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Complete description of the Maxwell strata in the generalized Dido problem

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Abstract. The generalized Dido problem is considered — a model of the nilpotent sub-Riemannian problem with the growth vector $(2, 3, 5)$. The Maxwell set is studied, that is, the locus of the intersection points of geodesics of equal length. A complete description is obtained for the Maxwell strata corresponding to the symmetry group of the exponential map generated by rotations and reflections. All the corresponding Maxwell times are found and located. The conjugate points that are limit points of the Maxwell set are also found. An upper estimate is obtained for the cut time (time of loss of optimality) on geodesics.

Bibliography: 12 titles.

§ 1. Introduction

1.1. Statement of the problem. The present paper is devoted to the study of the optimality of geodesics in the generalized Dido problem. This problem can be formulated as follows. Suppose that we are given two points (x_0, y_0), $(x_1, y_1) \in \mathbb{R}^2$ connected by some curve $\gamma_0 \subset \mathbb{R}^2$, a number $S \in \mathbb{R}$, and a point $c = (c_x, c_y) \in \mathbb{R}^2$. It is required to find a shortest curve $\gamma \subset \mathbb{R}^2$ connecting the points (x_0, y_0) and (x_1, y_1) such that the domain bounded by the two curves γ_0 and γ has the prescribed algebraic area S and centre of mass c.

In [1] we showed that this problem can be reformulated as the following optimal control problem in 5-dimensional space with a 2-dimensional control and an integral criterion:

$$\dot q = u_1 X_1 + u_2 X_2, \qquad q = (x, y, z, v, w) \in M = \mathbb{R}^5, \quad u = (u_1, u_2) \in U = \mathbb{R}^2,$$
$$q(0) = q_0 = 0, \qquad q(t_1) = q_1,$$
$$l = \int_0^{t_1} \sqrt{u_1^2 + u_2^2}\, dt \to \min,$$

where the vector fields at the controls have the form

$$X_1 = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z} - \frac{x^2+y^2}{2}\frac{\partial}{\partial w}, \qquad X_2 = \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z} + \frac{x^2+y^2}{2}\frac{\partial}{\partial v}.$$

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From the invariant viewpoint, this is a sub-Riemannian problem given by the distribution

$$\Delta_q = \operatorname{span}(X_1(q), X_2(q)), \qquad q \in M$$

with scalar product $\langle \cdot , \cdot \rangle$ with respect to which the fields X_1, X_2 form an orthonormal basis:

$$\langle X_i, X_j \rangle = \delta_{ij}, \qquad i, j = 1, 2.$$

The Lie algebra generated by the fields X_1, X_2 is a free nilpotent Lie algebra of length 3 with two generators. The distribution Δ has the flag

$$\Delta \subset \Delta^2 = [\Delta, \Delta] \subset \Delta^3 = [\Delta, \Delta^2] = TM$$

and the growth vector $(2, 3, 5) = (\dim \Delta_q, \dim \Delta_q^2, \dim \Delta_q^3)$.

Thus, $(\Delta, \langle \cdot , \cdot \rangle)$ is a nilpotent sub-Riemannian structure with the growth vector $(2, 3, 5)$. It is a local quasihomogeneous nilpotent approximation of an arbitrary sub-Riemannian structure on a 5-dimensional manifold with the growth vector $(2, 3, 5)$ (see [2], [3], as well as [4]). As shown in [5], such a nilpotent structure is unique. The generalized Dido problem is a model of the nilpotent sub-Riemannian problem with the growth vector $(2, 3, 5)$.

1.2. Known results. We continue the study of the generalized Dido problem started in [1], [5]–[8].

In [5] and [7] we found, respectively, the groups of continuous and discrete symmetries in this problem: there is a two-parameter continuous symmetry group (rotations and dilations), as well as a discrete symmetry group of order 4 (reflections).

In [1] we obtained a parametrization of sub-Riemannian geodesics (extremal trajectories) by the Jacobi elliptic functions. The abnormal geodesics are optimal up to infinity, and the normal ones, generally speaking, on finite time intervals. A point where a geodesic ceases to be optimal is called a *cut point*. It is known that a normal geodesic can cease to be optimal either because another geodesic with the same value of the functional hits some point on it (a Maxwell point), or because the family of geodesics has an envelope (a conjugate point).

In [8] we found the Maxwell strata MAX_i corresponding to the symmetry group preserving time on geodesics (rotations and reflections): the two hypersurfaces $z = 0$ and $V = 0$ that contain these Maxwell strata were produced in the state space M, the invariant meaning of these hypersurfaces was clarified in terms of the sub-Riemannian structure, as well as their graphical significance for Euler elastics (the projections of geodesics onto the (x, y)-plane).

1.3. Contents of the paper. The purpose of the present paper is a complete analysis of the roots of the equations $z = 0$ and $V = 0$ along geodesics.

We study solubility of these equations; in some cases they have no roots. In those cases where these equations are soluble, for each root we indicate an interval containing it, locating the roots. Moreover, on each of these intervals the corresponding root proves to be a zero of a certain monotonic function. This provides an effective algorithm for the approximate calculation of these roots.

On each geodesic we find the first point that belongs to the Maxwell strata MAX_i. On the geodesics that do not contain points of these strata we find the conjugate points that are limits of Maxwell points.

Thus, on each normal geodesic (apart from certain exceptional ones) we indicate either the first point of the strata MAX_i or the first of the conjugate points found before. But a normal geodesic cannot be optimal after Maxwell points and conjugate points. We thus obtain an upper estimate for the cut time along geodesics. On the exceptional geodesics this estimate is trivial ($+\infty$). The estimate obtained — Theorem 6.1 — is the main result of this paper.

Computer calculations show that our upper estimate is in fact equal to the cut time. So far this conjecture has been proved only for some of the geodesics.

We used the system "Mathematica" [9] to carry out complicated calculations and to produce the figures in this paper.

1.4. Information from the preceding papers. We recall some definitions and facts in [1], [5]–[8].

It follows from the Pontryagin maximum principle [10] that the extremals in the generalized Dido problem are the trajectories of the Hamiltonian system $\dot\lambda = \vec{H}(\lambda)$, $\lambda \in T^*M$, with Hamiltonian $H = (h_1^2+h_2^2)/2$, $h_i(\lambda) = \langle \lambda, X_i(q)\rangle$. The geodesics are the projections of extremals in the cotangent bundle T^*M onto the state space M: $q_t = \pi(\lambda_t)$, $\lambda_t = e^{t\vec{H}}(\lambda)$. Henceforth, $e^{t\vec{H}}$ denotes the flow of the Hamiltonian field $\vec{H}$ with Hamiltonian H.

Since the Hamiltonian H is homogeneous, it is sufficient to consider the restriction of the Hamiltonian flow to the level surface $H = 1/2$ and therefore to take initial covectors λ in the initial cylinder $C = \{H = 1/2\} \cap T^*_{q_0}M$. All the information about the geodesics is contained in the exponential map $\operatorname{Exp}\colon C \times \mathbb{R}_+ \to M$, $\operatorname{Exp}(\lambda, t) = \pi \circ e^{t\vec{H}}(\lambda) = q_t$.

The projections of the geodesics onto the plane (x, y) satisfy the differential equations

$$\dot x = \cos\theta, \quad \dot y = \sin\theta, \quad \ddot\theta = -\alpha\sin(\theta - \beta), \qquad \alpha, \beta = \mathrm{const};$$

such curves are called *Euler elastics.*

In [7] we defined and studied the reflections ε^i, $i = 1, 2, 3$; these are the discrete symmetries of the exponential map $\varepsilon^i\colon N \to N$, $\varepsilon^i\colon M \to M$, $\operatorname{Exp}\circ\varepsilon^i = \varepsilon^i \circ \operatorname{Exp}$. We denote $\nu = (\lambda, t) \in N = C \times \mathbb{R}_+$, $\nu^i = \varepsilon^i(\nu)$. Along with the discrete symmetry group $D_2 = \{\mathrm{Id}, \varepsilon^1, \varepsilon^2, \varepsilon^3\}$, the exponential map has the continuous two-parameter symmetry group $G_{\vec{h}_0, Z} = e^{\mathbb{R}\vec{h}_0} \circ e^{\mathbb{R}Z}$ (see [1]), where

$$h_0(\lambda) = \langle \lambda, X_0(q)\rangle, \qquad X_0 = -y\,\frac{\partial}{\partial x} + x\,\frac{\partial}{\partial y} - w\,\frac{\partial}{\partial v} + v\,\frac{\partial}{\partial w}\,,$$

$$Z = \vec{h}_Y + e, \qquad h_Y(\lambda) = \langle \lambda, Y(q)\rangle,$$

$$Y = x\,\frac{\partial}{\partial x} + y\,\frac{\partial}{\partial y} + 2z\,\frac{\partial}{\partial z} + 3v\,\frac{\partial}{\partial v} + 3w\,\frac{\partial}{\partial w}\,, \qquad e = \sum_{i=1}^{5} h_i\,\frac{\partial}{\partial h_i}\,.$$

The Maxwell strata generated by the rotations $\vec{h}_0$ and reflections ε^i are defined as follows:

$$\mathrm{MAX}_0 = \bigl\{\nu \in N \mid \exists\, \sigma \in \mathbb{R}\colon \widetilde{\nu} = e^{\sigma\vec{h}_0}(\nu) \neq \nu,\ \operatorname{Exp}(\widetilde{\nu}) = \operatorname{Exp}(\nu)\bigr\},$$

$$\mathrm{MAX}_i = \{\nu \in N \mid \exists\, \sigma \in \mathbb{R}\colon \widetilde{\nu} = e^{\sigma\vec{h}_0}(\nu^i) \neq \nu,\ \operatorname{Exp}(\widetilde{\nu}) = \operatorname{Exp}(\nu)\}, \quad i = 1, 2, 3.$$

A geodesic cannot be optimal after a Maxwell point. The points on geodesics which correspond to the Maxwell strata belong to the following sets ([8], Theorem 5.13):

$$\begin{aligned}
(\lambda, t) \in \mathrm{MAX}_0 \quad &\Rightarrow \quad r_t^2 + \rho_t^2 = 0, \\
(\lambda, t) \in \mathrm{MAX}_1 \quad &\Rightarrow \quad z_t = 0, \\
(\lambda, t) \in \mathrm{MAX}_2 \quad &\Rightarrow \quad V_t = 0, \\
(\lambda, t) \in \mathrm{MAX}_3 \quad &\Rightarrow \quad z_t = V_t = 0,
\end{aligned}$$

where $r^2 = x^2 + y^2$, $\rho^2 = v^2 + w^2$, $V = xv + yw - zr^2/2$. In this paper we solve the equations $z = 0$, $V = 0$, and $r^2 + \rho^2 = 0$, which define the Maxwell strata.

