\ x^5(s) &= \cos\theta(t)(x(t-s) - x(t)) + \sin\theta(t)(y(t-s) - y(t)), \\ y^5(s) &= -\sin\theta(t)(x(t-s) - x(t)) + \cos\theta(t)(y(t-s) - y(t)). \end{aligned}$$

Modulo rotations of the plane (x, y), the mapping ε^2 acts as reflection of the curve $(x(s), y(s))$ in the middle perpendicular to the segment that connects the points $(x(0), y(0))$ and $(x(t), y(t))$; the mapping ε^5 acts as reflection in the midpoint of this segment (see Fig. 5).

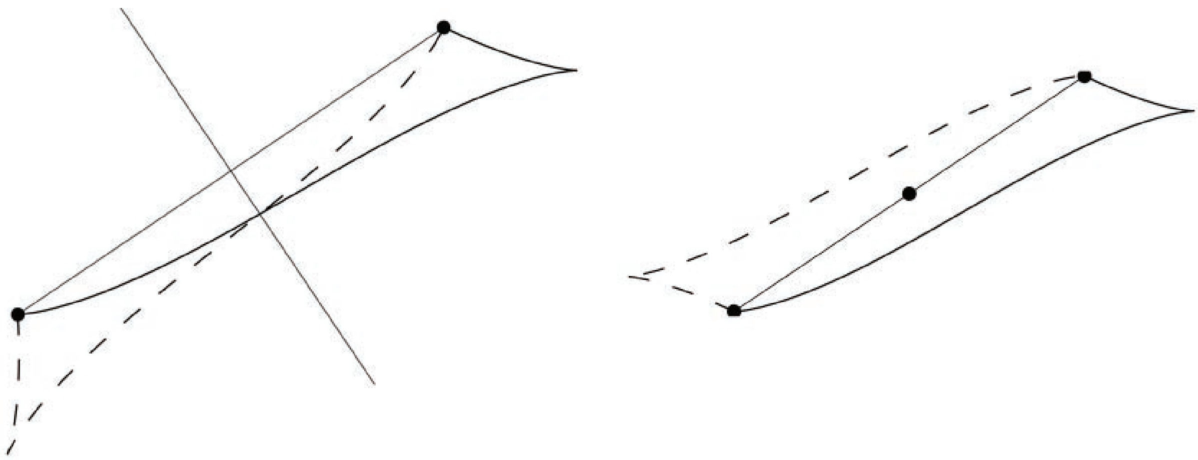

Figure 5: Action of ε^2 (left) and ε^5 (right) on $(x(s), y(s))$.

Definition 7 *A point* $\Gamma(t)$ *of a trajectory* $\Gamma(.)$ *is called a Maxwell point corresponding to a reflection* ε^i *if* $\Gamma(t) = \Gamma^i(t)$ *and* $\Gamma(.) \not\equiv \Gamma^i(.)$.

Examples of Maxwell points for the reflections ε^2 and ε^5 are shown at Fig. 6.

