Restoration of curves via variational principle

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Abstract

We describe an approach to the problem of restoration of curves via the following variational principle: the curve $(x(t), y(t))$ should minimize length in the space (x, y, θ), where θ is the angle of slope of the curve $(x(t), y(t))$.

1 Problem statement and method of solution

Consider a smooth curve in the plane

$$AB = \{(x(t), y(t)) \mid t \in [a, b]\},$$
$$A = (x(a), y(a)), \qquad B = (x(b), y(b)).$$

Assume that a portion of this curve

$$CD = \{(x(t), y(t)) \mid t \in [c, d]\},$$
$$C = (x(c), y(c)), \qquad D = (x(d), y(d)),$$
$$a < c < d < b,$$

is hidden or corrupted, see Fig. 1. One should restore the curve CD in some natural way.

In works [1], [2] the following method of restoration of the curve CD is considered. Construct the tangent T_C to the curve AC at the point C and the tangent T_D at the point D, see Fig. 2. Denote by θ_c, θ_d the angles of slope of these tangents:

$$\tan(\theta_c) = \left.\frac{d\,y}{d\,x}\right|_C = \frac{\dot{y}(c)}{\dot{x}(c)}, \qquad \tan(\theta_d) = \left.\frac{d\,y}{d\,x}\right|_D = \frac{\dot{y}(d)}{\dot{x}(d)}.$$

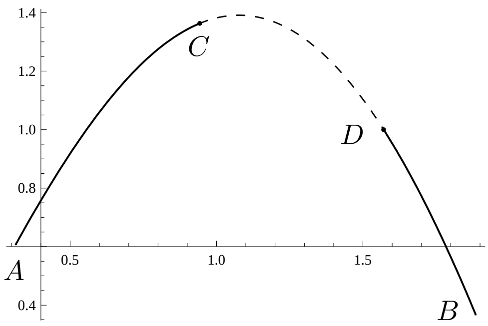


Figure 1: Initial curve AB with corrupted arc CD

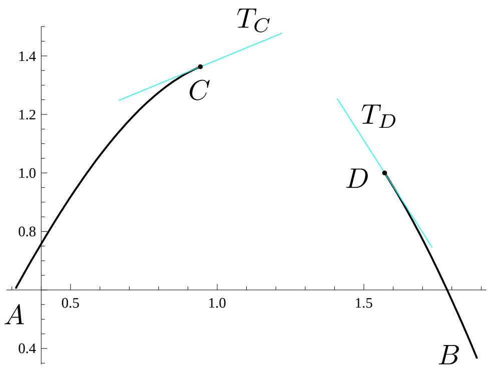


Figure 2: Boundary conditions for restoration of arc CD

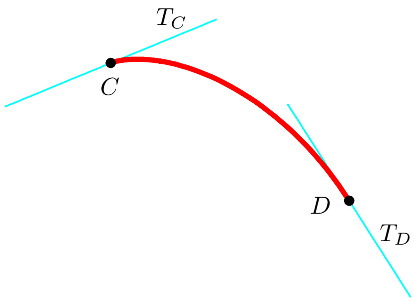


Figure 3: New arc $\widetilde{CD}$

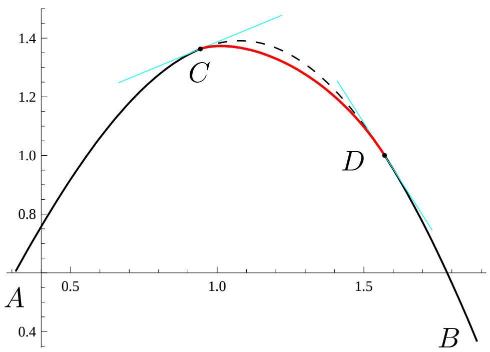


Figure 4: Initial curve AB with corrupted and new arcs

The required curve

$$\widetilde{CD} = \{(\tilde{x}(t), \tilde{y}(t)) \mid t \in [c, d]\}$$

should start at the point C at the angle θ_c:

$$(\tilde{x}(c), \tilde{y}(c)) = C, \qquad \left.\frac{d\,\tilde{y}}{d\,\tilde{x}}\right|_C = \frac{\dot{\tilde{y}}(c)}{\dot{\tilde{x}}(c)} = \tan(\theta_c), \tag{1}$$