§ 2. Maxwell strata in the domain N_1

By the equalities $X_0 V = 0$, $YV = 4V$ and $X_0 z = 0$, $Yz = 2z$ (see [8], (1), (2)) the functions z, V can be transformed by continuous symmetries as follows:

$$(e^{sX_0})^* z = z, \quad (e^{rY})^* z = e^{2r} z, \qquad (e^{sX_0})^* V = V, \quad (e^{rY})^* V = e^{4r} V.$$

Therefore the hypersurfaces $z = 0$, $V = 0$ are invariant under the symmetry group $G_{X_0,Y} = e^{\mathbb{R}X_0} \circ e^{\mathbb{R}Y}$. The existence of a two-parameter symmetry group of the exponential map enables one to reduce the procedure of solving the equations $z = 0$, $V = 0$. Namely, let $\nu = (\lambda, t) \in N$; then for any s, r we have

$$\widehat{\nu} = e^{s\vec{h}_0} \circ e^{rZ}(\nu) = (e^{s\vec{h}_0} \circ e^{rZ}(\lambda), t') \in N, \qquad t' = te^r.$$

Setting $\mathrm{Exp}(\widehat{\nu}) = \widehat{q}_{t'} = (\widehat{x}_{t'}, \widehat{y}_{t'}, \widehat{z}_{t'}, \widehat{v}_{t'}, \widehat{w}_{t'})$ we obtain

$$z_t = 0 \quad \Leftrightarrow \quad \widehat{z}_{t'} = 0, \qquad V_t = 0 \quad \Leftrightarrow \quad \widehat{V}_{t'} = 0. \tag{1}$$

Therefore we can first solve the equations $z = 0$, $V = 0$ for any representative $\nu'' \in N''$ and then obtain the solutions for any $\nu \in N$ by using relations (1).

The initial cylinder $C = \{\lambda \in T_{q_0}^* M \mid h_1^2(\lambda) + h_2^2(\lambda) = 1\}$ can be parametrized by the coordinates $(\theta, c, \alpha, \beta)$ where

$$h_1 = \cos\theta, \quad h_2 = \sin\theta, \quad h_3 = c, \quad h_4 = \alpha \sin\beta, \quad h_5 = -\alpha\cos\beta.$$

We also use the elliptic coordinates in the inverse image of the exponential map introduced in [7]: time along the pendulum φ and the reparametrized energy of the pendulum k (as well as $\psi = \varphi/k$).

Recall the partition of the cylinder C into subsets introduced in [1]:

$$C = \bigcup_{i=1}^{7} C_i, \qquad C_i \cap C_j = \varnothing, \quad i \neq j,$$

$$\begin{aligned}
&C_1 = \{\lambda \in C \mid \alpha \neq 0,\ E \in (-\alpha, \alpha)\}, \\
&C_2 = \{\lambda \in C \mid \alpha \neq 0,\ E \in (\alpha, +\infty)\}, \\
&C_3 = \{\lambda \in C \mid \alpha \neq 0,\ E = \alpha,\ \theta - \beta \neq \pi\}, \\
&C_4 = \{\lambda \in C \mid \alpha \neq 0,\ E = -\alpha\}, \\
&C_5 = \{\lambda \in C \mid \alpha \neq 0,\ E = \alpha, \theta - \beta = \pi\}, \\
&C_6 = \{\lambda \in C \mid \alpha = 0,\ c \neq 0\}, \\
&C_7 = \{\lambda \in C \mid \alpha = c = 0\},
\end{aligned}$$

where

$$E = \frac{c^2}{2} - \alpha\cos(\theta - \beta) \in [-\alpha, +\infty)$$

is the energy of the generalized pendulum $\ddot{\theta} = -\alpha\sin(\theta - \beta)$. In accordance with the partition of the cylinder $C = \bigcup_{i=1}^{7} C_i$ we have the partition of the inverse image of the exponential map $N = \bigcup_{i=1}^{7} N_i$, where $N_i = C_i \times \mathbb{R}_+$.

2.1. Roots of the equation $z = 0$ for $\nu \in N_1$. If $\nu = (k, \varphi, \alpha, \beta, t) \in N_1$, then $\widehat{\nu} = e^{-\beta\vec{h}_0} \circ e^{(-1/2\ln\alpha)Z}(\nu) = (k, \varphi, 1, 0, \delta) \in N_1$ can be taken as a representative $\nu'' \in N_1''$.

In [1] we showed that for $\nu \in N_1$, $\alpha = 1$, $\beta = 0$

$$\begin{gathered} \varphi_t = \varphi + t, \\ x = 2(\mathrm{E}(\varphi_t) - \mathrm{E}(\varphi)) - (\varphi_t - \varphi), \\ z = 2k(\operatorname{sn}\varphi_t \, \operatorname{dn}\varphi_t - \operatorname{sn}\varphi \, \operatorname{dn}\varphi) - k(\operatorname{cn}\varphi + \operatorname{cn}\varphi_t)x. \end{gathered}$$

Henceforth we use the Jacobi elliptic functions $\operatorname{sn}(u,k)$, $\operatorname{cn}(u,k)$, $\operatorname{dn}(u,k)$, $\mathrm{E}(u,k)$ (see [11], [12]).

We pass to the new coordinates

$$\begin{gathered} \tau = \frac{\varphi_t + \varphi}{2} = \varphi + \frac{t}{2}, \qquad p = \frac{\varphi_t - \varphi}{2} = \frac{t}{2}, \\ \varphi = \tau - p, \qquad \varphi_t = \tau + p. \end{gathered}$$

By the addition formulae for elliptic functions we obtain

$$\begin{aligned} x &= 2(\mathrm{E}(\tau + p) - \mathrm{E}(\tau - p)) - 2p \\ &= 4\mathrm{E}(p) - 2p - \frac{4k^2}{\Delta}\operatorname{sn}^2\tau \, \operatorname{sn}p \, \operatorname{cn}p \, \operatorname{dn}p, \\ \Delta &= 1 - k^2\operatorname{sn}^2\tau \, \operatorname{sn}^2 p, \\ z &= \frac{4k}{\Delta}\operatorname{cn}\tau \, f_z(p), \\ f_z(p) &= \operatorname{sn}p \, \operatorname{dn}p - (2\mathrm{E}(p) - p)\operatorname{cn}p. \end{aligned} \tag{2}$$

The following lemma and especially the constant k_0 introduced in it will be important for the description of roots of the equations $z = 0$, $V = 0$. Recall that

$$E(k) = \int_0^{\pi/2} \sqrt{1 - k^2\sin^2 t}\, dt, \qquad K(k) = \int_0^{\pi/2} \frac{dt}{\sqrt{1 - k^2\sin^2 t}}$$

are the complete elliptic integrals of the first and second kind, respectively; see [11], [12].

Lemma 2.1. *The equation*

$$2E(k) - K(k) = 0, \qquad k \in [0, 1),$$

has a unique root $k_0 \in (0, 1)$. Moreover,

$$\begin{aligned} k \in [0, k_0) \quad &\Rightarrow \quad 2E - K > 0, \\ k \in (k_0, 1) \quad &\Rightarrow \quad 2E - K < 0. \end{aligned}$$

Proof. It is obvious that $E(k)$ is decreasing and $K(k)$ is increasing; hence $2E(k) - K(k)$ is decreasing for $k \in [0, 1)$. This proves the uniqueness of the root k_0. Its existence follows from the values of the functions at the end-points of the interval:

$$\begin{array}{lllll} k = 0 & \Rightarrow & K(k) = E(k) = \dfrac{\pi}{2} & \Rightarrow & 2E(k) - K(k) = \dfrac{\pi}{2}, \\ k \to 1 - 0 & \Rightarrow & K(k) \to +\infty,\ E(k) \to 1 & \Rightarrow & 2E(k) - K(k) \to -\infty. \end{array}$$

Remark. The graph of the function $k \mapsto 2E(k) - K(k)$ is given in Fig. 1. Computer calculations show that $k_0 \approx 0.909$.

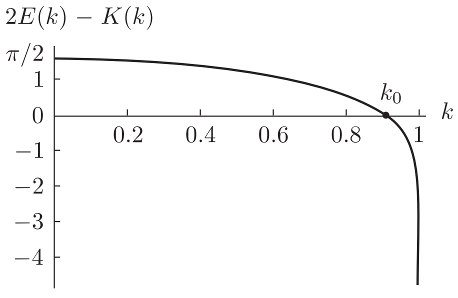

Figure 1. Definition of the number k_0

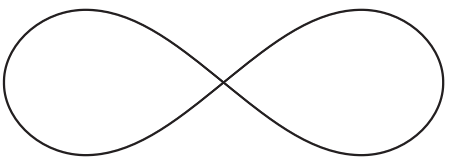

Figure 2. Periodic elastic, $k = k_0$

Corresponding to the value of the parameter $k = k_0$ there is a unique periodic Euler elastic (see Fig. 2).

It is clear from the factorization (2) that to investigate the roots of the equation $z = 0$ it is important to study the roots of the equation $f_z(p) = 0$.

Proposition 2.1. *For any* $k \in [0, 1)$ *the function*

$$f_z(p, k) = \operatorname{sn} p \,\operatorname{dn} p - (2\mathrm{E}(p) - p) \operatorname{cn} p$$

has denumerably many roots p_n^z, $n \in \mathbb{Z}$. *These roots are odd in* n:

$$p_{-n}^z = -p_n^z, \qquad n \in \mathbb{Z};$$

in particular, $p_0^z = 0$. *The roots* p_n^z *are located as follows*:

$$p_n^z \in (-K + 2Kn, K + 2Kn), \qquad n \in \mathbb{Z}.$$

In particular, the roots p_n^z *are monotonic in* n:

$$p_n^z < p_{n+1}^z, \qquad n \in \mathbb{Z}.$$

Moreover, for $n \in \mathbb{N}$

$$\begin{aligned} k \in [0, k_0) \quad &\Rightarrow \quad p_n^z \in (2Kn, K + 2Kn), \\ k = k_0 \quad &\Rightarrow \quad p_n^z = 2Kn, \\ k \in (k_0, 1) \quad &\Rightarrow \quad p_n^z \in (-K + 2Kn, 2Kn), \end{aligned}$$

where k_0 *is the unique root of the equation* $2E(k) - K(k) = 0$ (*see Lemma* 2.1).