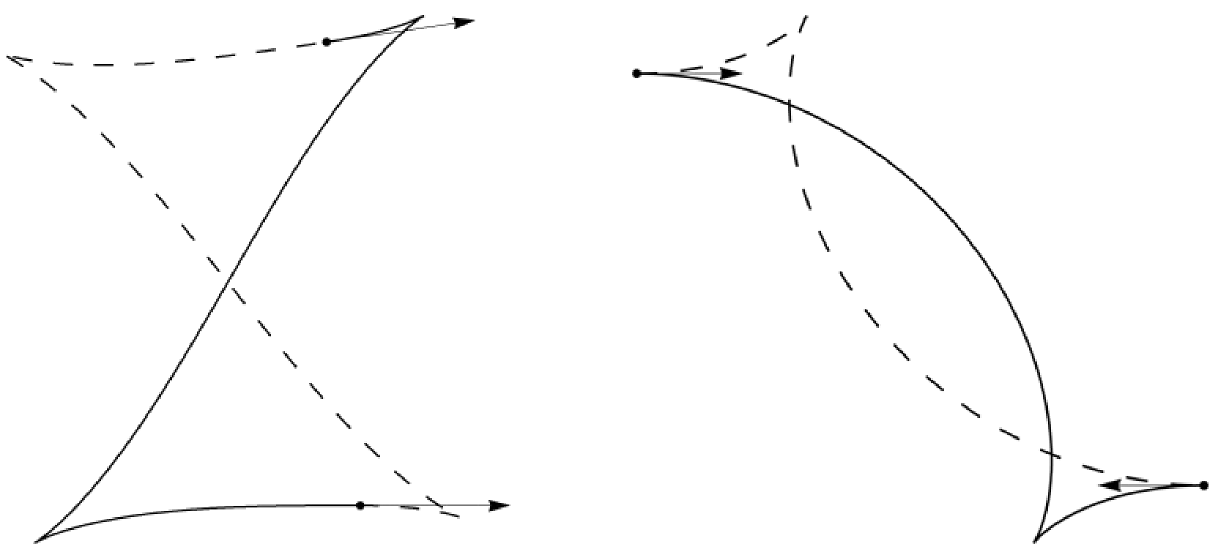

Figure 6: Maxwell point for reflection ε^2 (left) and ε^5 (right).

The following theorem proved in [13, 16, 17] describes optimality of geodesics.

Theorem 4 *A geodesic* $\Gamma(t)$*,* $t \in [0, T]$*, is optimal if and only if each point* $\Gamma(t)$*,* $t \in (0, T)$*, is neither a Maxwell points corresponding to* ε^2 *or* ε^5*, nor a limit of such Maxwell points.*

Notice that if a point $\Gamma(t)$ is a limit of Maxwell points then it is a conjugate point. See [13, 16, 17].

3.2 Internal cusps

In this section, we study the presence of internal cusps for solutions of $\mathbf{P_{MEC}}$. We first define them precisely.

Definition 8 *Let $\Gamma(.) = (x(.), y(.), \theta(.))$ be a geodesic parameterized by sR-arclength. We say that T_{cusp} is a cusp time for Γ (and $\Gamma(T_{cusp})$ a cusp point) if $\dot{x}(T_{cusp}) = \dot{y}(T_{cusp}) = 0$. We say that the restriction of $\Gamma(.)$ to an interval $[0, T]$ has no* **internal cusps** *if no $t \in]0, T[$ is a cusp time.*

The main result for internal cusps is the following.

Corollary 1 *Let Γ be a geodesic. Let T_{cusp}, and T_{cut} be the first cusp time and the cut time (possibly $+\infty$).Then $T_{cusp} \leq T_{cut}$.*

Proof. Take a geodesic Γ and consider the classification of Theorem 3. For cases 1,2,5, Γ has no cut time. For case 3, consider the reflection ε^5 giving the first Maxwell point. Due to Theorem 4, this is the cut point. Observe that there exists a time in which a cusp exists (Figure 6-Left), thus $T_{cusp} \leq T_{cut}$. The same holds for case 4, by using ε^2 (Figure 6-Right). □

Corollary 2 *Let Γ defined on $[0, T]$ be a minimizer having an internal cusp. Then any other minimizer between $\Gamma(0)$ and $\Gamma(T)$ has an internal cusp.*

Proof. Take $\tilde{\Gamma}$ a minimizer between $\Gamma(0)$ and $\Gamma(T)$ not coinciding with Γ. Since $\Gamma(T) = \tilde{\Gamma}(T)$, then $\tilde{\Gamma}$ has a cut time $\tilde{T}$, that is T or smaller. If $\tilde{T} = T$, then $\tilde{\Gamma}$ is given by a reflection ε^2 or ε^5 of Γ, thus it has an internal cusp. If $\tilde{T} < T$, then $\tilde{\Gamma}$ has a cusp time $t \leq \tilde{T} < T$, thus it has an internal cusp. □

4 Study of the problem of curve reconstruction

In this section we study the problem $\mathbf{P_{curve}}$. We first prove Theorem 1, that state that either there exists a global minimizer or there exists neither global nor local minimizer nor geodesic. We then show that the problem $\mathbf{P_{curve}}$ exhibits the Lavrentiev phenomenon.

4.1 Proof of Theorem 1

In this section we prove the main theorem of this paper, that is Theorem 1. Fix an initial and a final condition $q_{in} = (x_{in}, y_{in}, \theta_{in})$ and $q_{fin} = (x_{fin}, y_{fin}, \theta_{fin})$. Take a solution Γ of $\mathbf{P_{MEC}}$. If it has no cusps, the corresponding γ is a solution of $\mathbf{P_{curve}}$, as proved in Section 2.1. If it has cusps at boundaries, then the same reparametrization gives the corresponding γ that is a solution of $\mathbf{P_{curve}}$. For more details, see Section 4.2. The first part of Theorem 1 is now proved.