terminate at the point D at the angle θ_d:

$$(\tilde{x}(d), \tilde{y}(d)) = D, \qquad \left.\frac{d\,\tilde{y}}{d\,\tilde{x}}\right|_D = \frac{\dot{\tilde{y}}(d)}{\dot{\tilde{x}}(d)} = \tan(\theta_d), \tag{2}$$

and have the minimum length in the space (x, y, θ):

$$\int_c^d \sqrt{\dot{\tilde{x}}^2 + \dot{\tilde{y}}^2 + \dot{\tilde{\theta}}^2}\, dt = \min. \tag{3}$$

Conditions (1), (2) mean smooth attachment of the new curve $\widetilde{CD}$ to the known arcs AC and DB of the initial curve. Condition (3) formalizes the condition that the new curve $\widetilde{CD}$ should be natural: for this curve, both big deviations for coordinates (x, y), and for the angle θ are penalized. The new curve $\widetilde{CD}$ is shown at Fig. 3. The initial and restored curves are shown at Fig. 4.

Two more examples of curves restored via the method described are shown at Figs. 5, 6.

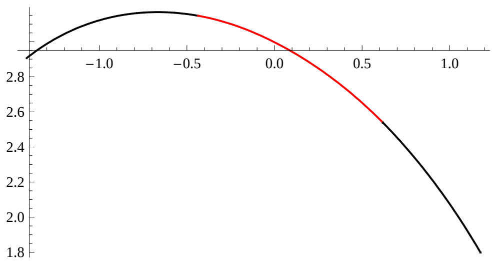


Figure 5: Restored curve

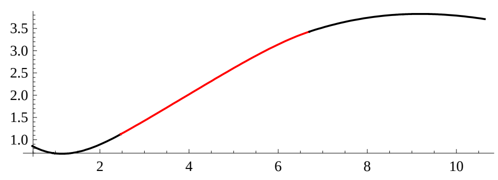


Figure 6: Restored curve

Problem (1), (2), (3) is formalized as the following optimal control problem:

$$\begin{aligned}
&\dot{x} = u\cos\theta,\\
&\dot{y} = u\sin\theta,\\
&\dot{\theta} = v,\\
&(x,y)\in\mathbb{R}^2, \qquad \theta\in\mathbb{R}/(\pi\mathbb{Z}),\\
&(x(c),y(c)) = C, \quad \theta(c)=\theta_c,\\
&(x(d),y(d)) = D, \quad \theta(d)=\theta_d,\\
&\int_c^d \sqrt{\dot{x}^2+\dot{y}^2+\dot{\theta}^2}\,dt = \int_c^d \sqrt{u^2+v^2}\,dt \to \min .
\end{aligned}$$

This is a left-invariant sub-Riemannian problem on the group of roto-translations of a plane, factorized by the equivalence relation $\theta \sim \theta+\pi$. In works [3], [4] this problem was reduced to solving systems of algebraic equations in Jacobi's functions. A software in Mathematica [5] was developed for solving these systems of equations.

In Sec. 2 we present solutions of this problem for the initial condition $(x_0, y_0, \theta_0) = (0, 0, 0)$ and certain explicit terminal conditions (x_1, y_1, θ_1). In Sec. 3 we present solutions for random boundary conditions. In the most cases a solution is given by a smooth curve $(x(t), y(t))$, but sometimes this curve has cusps. Such solutions are shown in Sec. 4, such cases should be studied additionally.

Question: are the approach described and the solutions obtained applicable to problems of restoration of images (individual curves and their families)?