Proof. We calculate the values of $f_z(p)$ at the points that are multiples of K. Let $p = 4Kn$; then $\operatorname{sn} p = 0$, $\operatorname{cn} p = 1$, $\operatorname{dn} p = 1$, $\mathrm{E}(p) = 4nE$, and therefore $f_z(p) = -4n(2E - K)$. Proceeding similarly, we obtain a table of values of the function $f_z(p)$ at the quarters of the period of the standard pendulum:

p	$4Kn$	$K + 4Kn$	$2K + 4Kn$	$3K + 4Kn$
$f_z(p)$	$-4n(2E - K)$	k'	$(2 + 4n)(2E - K)$	$-k'$

Here $k' = \sqrt{1 - k^2} \in (0, 1]$ is the complementary modulus of the elliptic functions.

Next, we define the function

$$g_z(p) = \frac{f_z(p)}{\operatorname{cn} p} = \frac{\operatorname{sn} p \operatorname{dn} p}{\operatorname{cn} p} - 2\mathrm{E}(p) + p, \qquad p \neq K + 2Kn.$$

A straightforward calculation shows that

$$g_z'(p) = \frac{\operatorname{sn}^2 p \operatorname{dn}^2 p}{\operatorname{cn}^2 p}\,.$$

We calculate the limits at the end-points of the intervals:

$$\begin{aligned} p \to -K + 4Kn \pm 0 \quad &\Rightarrow \quad f_z(p) \to -k', \ \operatorname{cn} p \to \pm 0 \quad \Rightarrow \quad g_z(p) \to \mp\infty, \\ p \to K + 4Kn \pm 0 \quad &\Rightarrow \quad f_z(p) \to k', \quad \operatorname{cn} p \to \mp 0 \quad \Rightarrow \quad g_z(p) \to \mp\infty. \end{aligned}$$

This means that the function $g_z(p)$ increases from $-\infty$ to $+\infty$ on each interval

$$(-K + 2Kn, K + 2Kn), \qquad n \in \mathbb{Z}, \quad n \neq 0;$$

hence it has a unique root

$$p_n^z \in (-K + 2Kn, K + 2Kn), \qquad n \in \mathbb{Z}.$$

At points of the form $p = K + 2Kn$, $n \in \mathbb{Z}$, where the function $g_z(p)$ is undefined, we have $f_z(p) = \pm k' \neq 0$. Consequently, the function $f_z(p)$ vanishes only at the points p_n^z, $n \in \mathbb{Z}$.

The fact that the roots p_n^z are odd in n follows from the fact that the function $f_z(p)$ is odd in p. The fact that the roots p_n^z are monotonic in n follows from the fact that each of the mutually disjoint intervals $(-K+2Kn, K+2Kn)$ contains exactly one root p_n^z.

It remains to locate the positive roots p_n^z with respect to the midpoints $2Kn$. As calculated above, $f_z(2Kn) = (-1)^{n-1}2n(2E-K)$, $n \in \mathbb{Z}$.

Let $k < k_0$; then $2E(k) - K(k) > 0$. First we consider the case $n = 2m \in \mathbb{N}$. Then $f_z(4Km) < 0$, $g_z(4Km) < 0$, and therefore $p_n^z > 2Kn$. If $n = 2m-1 \in \mathbb{N}$, then $f_z(4Km-2K) > 0$, $g_z(4Km-2K) < 0$, and again $p_n^z > 2Kn$.

For $k > k_0$ we have $2E(k) - K(k) > 0$ and therefore $p_n^z < 2Kn$.

Finally, for $k = k_0$ we obtain $2E(k) - K(k) = 0$ and $p_n^z = 2Kn$.

The graphs of the function $f_z(p)$ for the different values of k are given in Figs. 3–5.

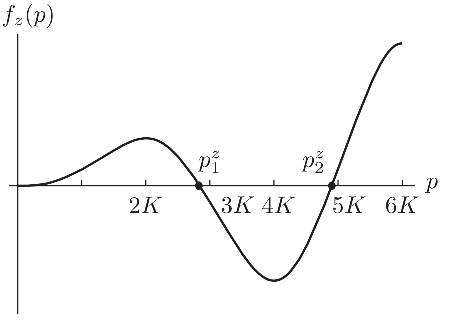

Figure 3. $p \mapsto f_z(p)$, $k \in [0, k_0)$, $\lambda \in C_1$

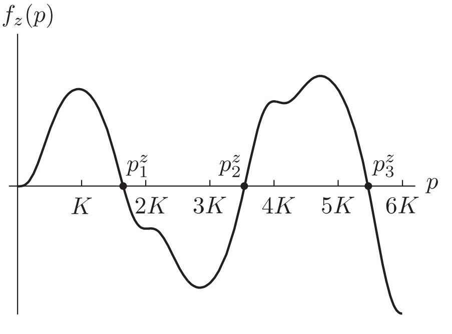

Figure 4. $p \mapsto f_z(p)$, $k \in (k_0, 1)$, $\lambda \in C_1$

Proposition 2.1 asserts that the algebraic area of the segment of an inflectional elastic changes sign infinitely many times (see [8], § 3.2).

Corollary 2.1. *The first positive root $p = p_1^z$ of the equation $f_z(p) = 0$ is located as follows*:

$$\begin{aligned} k \in [0, k_0) \quad &\Rightarrow \quad p_1^z \in (2K, 3K), \\ k = k_0 \quad &\Rightarrow \quad p_1^z = 2K, \\ k \in (k_0, 1) \quad &\Rightarrow \quad p_1^z \in (K, 2K). \end{aligned}$$

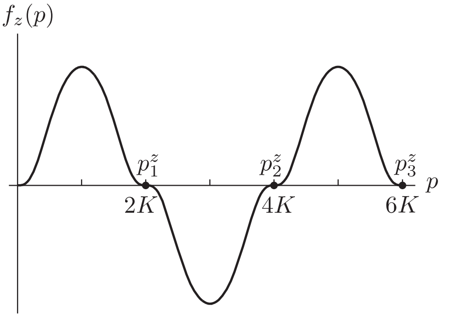

Figure 5. $p \mapsto f_z(p), \;\; k = k_0, \;\; \lambda \in C_1$

In Fig. 6 we give the graph of the function $k \mapsto p_1^z(k)$, and in Fig. 7, the graph of the function $k \mapsto p_1^z(k)/K(k)$. Recall that the value $p = 2K$ corresponds to a complete revolution of the pendulum; this value is marked on the ordinate axis in Fig. 7.

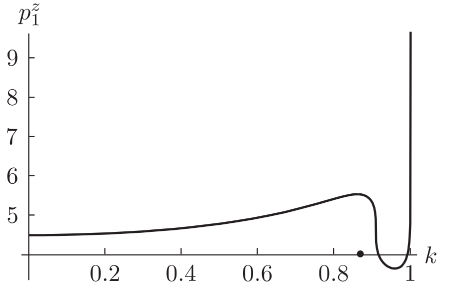

Figure 6. $k \mapsto p_1^z, \;\; \lambda \in C_1$

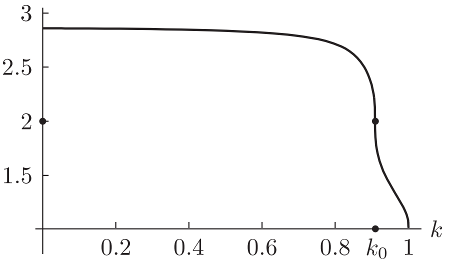

Figure 7. $k \mapsto p_1^z/K, \;\; \lambda \in C_1$

From equality (2) and Proposition 2.1 we obtain the following assertion.

Corollary 2.2. *Let $\nu \in N_1 \cap \{\alpha = 1,\ \beta = 0\}$. Then*

$$z_t = 0 \quad \Leftrightarrow \quad \begin{cases} \operatorname{cn} \tau = 0, \quad \tau = \varphi + \dfrac{t}{2}\,, \\ \ \textit{or} \\ t = 2p_n^z, \quad n \in \mathbb{Z}, \end{cases}$$

where the p_n^z are the roots of the function $f_z(p)$ described in Proposition 2.1.

2.2. Roots of the equation $V = 0$ for $\nu \in N_1$. Let $\nu \in N_1$, $\alpha = 1$, $\beta = 0$. In the coordinates

$$\tau = \varphi + \frac{t}{2}\,, \qquad p = \frac{t}{2}$$

we have

$$x = \frac{1}{\Delta}(2\Delta(2\mathrm{E} - p) - 4k^2 \operatorname{sn}^2 \tau \operatorname{cn} p \operatorname{sn} p \operatorname{dn} p), \tag{3}$$

$$y = \frac{1}{\Delta}\, 4k \operatorname{dn} \tau \operatorname{sn} \tau \operatorname{sn} p \operatorname{dn} p, \tag{4}$$

$$z = \frac{4k \operatorname{cn} \tau}{\Delta}(\operatorname{sn} p \operatorname{dn} p - (2\mathrm{E} - p) \operatorname{cn} p),$$

$$V = \frac{2k \operatorname{sn} \tau \operatorname{dn} \tau}{\Delta} f_V(p), \tag{5}$$

$$f_V(p) = \frac{4}{3} \operatorname{sn} p \operatorname{dn} p(-p - 2(1 - 2k^2 + 6k^2 \operatorname{cn}^2 p)(2\mathrm{E} - p) + (2\mathrm{E} - p)^3 + 8k^2 \operatorname{cn} p \operatorname{sn} p \operatorname{dn} p) + 4 \operatorname{cn} p(1 - 2k^2 \operatorname{sn}^2 p)(2\mathrm{E} - p)^2,$$

$$\Delta = 1 - k^2 \operatorname{sn}^2 \tau \operatorname{sn}^2 p.$$

Proposition 2.2. *For any $k \in [0, 1)$ the equation $f_V(p, k) = 0$ has denumerably many roots p_n^V, $n \in \mathbb{Z}$. These roots are odd and monotonic in n. For $n \in \mathbb{N}$ the roots p_n^V are located as follows:*

$$p_n^V \in [2Kn, 2K(n + 1)).$$

Moreover,

$$k \neq k_0 \quad \Rightarrow \quad p_n^V \in (2Kn, 2K(n + 1)),$$
$$k = k_0 \quad \Rightarrow \quad p_n^V = 2Kn.$$

Proof. We define the function

$$g_V(p) = \frac{f_V(p)}{\operatorname{sn} p \operatorname{dn} p}, \qquad p \neq 2Kn. \tag{6}$$

A straightforward calculation shows that

$$(g_V(p))' = -4\frac{(f_z(p))^2}{\operatorname{sn}^2 p \operatorname{dn}^2 p}; \tag{7}$$

hence the function $g_V(p)$ decreases on each interval $p \in (2K(n - 1), 2Kn)$.