We prove the second part by contradiction. If Γ has an internal cusp, then any other solution of $\mathbf{P_{MEC}}$ has an internal cusp, as proved in Corollary 2. By contradiction, assume that there exists $\tilde{\gamma}$, either a solution of $\mathbf{P_{curve}}$, or a local minimizer, or a geodesic. In the three cases, $\tilde{\gamma}$ has no cusps. We study the three cases:
1) If $\tilde{\gamma}$ is a **solution** of $\mathbf{P_{curve}}$, i.e. a global minimizer, then its lift is a solution of $\mathbf{P_{MEC}}$ between the same boundary conditions of Γ. Then $\tilde{\gamma}$ has cusps. Contradiction.
2) If $\tilde{\gamma}$ is a **geodesic** for $\mathbf{P_{curve}}$ but not a global minimizer, then it exists a cut time t_{cut} for $\tilde{\gamma}$. Then its lift $\tilde{\Gamma}$ is a geodesic for $\mathbf{P_{MEC}}$. Due to Corollary 1, there exists $t_{cusp} \leq t_{cut}$ for which $\tilde{\Gamma}$ has a cusp. Contradiction.
3) If $\tilde{\gamma}$ is a local minimizer for $\mathbf{P_{curve}}$. Using the reparametrization method used in Section 2.1, one can prove that the lift of $\tilde{\gamma}$ is a local minimizer for $\mathbf{P_{MEC}}$. Then, it is a geodesic for $\mathbf{P_{MEC}}$, by definition (see Section 2.3). Apply now case 2.

4.2 The Lavrentiev phenomenon

We now show that the problem $\mathbf{P_{curve}}$ exhibits the Lavrentiev phenomenon, that means that there exist absolutely continuous minimizers that are not Lipschitz. Also in this case, we show an example starting from $\mathbf{P_{MEC}}$.

Take a minimizer $\tilde{\Gamma}$ of $\mathbf{P_{MEC}}$ with a cusp (i.e. of the kind 3,4, or 5 in Theorem 3), parametrized by sR-arclength. In particular, it means that $\xi^2 u(s)^2 + v(s)^2 \equiv 1$ for all times s. Let $\tilde{T}$ be the cusp time for $\tilde{\Gamma}$. Observe that $u(\tilde{T}) = 0$. Restrict $\tilde{\Gamma}$ to the interval $\left[0, \tilde{T}\right]$ and denote it with Γ.

Observe that Γ has no internal cusp and it has finite length $l(\Gamma)$. Reparametrize it by plane-arclength, that we denote with $\sigma = \sigma(s)$. Observe that $\int_0^{\tilde{T}} \sqrt{\xi^2 u(s)^2 + v(s)^2}\, ds = \int_0^{l(\gamma)} \sqrt{\xi^2 + K(\sigma)^2}\, d\sigma$. By construction of the parametrization, one has $K(\sigma(s)) = \frac{v(s)}{u(s)}$. In particular,

$$\lim_{\sigma \to l(\gamma)} K(\sigma)^2 = \lim_{s \to \tilde{T}} \frac{v(s)^2}{u(s)^2} = \lim_{s \to \tilde{T}} \frac{1 - \xi^2 u(s)^2}{u(s)^2} = \frac{1}{0^+} = +\infty.$$

Hence, the corresponding γ satisfies $\sqrt{\xi^2 + K(.)^2} \in L^1([0,\tilde{T}],\mathbf{R}) \setminus L^\infty([0,\tilde{T}],\mathbf{R})$. As a consequence, $\gamma \in AC([0,\tilde{T}],\mathbf{R}^2) \setminus Lip([0,\tilde{T}],\mathbf{R}^2)$.

This is exactly the Lavrentiev phenomenon. This shows that a direct application of standard techniques for the computation of local minimizers, like PMP, would provide local minimizers in the "too small" set of control $L^\infty([0,T],\mathbf{R})$. In our case, the auxiliary problem $\mathbf{P_{MEC}}$ does not present this phenomenon, since by reparametrization one can always reduce to the set $L^\infty([0,T],\mathbf{R})$. With this technique we can find all minimizing controls for $\mathbf{P_{curve}}$, even those in $L^1([0,T],\mathbf{R}) \setminus L^\infty([0,T],\mathbf{R})$.

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