2 Solutions with prescribed boundary conditions

In all examples of this section we use the initial condition

$$(x_0, y_0, \theta_0) = (0, 0, 0).$$

Figure 7: $(x_1, y_1, \theta_1) = (2, 0, 0)$

Figure 8: $(x_1, y_1, \theta_1) = (2, 0, \pi/8)$

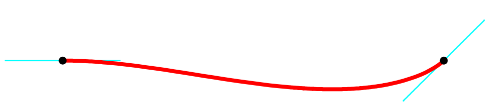


Figure 9: $(x_1, y_1, \theta_1) = (2, 0, \pi/4)$

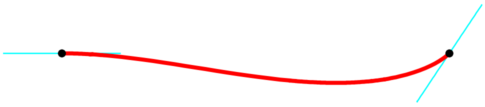


Figure 10: $(x_1, y_1, \theta_1) = (2, 0, 5\pi/16)$

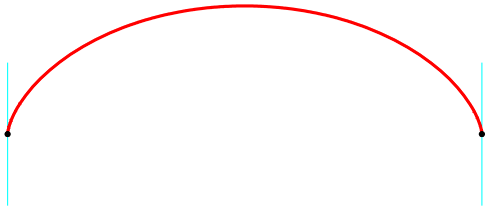


Figure 11: $(x_1, y_1, \theta_1) = (0, 2, 0)$; rotated by $\pi/2$

Figure 12: $(x_1, y_1, \theta_1) = (0, 2, \pi/4)$; rotated by $\pi/2$

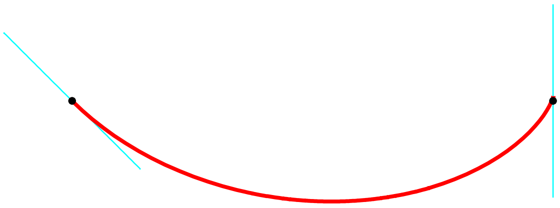


Figure 13: $(x_1, y_1, \theta_1) = (0, 1.5, \pi/4)$; rotated by $\pi/2$

Figure 14: $(x_1, y_1, \theta_1) = (0, 1.5, \pi/8)$; rotated by $\pi/2$

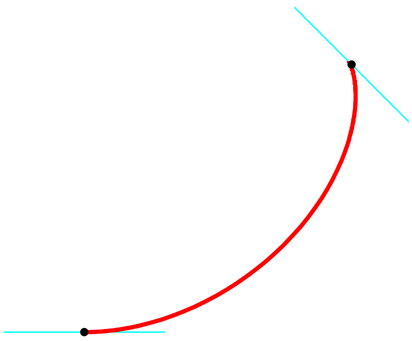


Figure 15: $(x_1, y_1, \theta_1) = (1, 1, 3\pi/4)$

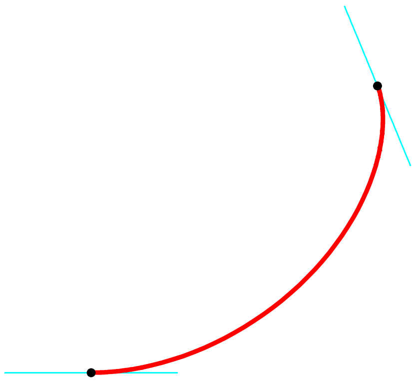


Figure 16: $(x_1, y_1, \theta_1) = (1, 1, 5\pi/8)$

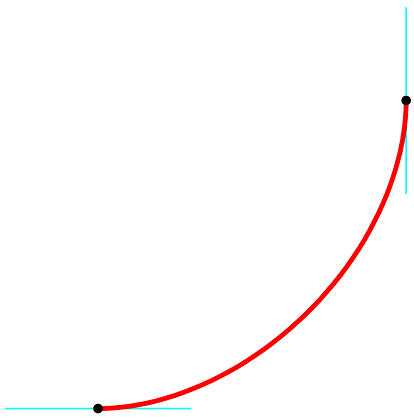