First we consider the case $k \neq k_0$. We calculate the limits at the end-points of the intervals:

$$\begin{aligned} &p \to 4Kn \pm 0, \quad n \neq 0 && \Rightarrow \quad f_V \to 64n^2(2E-K)^2 > 0, \quad \operatorname{sn} p \to \pm 0 \\ &&& \quad\ \Rightarrow \quad g_V \to \pm\infty, \\ &p \to \pm 0 && \Rightarrow \quad f_V \to 0, \quad g_V \to 0, \\ &p \to 2K + 4Kn \pm 0 && \Rightarrow \quad f_V \to -4(4n+2)^2(2E-K)^2 < 0, \quad \operatorname{sn} p \to \mp 0 \\ &&& \quad\ \Rightarrow \quad g_V \to \pm\infty. \end{aligned}$$

Let $n \in \mathbb{N}$. Then

$$p \to 2Kn \pm 0 \quad \Rightarrow \quad g_V(p) \to \pm\infty,$$

on each interval $(2Kn, 2K(n+1))$ the function $g_V(p)$ decreases from $+\infty$ to $-\infty$ and therefore has one root p_n^V. If $n = 0$, then

$$p \to +0 \quad \Rightarrow \quad g_V(p) \to 0, \qquad p \to 2K - 0 \quad \Rightarrow \quad g_V(p) \to -\infty,$$

on the interval $(0, 2K)$ the function $g_V(p)$ decreases from 0 to $-\infty$ and therefore has no roots.

We return to the function $f_V(p)$. For $n \in \mathbb{N}$ we have

$$f_V(2Kn) = (-1)^n 16n^2(2E-K) \neq 0;$$

in addition, $f_V(0) = 0$. Therefore all the non-negative roots of the function $f_V(p)$ are given by $p = p_n^V$, $n = 0, 1, 2, \dots$.

We now consider the case $k = k_0$; then $2E(k) - K(k) = 0$. The function $f_V(p)$ vanishes at the points $p = 2Kn$, $n = 0, 1, 2, \dots$; we claim that there are no other non-negative roots. If $p \to 2Kn$, $n = 0, 1, 2, \dots$, then $g_V(p) \to -(8/3)Kn$. If we extend the function $g_V(p)$ by continuity to the points $p = 2Kn$, then $g_V(p)$ decreases from 0 to $-\infty$ for $p \in [0, +\infty)$. Therefore $g_V(p) < 0$ for $p > 0$. Consequently, in the case $k = k_0$ the function $f_V(p)$ vanishes only at the points $p = p_n^V = 2Kn$.

The fact that the roots p_n^V are odd in n follows from the fact that the function $f_V(p)$ is even in p. The monotonicity of p_n^V in n follows from the fact that the intervals $[2Kn, 2K(n+1))$, $n \in \mathbb{N}$, are disjoint for different n.

Proposition 2.2 asserts that the centre of mass of the segment of an inflectional elastic crosses the perpendicular bisector of the chord infinitely many times (see [8], § 3.2).

The graphs of the function $p \mapsto f_V(p)$ for various k are given in Figs. 8, 9.

Corollary 2.3. *For any $k \in [0,1)$ the first positive root $p = p_1^V$ of the equation $f_V(p) = 0$ is located as follows:*

$$\begin{aligned} k \neq k_0 \quad &\Rightarrow \quad p_1^V \in (2K, 4K), \\ k = k_0 \quad &\Rightarrow \quad p_1^V = 2K. \end{aligned}$$

In Figs. 10, 11 we give the graphs of the functions $k \mapsto p_1^V(k)$, $k \mapsto p_1^V(k)/K(k)$. On the ordinate axis in Fig. 11 the points $p_1^V/K = 2$ and 4 are marked, which correspond to one and two complete revolutions of the pendulum.

From Proposition 2.2 and equality (5) we obtain the following assertion.

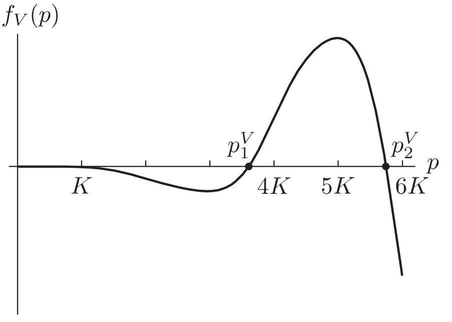

Figure 8. $p \mapsto f_V(p)$, $k \neq k_0$, $\lambda \in C_1$

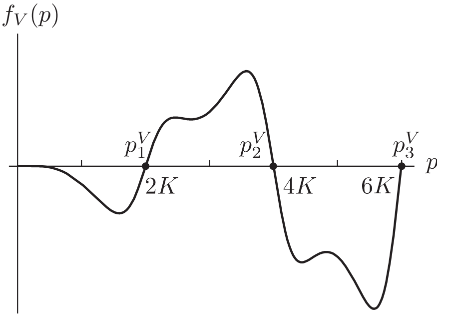

Figure 9. $p \mapsto f_V(p)$, $k = k_0$, $\lambda \in C_1$

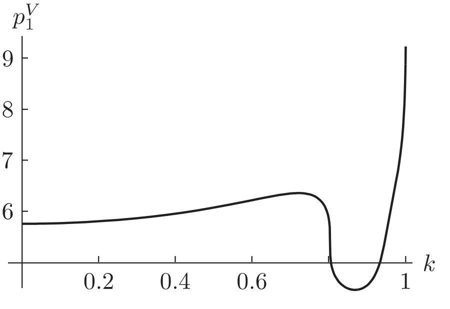

Figure 10. $k \mapsto p_1^V$, $\lambda \in C_1$

Corollary 2.4. *Let $\nu \in N_1$, $\alpha = 1$, $\beta = 0$. Then*

$$V_t = 0 \quad \Leftrightarrow \quad \begin{cases} \operatorname{sn} \tau = 0, \quad \tau = \varphi + \dfrac{t}{2}, \\ \text{or} \\ t = 2p_n^V, \quad n \in \mathbb{Z}, \end{cases}$$

where the p_n^V are the roots of the function $f_V(p)$ described in Proposition 2.2.

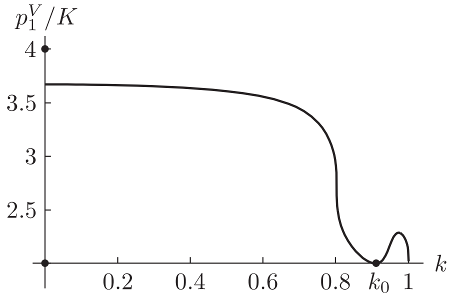

Figure 11. $k \mapsto p_1^V/K,\ \ \lambda \in C_1$

2.3. Relative positions of the roots of the equations $z = 0$ and $V = 0$ for $\nu \in N_1$. In order to determine the first Maxwell time along a geodesic it is important to know which of the equations $z_t = 0$, $V_t = 0$ has a root that is the first one occurring on this geodesic. In this subsection we answer this question.

First we describe the curve $\{f_z = 0\}$.

Lemma 2.2. *The curve $\{(p,k) \in \mathbb{R} \times [0,1) \mid f_z(p) = 0,\ p \neq 0\}$ is smooth. It has tangent parallel to the p-axis only at the points $(p,k) = (2Kn, k_0)$, $n \neq 0$. At the points $(p,k) = (p_n^z(0), 0)$ this curve has tangent parallel to the k-axis.*

Proof. We have $\partial f_z/\partial p = (2\mathrm{E} - p)\,\mathrm{dn}\,p\,\mathrm{sn}\,p$. Therefore

$$\begin{cases} f_z = 0, \\ \dfrac{\partial f_z}{\partial p} = 0 \end{cases} \Leftrightarrow \begin{cases} 2\mathrm{E} - p = \dfrac{\mathrm{sn}\,p\ \mathrm{dn}\,p}{\mathrm{cn}\,p}, \\ \dfrac{\mathrm{sn}^2 p\ \mathrm{dn}^2 p}{\mathrm{cn}\,p} = 0 \end{cases} \Leftrightarrow \begin{cases} 2\mathrm{E} - p = 0, \\ p = 2Kn \end{cases}$$

$$\Leftrightarrow \begin{cases} 2n(2E - K) = 0, \\ p = 2Kn \end{cases} \Leftrightarrow \left[\begin{array}{l} n = 0 \\ \text{or} \\ k = k_0,\ p = 2Kn. \end{array}\right.$$

Therefore for $(p,k) \neq (2Kn, k_0)$ the curve $\{f_z = 0\}$ is smooth and its tangent is not parallel to the p-axis.

We now calculate the other partial derivative:

$$\begin{aligned} \frac{\partial f_z}{\partial k} = -\frac{1}{kk'^2}\big\{&\mathrm{dn}\,p\ \mathrm{sn}\,p[k^2 + (2\mathrm{E} - p)(\mathrm{E} - (1-k^2)p)] \\ &+ \mathrm{cn}\,p[\mathrm{E}(1 - 2k^2(1 + \mathrm{sn}^2 p)) + p(-1 + k^2(1+s^2))]\big\}. \end{aligned}$$

For $k = k_0$, $p = 2Kn$ we have $\mathrm{sn}\,p = 0$, $\mathrm{cn}\,p = \pm 1$, $\mathrm{dn}\,p = 1$, $\mathrm{E} = p/2$; therefore

$$\left.\frac{\partial f_z}{\partial k}\right|_{k=k_0,\,p=2Kn} = \pm\frac{Kn}{k_0 k_0'^2} \neq 0 \qquad \text{for } n \neq 0.$$

Therefore at the points $(p,k) = (2Kn, k_0)$ the curve $\{f_z = 0\}$ is smooth and has tangent parallel to the p-axis.

Finally, for $k = 0$ we have $\partial f_z/\partial k = 0$; therefore at the points $(p,k) = (p_n^z, 0)$, $n \neq 0$, the curve $\{f_z = 0\}$ has tangent parallel to the k-axis.

Remark. Naturally, the component $\{f_z = 0,\ p = 0\} = \{p = 0\}$ is a smooth one-dimensional manifold, although along this component we have

$$f_z(0,k) = \frac{\partial f_z}{\partial p}(0,k) = \frac{\partial f_z}{\partial k}(0,k) = 0.$$

Proposition 2.3. *For every $n \in \mathbb{N}$ the function $p = p_n^z(k)$ is continuous for $k \in [0,1)$, and smooth for $k \in [0,k_0) \cup (k_0,1)$. If $k = k_0$, then $dp_n^z/dk = \infty$. If $k = 0$, then $dp_n^z/dk = 0$. If $k \to 1-0$, then $p_n^z \to +\infty$.*

Proof. The first three assertions follow from Lemma 2.2, and the fourth from the inclusion $p_n^z \in (-K + 2Kn, K + 2Kn)$ (see Proposition 2.1) and the limit $\lim_{k\to 1-0} K = +\infty$.

Proposition 2.4. *For every $n \in \mathbb{N}$ there exists a number $k_n \in (0,k_0)$ such that*

$$\begin{aligned}
k &\in [0,k_n) && \Rightarrow && p_n^z(k) < p_n^V(k),\\
k &= k_n && \Rightarrow && p_n^z(k) = p_n^V(k),\\
k &\in (k_n,k_0) && \Rightarrow && p_n^z(k) > p_n^V(k).
\end{aligned}$$

Proof. Suppose that $k \in [0,k_0)$ and therefore

$$2E(k) - K(k) > 0.$$

Let $n = 2m+1$; the case of even n is quite similar. We have

$$\begin{aligned}
f_z(2Kn) &= 2n(2E-K) > 0,\\
f_z(2K(n+1)) &= -2(n+1)(2E-K) < 0,\\
f_V(2Kn) &= -4n^2(2E-K)^2 < 0,\\
f_V(2K(n+1)) &= 4(n+1)^2(2E-K)^2 > 0.
\end{aligned}$$

From this and the fact that each function $f_z(p)$, $f_V(p)$ has a unique zero in the interval $(2Kn, 2K(n+1))$ it follows that

$$p_n^z(k) < p_n^V(k) \quad \Leftrightarrow \quad f_V(p_n^z(k),k) < 0. \tag{8}$$

Next,

$$\begin{aligned}
f_V\big|_{f_z=0} &= f_V\big|_{\mathrm{E}=(p\,\mathrm{cn}\,p+\mathrm{sn}\,p\ \mathrm{dn}\,p)/(2\,\mathrm{cn}\,p)}\\
&= \mathrm{sn}\,p\ \mathrm{dn}\,p(\mathrm{dn}^2\,p - k^2\,\mathrm{sn}^2\,p\ \mathrm{cn}^2\,p) - p\,\mathrm{cn}^3\,p =: h_V(p).
\end{aligned}$$

Therefore relation (8) can be rewritten in the form

$$p_n^z(k) < p_n^V(k) \quad \Leftrightarrow \quad \alpha_n(k) < 0, \tag{9}$$

where

$$\alpha_n(k) := h_V(p_n^z(k),k).$$

The function $\alpha_n(k)$ is continuous for $k \in [0,k_0]$ and differentiable for $k \in (0,k_0)$. We now calculate its values at the end-points of this interval.

We have $\alpha_n(0) = h_V(p_n^z(0), 0)$, $h_V(p, 0) = \operatorname{sn} p - p \operatorname{cn}^3 p$, $f_z(p, 0) = \operatorname{sn} p - p \operatorname{cn} p$. Therefore

$$p = p_n^z(0) \quad \Rightarrow \quad \operatorname{sn} p = p \operatorname{cn} p \quad \Rightarrow \quad h_V(p, 0) = p \operatorname{cn} p \operatorname{sn}^2 p. \tag{10}$$

But $p_n^z = p_{2m+1}^z \in ((4m+2)K, (4m+3)K)$; therefore $\operatorname{cn} p < 0$. Consequently, the last equality in (10) yields

$$\alpha_n(0) = h_V(p_n^z(0), 0) = p \operatorname{cn} p \operatorname{sn}^2 p < 0.$$

At the other end-point of the interval we have

$$\alpha_n(k_0) = h_V(p_n^z(k_0), k_0) = h_V(2nK(k_0), k_0).$$

If $p = 2nK = (4m+2)K$, then $\operatorname{sn} p = 0$, $\operatorname{cn} p = -1$, $\operatorname{dn} p = 1$, and therefore $h_V(2nK, k) = p = 2nK > 0$.

Thus, $\alpha_n(0) < 0$, $\alpha_n(k_0) > 0$. We shall prove that $\alpha_n'(k) > 0$ for $k \in (0, k_0)$; then $\alpha_n(k)$ has a unique zero $k_n \in (0, k_0)$ and this proposition will be proved (see relation (9)).

We have

$$\frac{d\alpha_n}{dk} = \frac{d}{k} h_V(p_n^z(k), k) = \left(\frac{\partial h_V}{\partial k}\frac{\partial f_z}{\partial p} - \frac{\partial h_V}{\partial p}\frac{\partial f_z}{\partial k}\right)\left(\frac{\partial f_z}{\partial p}\right)^{-1},$$

since $f_z(p_n^z(k), k) \equiv 0$. Next,

$$\left.\frac{\partial f_z}{\partial p}\right|_{f_z=0} = (2\mathrm{E} - p) \operatorname{sn} p \operatorname{dn} p\big|_{2\mathrm{E}-p=\operatorname{sn} p \operatorname{dn} p/\operatorname{cn} p} = \left.\frac{\operatorname{sn}^2 p \operatorname{dn}^2 p}{\operatorname{cn} p}\right|_{p=p_n^z} < 0;$$

therefore

$$\frac{d\alpha_n}{dk} > 0 \quad \Leftrightarrow \quad \beta := \left.\left(\frac{\partial h_V}{\partial k}\frac{\partial f_z}{\partial p} - \frac{\partial h_V}{\partial p}\frac{\partial f_z}{\partial k}\right)\right|_{f_z=0} < 0.$$

A straightforward calculation gives

$$\begin{aligned} \beta &= \frac{3 \operatorname{sn} p \operatorname{dn} p}{2kk'^2 \operatorname{cn} p}\beta_1, \\ \beta_1 &:= \operatorname{cn}^3 p \operatorname{sn} p \operatorname{dn} p\,(1 + 2k^2 \operatorname{cn}^2 p)p - p^2 \operatorname{cn}^4 p \\ &\qquad - 2k^2 \operatorname{sn}^2 p \operatorname{dn}^2 p(\operatorname{dn}^4 p + k^2 k'^2 \operatorname{sn}^4 p). \end{aligned}$$

By the inclusion $p_n^z = p_{2m+1}^z \in (2K + 4Km, 3K + 4Km)$ we obtain the inequalities $\operatorname{sn}(p_n^z) < 0$, $\operatorname{cn}(p_n^z) < 0$, whence

$$\beta\big|_{f_z=0} < 0 \quad \Leftrightarrow \quad \beta_1\big|_{f_z=0} < 0.$$

Next,

$$\begin{aligned} \beta_1 &= \beta_2 - 2k^2 \operatorname{sn}^2 p \operatorname{dn}^2 p(\operatorname{dn}^4 p + k^2 k'^2 \operatorname{sn}^4 p) \\ &< \beta_2 := p \operatorname{cn}^3 p[\operatorname{sn} p \operatorname{dn} p(1 + 2k^2 \operatorname{cn}^2 p) - p \operatorname{cn} p]. \end{aligned}$$

It remains to prove that $\beta_2|_{f_z=0} < 0$: if this is the case, then $\beta_1|_{f_z=0} < 0$ and therefore $\alpha_n'(k) > 0$ for $k \in (0, k_0)$.

We have

$$\beta_2 = p \,\mathrm{cn}^3 p \,\beta_3, \qquad \beta_3 := \mathrm{sn}\, p \;\mathrm{dn}\, p(1 + 2k^2 \,\mathrm{cn}^2 p) - p \,\mathrm{cn}\, p.$$

Since $\mathrm{cn}\, p|_{f_z=0} = \mathrm{cn}(p_n^z) < 0$, it remains to prove that $\beta_3|_{f_z=0} > 0$. Next,

$$\beta_3 = f_z - 2 \,\mathrm{cn}\, p \,\beta_4, \qquad \beta_4 := p - \mathrm{E} - k^2 \,\mathrm{cn}\, p \;\mathrm{sn}\, p \;\mathrm{dn}\, p;$$

therefore to complete the proof of this proposition it is sufficient to show that $\beta_4(p) > 0$ for $p \in (2nK, (2n+1)K) \ni p_n^z$.

We have

$$\frac{\partial \beta_4}{\partial p} = k^2(-1 + \mathrm{sn}^2 p(2k^2 + 3 \,\mathrm{dn}^2 p)) > -k^2;$$

therefore for $p \in (2nK, (2n+1)K)$ we obtain

$$\begin{aligned} \beta_4(p) &= \beta_4(2nK) + \int_{2nK}^{p} \frac{\partial \beta_4}{\partial p}\, dp > 2(K - E) - k^2(p - 2K) \\ &> 2(K - E) - k^2 K = (2 - k^2)K - 2E(k) = -\varphi_V(k). \end{aligned}$$

It is sufficient to prove that $\varphi_V(k) = 2E(k) - (2 - k^2)K(k) < 0$ for $k \in (0, 1)$. This is shown in the following Lemma 2.3, which completes the proof of this proposition.