Figure 17: $(x_1, y_1, \theta_1) = (1, 1, \pi/2)$

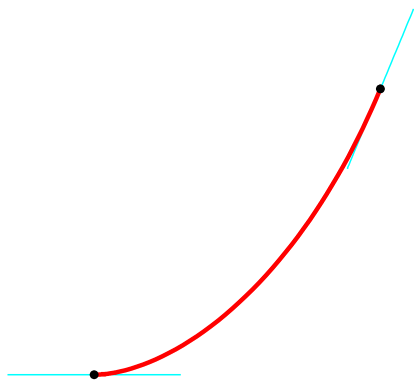


Figure 18: $(x_1, y_1, \theta_1) = (1, 1, 3\pi/8)$

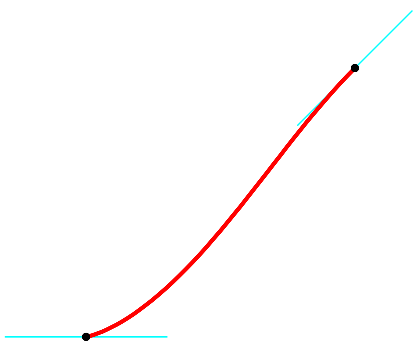


Figure 19: $(x_1, y_1, \theta_1) = (1, 1, \pi/4)$

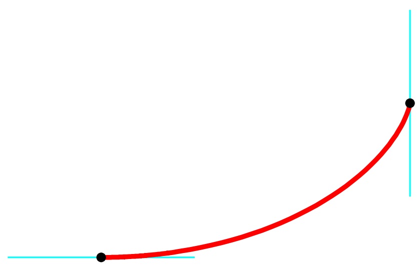
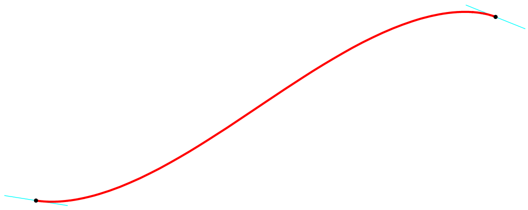


Figure 20: $(x_1, y_1, \theta_1) = (1, 0.5, \pi/2)$

3 Solutions with randomly chosen boundary conditions



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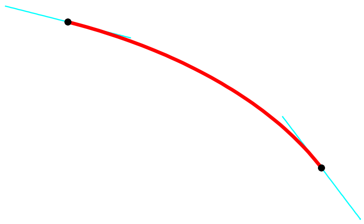

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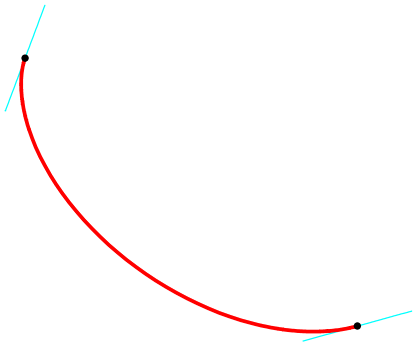

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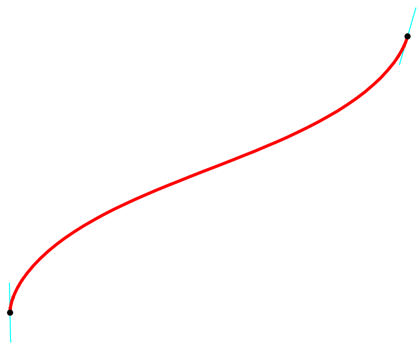
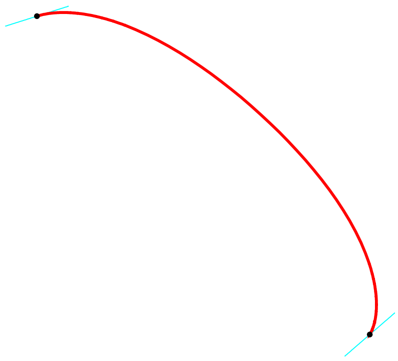


Figure 24:

Figure 25:



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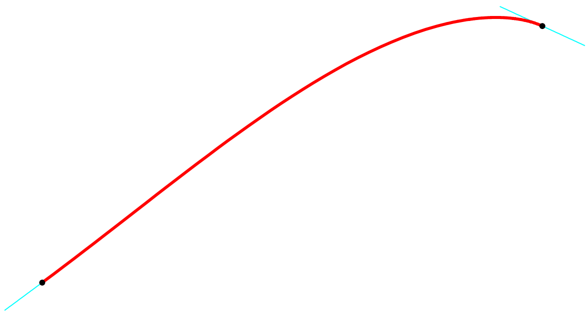

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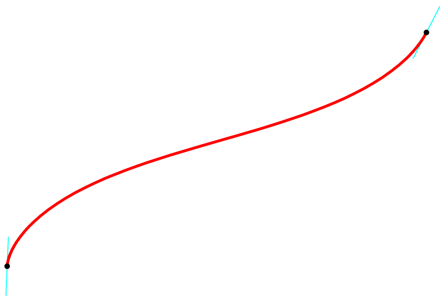
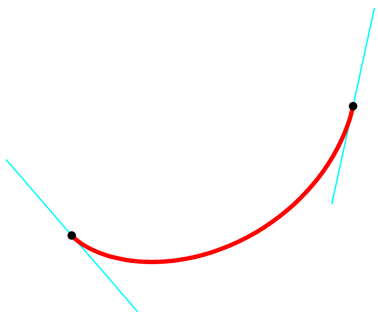