Lemma 2.3. *The function*

$$\varphi_V(k) = 2E(k) - (2 - k^2)K(k)$$

is negative for $k \in (0, 1)$.

Proof. We have $\varphi_V(0) = 0$ and

$$\frac{d\varphi_V}{dk} = -\frac{k}{1 - k^2}\, \varphi_V^1(k), \qquad \varphi_V^1(k) := E - (1 - k^2)K.$$

Therefore it is sufficient to show that $\varphi_V^1(k) > 0$ for $k \in (0, 1)$. But this follows from the fact that $\varphi_V^1(0) = 0$ and $d\varphi_V^1/dk = kK > 0$.

Fig. 12 depicts the graph of the sequence k_n, $n = 0, 1, \ldots$. Computer calculations show that $k_n \to k_0$ monotonically as $n \to \infty$; therefore it would be more natural to denote $k_0 = k_\infty$. We point out that $k_1 \approx 0.802$.

We obtain the following description of the relative positions of the roots $p = p_n^z$, p_n^V of the equations $f_z = 0$, $f_V = 0$ for various k.

Proposition 2.5. *For every* $n \in \mathbb{N}$

$$\begin{aligned} k &\in [0, k_n) &&\Rightarrow \quad p_n^z(k) < p_n^V(k), \\ k &= k_n &&\Rightarrow \quad p_n^z(k) = p_n^V(k), \\ k &\in (k_n, k_0) &&\Rightarrow \quad p_n^z(k) > p_n^V(k), \\ k &= k_0 &&\Rightarrow \quad p_n^z(k) = p_n^V(k) = 2Kn, \\ k &\in (k_0, 1) &&\Rightarrow \quad p_n^z(k) < p_n^V(k). \end{aligned}$$

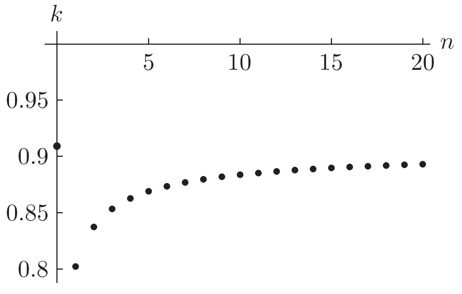

Figure 12. The graph of $n \mapsto k_n$, $\lambda \in C_1$

Proof. The disposition of the roots on the interval $k \in [0, k_0)$ was proved in Proposition 2.4. It follows from Propositions 2.1, 2.2 that $p_n^z(k) < 2Kn < p_n^V(k)$ for $k \in (k_0, 1)$. The equality of roots for $k = k_0$ follows from the same propositions.

In Fig. 13 we present the graphs of the functions $k \mapsto p_1^z(k)$, $k \mapsto p_1^V(k)$, and in Fig. 14, of the functions $k \mapsto p_1^z(k)/K(k)$, $k \mapsto p_1^V(k)/K(k)$. On the ordinate axis in Fig. 14 we have marked the points corresponding to a whole number of revolutions of the pendulum.

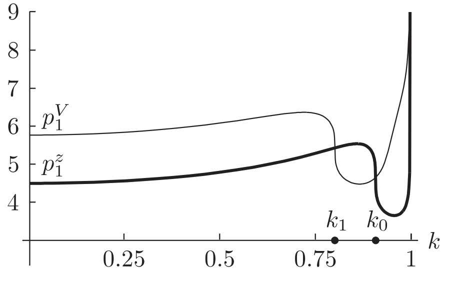

Figure 13. $k \mapsto p_1^z, p_1^V$, $\lambda \in C_1$

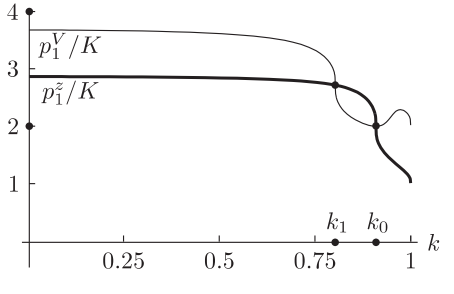

Figure 14. $k \mapsto p_1^z/K, p_1^V/K$, $\lambda \in C_1$

We can now give a description of the curve $\{f_V = 0\}$.

Lemma 2.4. *The curve* $\gamma_V = \{(p,k) \in (0,+\infty) \times [0,1) \mid f_V(p) = 0\}$ *is smooth. It has tangent parallel to the* p*-axis only at the points* $(p,k) = (p_n^V(k_n), k_n)$, $n \in \mathbb{N}$.

Proof. It follows from Proposition 2.2 that the curve γ_V decomposes into infinitely many connected components

$$\gamma_V \cap \{p \in [2Kn, 2K(n+1))\} = \{p = p_n^V(k) \mid k \in [0,1)\}, \qquad n \in \mathbb{N}.$$

We fix any $n \in \mathbb{N}$, and let $p \in [2Kn, 2K(n+1))$.

First we consider the case $k \in [0,1) \setminus \{k_n, k_0\}$. According to Propositions 2.2 and 2.4 we have $p \in (2Kn, 2K(n+1))$ on the curve γ_V. For $p \neq 2Kn$, taking equality (6) into account we obtain $f_V(p) = g_V(p) \operatorname{sn} p \operatorname{dn} p$ and therefore $f_V' = g_V' \operatorname{sn} p \operatorname{dn} p + g_V (\operatorname{sn} p \operatorname{dn} p)'$. The identity $g_V \equiv 0$ holds on the curve γ_V. Taking equality (7) into account we obtain

$$f_V'\big|_{\gamma_V} = g_V' \operatorname{sn} p \operatorname{dn} p = -\frac{4f_z^2}{\operatorname{sn} p \operatorname{dn} p},$$

which is non-zero for $k \neq k_n, k_0$. Therefore the curve γ_V is smooth for $k \neq k_n, k_0$.

If $(p,k) = (2Kn, k_0)$, then by a straightforward calculation we obtain

$$f_V'(p) = \pm \frac{8}{3} Kn \neq 0;$$

therefore the curve γ_V is also smooth for $k = k_0$.

For $k \neq k_n$ we have $f_V'(p) \neq 0$; consequently, the tangent to the curve γ_V at these points is not parallel to the p-axis.

Finally, consider the point $(p,k) = (p_n^V(k_n), k_n)$. From the equations $f_z = 0$, $f_V = 0$ at this point we obtain the equalities

$$\mathrm{E} = \frac{p \operatorname{cn} p + \operatorname{sn} p \operatorname{dn} p}{2 \operatorname{cn} p}, \qquad p = \frac{\operatorname{sn} p \operatorname{dn} p}{\operatorname{cn}^3 p}(\operatorname{dn}^2 p - k^2 \operatorname{sn}^2 p \operatorname{cn}^2 p).$$

Using these equalities we calculate the derivatives at this point:

$$\frac{\partial f_V}{\partial p} = 0, \qquad \frac{\partial f_V}{\partial k} = -\frac{2 \operatorname{sn}^4 p \operatorname{dn}^4 p (1 - (1 - \operatorname{cn}^4 p) k^2)}{k(1-k^2) \operatorname{cn}^5 p} \neq 0.$$

Therefore for $k = k_n$ the curve γ_V is smooth and has tangent parallel to the p-axis.

Proposition 2.6. *For every* $n \in \mathbb{N}$ *the function* $p = p_n^V(k)$ *is continuous for* $k \in [0,1)$, *and smooth for* $k \in [0, k_n) \cup (k_n, 1)$. *If* $k = k_n$, *then* $dp_n^V/dk = \infty$. *If* $k \to 1-0$, *then* $p_n^V \to +\infty$.

Proof. The first two assertions follow from Lemma 2.4, and the third from the inclusion $p_n^V \in [2Kn, 2K(n+1))$ (see Proposition 2.2) and the fact that $\lim_{k \to 1-0} K = +\infty$.

Remark. Lemma 2.2 characterizes the point $k = k_0$ as the unique value $k \in [0, 1)$ at which the smooth curve $\{f_z = 0\}$ has tangent parallel to the p-axis, that is,

$$\frac{dp_n^z}{dk}(k_0) = \infty.$$

Similarly, Lemma 2.4 characterizes the values $k = k_n$ for the curve $\{f_V = 0\}$: this curve has tangent parallel to the p-axis only at these points, that is,

$$\frac{dp_n^V}{k}(dk_n) = \infty.$$

As mentioned above, the value $k = k_0$ has a clear graphical meaning for elastics: to this value there corresponds the unique periodic elastic-eight (see Fig. 2). But the graphical meaning of the values $k = k_n$, $n \in \mathbb{N}$, is not at all obvious. To these values there correspond non-periodic elastics with $z = V = 0$. In other words, according to § 3.2 in [8] the segment of the elastic has zero algebraic area $z = 0$, while its centre of mass $(c_x, c_y) = (v - r^2 y/6, w + r^2 x/6)/z$ tends to infinity in the direction $(c_x^\infty, c_y^\infty) = (v - r^2 y/6, w + r^2 x/6)$, orthogonal to the chord:

$$V = xv + yw = xc_x^\infty + yc_y^\infty = 0 \quad \Leftrightarrow \quad (x, y) \perp (c_x^\infty, c_y^\infty).$$

The shortest such elastic (for $k = k_1$) is depicted in Fig. 2 in [8].

In the domain C_1 we define the function that determines the first Maxwell time:

$$p_1(k) = \min\{p_1^z(k), p_1^V(k)\}, \qquad k \in (0, 1), \quad \lambda \in C_1. \tag{11}$$

According to Proposition 2.5, in the domain C_1 we have

$$p_1(k) = \begin{cases} p_1^z(k), & k \in (0, k_1] \cup [k_0, 1); \\ p_1^V(k), & k \in [k_1, k_0]. \end{cases}$$

From Propositions 2.3, 2.6 we obtain the following description of the regularity properties of the function $p_1(k)$.