Figure 28:

Figure 29:



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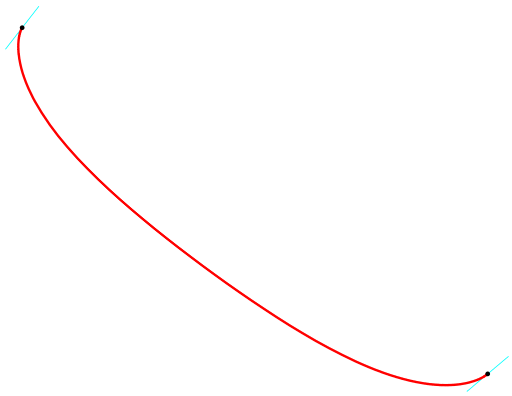
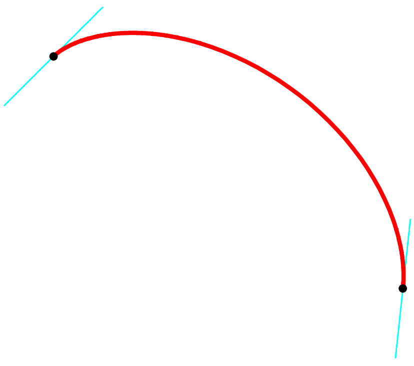


Figure 31:

Figure 32:



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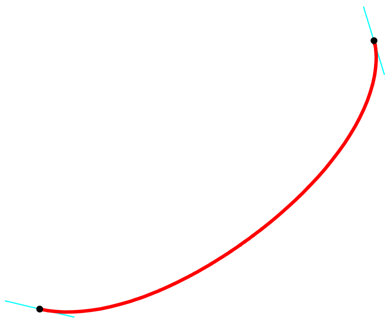
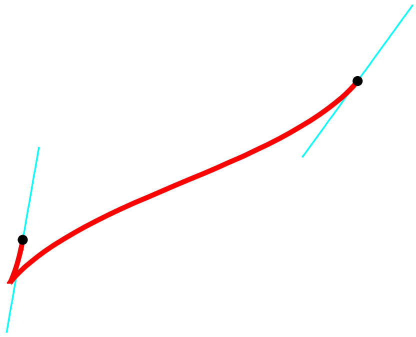


Figure 34:

4 Nonsmooth solutions

Figure 35: $(x_0, y_0, \theta_0) = (0, 0, 0)$, $(x_1, y_1, \theta_1) = (2, 0, \pi/2)$



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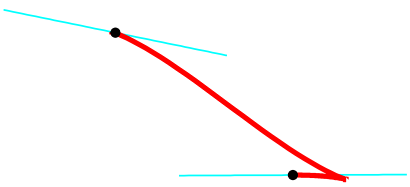

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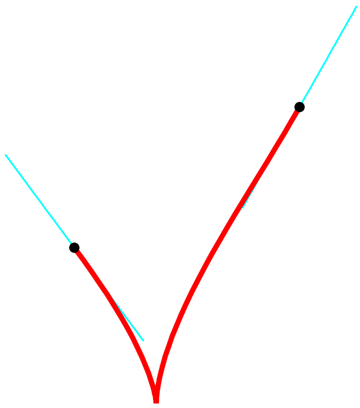

Figure 38:

References

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[3] I. Moiseev, Yu. L. Sachkov, Maxwell strata in sub-Riemannian problem on the group of motions of a plane, *ESAIM: COCV*, 2009, accepted, http://arxiv.org/abs/0807.4731v1.

[4] Yu. L. Sachkov, Cut time and optimal synthesis in sub-Riemannian problem on the group of motions of a plane, http://arxiv.org/abs/0903.0727v1, submitted.

[5] S. Wolfram, *Mathematica: a system for doing mathematics by computer*, Addison-Wesley, Reading, MA 1991.