Corollary 2.5. *Let* $\lambda \in C_1$. *The function* $p_1(k)$ *is continuous on the interval* $(0, 1)$, *while at the end-points*, $\lim_{k\to+0} p_1(k) = p_1^z(0)$ *and* $\lim_{k\to1-0} p_1(k) = +\infty$. *This function is smooth at all points of the interval* $(0, 1)$ *except for* k_1 *and* k_0, *where its one-sided derivatives are calculated as follows*:

$$\begin{aligned} (p_1)'_-(k_1) &= (p_1^z)'(k_1) < \infty, \qquad & (p_1)'_+(k_1) &= (p_1^V)'(k_1) = \infty, \\ (p_1)'_-(k_0) &= (p_1^V)'(k_0) < \infty, \qquad & (p_1)'_+(k_0) &= (p_1^z)'(k_0) = \infty. \end{aligned}$$

If we extend $p_1(k)$ *by continuity to* $k = 0$ *by the value* $p_1^z(0)$, *then we obtain* $(p_1)'_+(0) = 0$.

2.4. Roots of the equation $r^2 + \rho^2 = 0$ for $\nu \in N_1$.

Lemma 2.5. *If* $\nu = (\lambda, t) \in N_1$, *then* $r_t^2 + \rho_t^2 \neq 0$.

Proof. Suppose that $r_t^2 + \rho_t^2 = 0$. Then

$$y_t = \frac{4k}{\Delta} \operatorname{sn} p \operatorname{dn} p \operatorname{sn} \tau \operatorname{dn} \tau = 0 \quad \Rightarrow \quad \operatorname{sn} p = 0 \quad \text{or} \quad \operatorname{sn} \tau = 0.$$

First consider the case $\operatorname{sn} p = 0 \;\Leftrightarrow\; p = 2Kn$, $n \in \mathbb{N}$. Then

$$x_t\big|_{p=2Kn} = 2(2\mathrm{E} - p)\big|_{p=2Kn} = 0.$$

But

$$w_t\big|_{\operatorname{sn} p = 0,\, 2\mathrm{E} - p = 0} = -\frac{2}{3} p \neq 0.$$

Thus, the equality $r_t^2 + \rho_t^2 = 0$ is impossible in the case $\operatorname{sn} p = 0$.

Now consider the case $\operatorname{sn} \tau = 0 \;\Leftrightarrow\; \tau = 2Kn$, $n \in \mathbb{N}$. Then

$$x_t\big|_{\tau=2Kn} = 2(2\mathrm{E} - p) = 0,$$
$$w_t\big|_{\tau=2Kn} = \frac{2}{3}(-p + 8k^2 \operatorname{cn} p \operatorname{sn} p \operatorname{dn} p) = 0.$$

The assertion now follows from the fact that this system of equations has no roots (this is proved in Lemma 2.6, which follows).

Lemma 2.6. *The system of equations*

$$f_1(p,k) := 2\mathrm{E} - p = 0,$$
$$f_2(p,k) := -p + 8k^2 \operatorname{cn} p \operatorname{sn} p \operatorname{dn} p = 0$$

has no solutions for $p > 0$, $k \in (0,1)$.

We define the functions

$$g_1(u,k) := 2E(u,k) - F(u,k),$$
$$g_2(u,k) := 8k^2 \cos u \sin u \sqrt{1 - k^2 \sin^2 u} - F(u,k),$$

where

$$F(u,k) = \int_0^u \frac{dt}{\sqrt{1 - k^2 \sin^2 t}}, \qquad E(u,k) = \int_0^u \sqrt{1 - k^2 \sin^2 t}\, dt$$

are elliptic integrals of the first and second kind, respectively. In view of the equalities $g_1(\operatorname{am} p, k) = f_1(p,k)$ and $g_2(\operatorname{am} p, k) = f_2(p,k)$, to prove Lemma 2.6 it is required to show that the system $g_1 = g_2 = 0$ is inconsistent for $u > 0$, $k \in (0,1)$. Recall that the amplitude $u = \operatorname{am}(p,k)$ is the inverse function of the elliptic integral $p = F(u,k)$; see [11].

Lemma 2.7. *The curve* $\{g_1(u,k) = 0\}$ *is smooth and is entirely contained in the domain* $\{k > 1/\sqrt{2},\ u \geqslant u_*\}$, *where* $u_* \in (2\pi/5, \pi/2)$ *is the unique root of the equation* $g_1(u,1) = 0$ *on the interval* $u \in (0, \pi/2)$. *Moreover,*

$$g_1(u,k) = 0, \quad u > 0, \quad k \in (0,1] \quad \Leftrightarrow \quad k = k_{g_1}(u), \quad u \in [u_*, +\infty),$$

where $k = k_{g_1}(u)$ *is a smooth function,* $k_{g_1}(u_*) = 1$.

Proof. We note the limit values of the elliptic integrals of the first and second kind:

$$F(u,1) = \frac{1}{2}\ln\frac{1+\sin u}{1-\sin u}, \quad E(u,1) = \sin u, \qquad u \in \left[0, \frac{\pi}{2}\right).$$

Therefore,

$$g_1(u,1) = 2\sin u - \frac{1}{2}\ln\frac{1+\sin u}{1-\sin u}, \qquad u \in \left[0, \frac{\pi}{2}\right).$$

It is easy to see that the function $g_1(u,1)$ has a unique zero u_* on the interval $u \in (0, \pi/2)$. Indeed, its derivative

$$\frac{\partial}{\partial u} g_1(u,1) = \frac{\cos 2u}{\cos u}$$

is positive for $u \in [0, \pi/4)$, and negative for $u \in (\pi/4, \pi/2)$. We have $g_1(0,1) = 0$; therefore $g_1(u,1) > 0$ for $u \in (0, \pi/4)$. But

$$\lim_{u \to \pi/2+0} g_1(u,1) = -\infty;$$

therefore the function $g_1(u,1)$ is strictly decreasing from $g_1(\pi/4,1) > 0$ to $-\infty$ for $u \in [\pi/4, \pi/2)$. The existence of a unique zero $u_* \in (\pi/4, \pi/2)$ is proved. Calculating the value $g_1(2/5\pi, 1) \approx 0.005 > 0$ we conclude that $u_* \in (2\pi/5, \pi/2)$.

Next,

$$\frac{\partial g_1}{\partial k} = -2k \int_0^u \frac{\sin^2 t}{\sqrt{1-k^2\sin^2 t}}\,dt - k\int_0^u \frac{\sin^2 t}{(1-k^2\sin^2 t)^{3/2}}\,dt < 0. \qquad (12)$$

Therefore the function $g_1(u,k)$ is decreasing in k, and the curve $\{g_1 = 0,\ k \in (0,1),\ u > 0\}$ is smooth.

To locate this curve we calculate the values of the function g_1 at the boundary of the strip:

$$g_1(u,0) = u, \qquad u \in [0, +\infty),$$

$$\lim_{k \to 1-0} g_1(u,k) = \begin{cases} g_1(u,1) > 0, & u \in (0, u_*); \\ g_1(u_*,1) = 0, & u = u_*; \\ g_1(u,1) < 0, & u \in \left(u_*, \dfrac{\pi}{2}\right); \\ -\infty, & u \in \left[\dfrac{\pi}{2}, +\infty\right). \end{cases}$$

Taking into account the sign of the derivative (12) we conclude that $g_1(u,k) > 0$ for $u \in (0, u_*)$, and

$$g_1(u,k) = 0 \quad \Leftrightarrow \quad k = k_{g_1}(u)$$

for $u \in [u_*, +\infty)$, $k \in (0,1]$, where the function $k = k_{g_1}(u)$ is continuous for $u \geqslant u_*$, and smooth for $u > u_*$ (see a sketch of the curve $g_1 = 0$ in Fig. 15).

It remains to prove that $k > 1/\sqrt{2}$ on the curve $g_1(u,k) = 0$. We have

$$\frac{\partial g_1}{\partial u} = \frac{1 - 2k^2\sin^2 u}{\sqrt{1-k^2\sin^2 u}} > 0 \qquad \text{for } k < \frac{1}{\sqrt{2}} \quad \text{and for } k = \frac{1}{\sqrt{2}}, \quad u \neq \pi n;$$

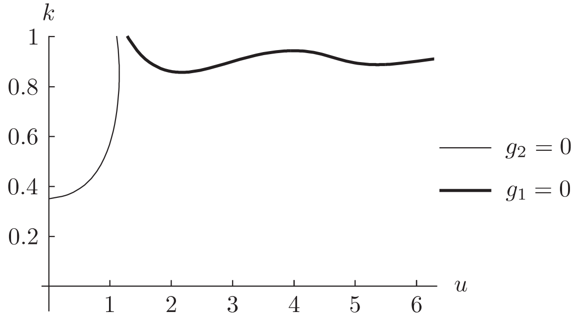

Figure 15. The curves $g_1 = 0$ and $g_2 = 0$; $\nu \in N_1$

therefore g_1 increases in u for $k \leqslant 1/\sqrt{2}$. But $g_1(0, k) = 0$; consequently, $g_1(u, k) > 0$ for $k \leqslant 1/\sqrt{2}$.

Lemma 2.8. *If* $u \geqslant 2\pi/5$, $k \geqslant 1/\sqrt{2}$, *then* $g_2(u, k) < 0$.

Proof. a) Let $u \in [\pi/2 + \pi n, \pi + \pi n]$, $n = 0, 1, 2, \ldots$, and $u > 0$. Then $\cos u \sin u \leqslant 0$, $F(u, k) > 0$, and therefore $g_2(u, k) < 0$.

b) Let $u \in [\pi + \pi n, 3\pi/2 + \pi n]$, $n \in \mathbb{N}$, $k \geqslant 1/\sqrt{2}$. Then

$$F(u, k) \geqslant F\left(\pi, \frac{1}{\sqrt{2}}\right) = 2K\left(\frac{1}{\sqrt{2}}\right) = 2\frac{(\Gamma(1/4))^2}{4\sqrt{\pi}} = 3.7\ldots > 3.7,$$

$$8k^2 \cos u \, \sin u \sqrt{1 - k^2 \sin^2 u} \leqslant 4 \sin 2u \sqrt{1 - \frac{1}{2}\sin^2 u} =: 4\alpha(u).$$

By using standard analytical methods one can prove that $\alpha(u) < 0.9$ for $u \in [\pi + \pi n, 3\pi/2 + \pi n]$; then $g_2(u, k) < 4 \cdot 0.9 - 3.7 < 0$ for $u \in [\pi + \pi n, 3\pi/2 + \pi n]$, $k \geqslant 1/\sqrt{2}$.

c) Let $u \in [2/5\pi, \pi/2]$, $k \geqslant 1/\sqrt{2}$. We consider the function

$$\alpha_1(k) = k^2 \sqrt{1 - k^2 \sin^2 u}\,, \qquad k \in \left[\frac{1}{\sqrt{2}}\,, 1\right].$$

It is easy to see that

$$\alpha_1(k) \leqslant \alpha\left(\frac{\sqrt{2}}{\sqrt{3}\,\sin u}\right) = \frac{2}{3\sqrt{3}\,\sin^2 u}$$

and therefore

$$\alpha_1(k) \leqslant \frac{2}{3\sqrt{3}\,\sin^2(2\pi/5)} = \frac{16}{3\sqrt{3}\,(5 + \sqrt{5})} = 0.42\ldots \leqslant 0.43.$$

Next,

$$\sin 2u \leqslant \sin\left(\frac{4\pi}{5}\right) = \frac{\sqrt{5 - \sqrt{5}}}{2\sqrt{2}} = 0.58\ldots < 0.59.$$

Thus,

$$4k^2 \sin 2u \sqrt{1 - k^2 \sin^2 u} \leqslant 4 \times 0.43 \times 0.59 = 1.01\ldots < 1.02.$$

Furthermore, $F(u,k) \geqslant F(2\pi/5, 1/\sqrt{2}\,) = 1.52\ldots > 1.52$. Therefore $g_2(u,k) \leqslant 1.02 - 1.52 < 0$ for $u \in [2\pi/5, \pi/2]$, $k \geqslant 1/\sqrt{2}$.

We can now prove Lemma 2.6.

Proof of Lemma 2.6. By Lemma 2.7 the curve $g_1(u,k) = 0$ is entirely contained in the domain $\{k > 1/\sqrt{2},\ u \geqslant 2\pi/5\}$. But Lemma 2.8 implies that the curve $g_2(u,k) = 0$ does not intersect this domain. Therefore the system of equations $g_1(u,k) = g_2(u,k) = 0$ is inconsistent, as is the system $f_1(p,k) = f_2(p,k) = 0$ given in the hypothesis of Lemma 2.6. The graphs of $g_1(u,k) = 0$ and $g_2(u,k) = 0$ are given in Fig. 15.

2.5. Complete description of the Maxwell strata in the domain N_1. From Theorem 5.13 in [8] and Lemma 2.5 we obtain a general description of the Maxwell strata in the domain N_1.

Theorem 2.1. *The following hold*:

0) $\mathrm{MAX}_0 \cap N_1 = \varnothing$;
1) $\mathrm{MAX}_1 \cap N_1 = \{(\lambda, t) \in N_1 \mid z_t = 0,\ \mathrm{cn}\,\tau \neq 0\}$;
2) $\mathrm{MAX}_2 \cap N_1 = \{(\lambda, t) \in N_1 \mid V_t = 0,\ \mathrm{sn}\,\tau \neq 0\}$;
3) $\mathrm{MAX}_3 \cap N_1 = \{(\lambda, t) \in N_1 \mid z_t = V_t = 0\}$,

where $\tau = \sqrt{\alpha}\,(\varphi + t/2)$.

From this theorem and Corollaries 2.2, 2.4 we obtain a complete description of the Maxwell strata in N_1.

Theorem 2.2. *The following hold*:

$$1)\ \mathrm{MAX}_1 \cap N_1 = \left\{(\lambda, t) \in N_1 \;\middle|\; t = \frac{2p_n^z}{\sqrt{\alpha}},\ n \in \mathbb{N},\ \mathrm{cn}\,\tau \neq 0\right\};$$

$$2)\ \mathrm{MAX}_2 \cap N_1 = \left\{(\lambda, t) \in N_1 \;\middle|\; t = \frac{2p_n^V}{\sqrt{\alpha}},\ n \in \mathbb{N},\ \mathrm{sn}\,\tau \neq 0\right\};$$

$$3)\ \mathrm{MAX}_3 \cap N_1 = \left\{(\lambda, t) \in N_1 \;\middle|\; (k,t) = \left(k_0, \frac{4Kn}{\sqrt{\alpha}}\right) \text{ or } (k,t) = \left(k_n, \frac{2p_n^z}{\sqrt{\alpha}}\right) = \left(k_n, \frac{2p_n^V}{\sqrt{\alpha}}\right),\ n \in \mathbb{N}\right\},$$

where $\tau = \sqrt{\alpha}\,(\varphi + t/2)$ *and the roots* p_n^z, p_n^V *are defined in Propositions* 2.1, 2.2.

We define the first Maxwell time corresponding to the stratum MAX_i:

$$t_1^{\mathrm{MAX}_i}(\lambda) = \inf\{t > 0 \mid (\lambda, t) \in \mathrm{MAX}_i\}, \qquad i = 0, 1, 2, 3,$$

and the first Maxwell time corresponding to all the strata MAX_i:

$$t_1^{\mathrm{MAX}}(\lambda) = \min\{t_1^{\mathrm{MAX}_i},\ i = 0, 1, 2, 3\}.$$

Theorem 2.3. *Let* $\lambda \in C_1$. *Then*

$$
\begin{aligned}
&k \in (0,k_1) \cup (k_0,1), \quad \operatorname{cn}\tau \neq 0 &&\Rightarrow && t_1^{\mathrm{MAX}} = t_1^{\mathrm{MAX}_1} = \frac{2}{\sqrt{\alpha}}\, p_1 = \frac{2}{\sqrt{\alpha}}\, p_1^z,\\
&k \in (0,k_1) \cup (k_0,1), \quad \operatorname{cn}\tau = 0 &&\Rightarrow && t_1^{\mathrm{MAX}} = t_1^{\mathrm{MAX}_2} = \frac{2}{\sqrt{\alpha}}\, p_1^V,\\
&k \in (k_1,k_0), \quad \operatorname{sn}\tau \neq 0 &&\Rightarrow && t_1^{\mathrm{MAX}} = t_1^{\mathrm{MAX}_2} = \frac{2}{\sqrt{\alpha}}\, p_1 = \frac{2}{\sqrt{\alpha}}\, p_1^V,\\
&k \in (k_1,k_0), \quad \operatorname{sn}\tau = 0 &&\Rightarrow && t_1^{\mathrm{MAX}} = t_1^{\mathrm{MAX}_1} = \frac{2}{\sqrt{\alpha}}\, p_1^z,\\
&k = k_1, k_0, \quad \operatorname{sn}\tau \neq 0 &&\Rightarrow && t_1^{\mathrm{MAX}} = t_1^{\mathrm{MAX}_1} = t_1^{\mathrm{MAX}_3}\\
&&&&& = \frac{2}{\sqrt{\alpha}}\, p_1 = \frac{2}{\sqrt{\alpha}}\, p_1^z = \frac{2}{\sqrt{\alpha}}\, p_1^V,\\
&k = k_1, k_0, \quad \operatorname{cn}\tau \neq 0 &&\Rightarrow && t_1^{\mathrm{MAX}} = t_1^{\mathrm{MAX}_2} = t_1^{\mathrm{MAX}_3}\\
&&&&& = \frac{2}{\sqrt{\alpha}}\, p_1 = \frac{2}{\sqrt{\alpha}}\, p_1^z = \frac{2}{\sqrt{\alpha}}\, p_1^V,
\end{aligned}
$$

where $\tau = \sqrt{\alpha}\,(\varphi + t/2)$ *and the roots* p_1^z, p_1^V *are defined in Propositions* 2.1, 2.2.

Proof. This follows from Theorem 2.2 and the estimates of the roots p_1^z, p_1^V in Proposition 2.5.

§ 3. Maxwell strata in the domain N_2

3.1. Roots of the equation $z = 0$ for $\nu \in N_2$. Let $\nu = (\lambda, t) \in N_2$, $\alpha = 1$, $\beta = 0$. Then (see [1])

$$
\begin{gathered}
x = \frac{2}{k}\left(\mathrm{E}(\psi_t) - \mathrm{E}(\psi) - \frac{2-k^2}{2}(\psi_t - \psi)\right),\\
z = 2(\operatorname{sn}\psi_t \, \operatorname{cn}\psi_t - \operatorname{sn}\psi \, \operatorname{cn}\psi) - \frac{1}{k}(\operatorname{dn}\psi + \operatorname{dn}\psi_t)x,\\
\psi_t = \psi + \frac{t}{k}\,.
\end{gathered}
$$

We pass to the variables

$$
\tau = \frac{\psi + \psi_t}{2} = \psi + \frac{t}{2k}\,, \qquad p = \frac{\psi_t - \psi}{2} = \frac{t}{2k}\,.
$$

From the addition formula for elliptic functions we obtain

$$
\begin{gathered}
x = \frac{4}{k}\,\mathrm{E} - \frac{4k}{\Delta}\,\operatorname{sn}^2\tau\,\operatorname{sn}p\,\operatorname{cn}p\,\operatorname{dn}p - \frac{2(2-k^2)}{k}\,p,\\
z = \frac{2}{k\Delta^2}\operatorname{dn}\tau\,(2k\operatorname{sn}p\,\operatorname{cn}p(\operatorname{cn}^2\tau - \operatorname{sn}^2\tau\,\operatorname{dn}^2 p) - \Delta\,\operatorname{dn}p\,x) = \frac{2\operatorname{dn}\tau}{k\Delta}\,f_z(p), \qquad (13)\\
f_z(p) = \frac{2}{k}[\operatorname{dn}p((2-k^2)p - 2\mathrm{E}) + k^2\operatorname{sn}p\,\operatorname{cn}p],\\
\Delta = 1 - k^2\operatorname{sn}^2\tau\,\operatorname{sn}^2 p.
\end{gathered}
$$

Proposition 3.1. *The function $f_z(p)$ has no root $p \neq 0$.*

Proof. The function

$$g_z(p) := \frac{f_z(p)}{\operatorname{dn} p}$$

has the same zeros as $f_z(p)$. But

$$g_z'(p) = \frac{k^4 \operatorname{cn}^2 p \operatorname{sn}^2 p}{\operatorname{dn}^2 p} \geqslant 0$$

(equality holds only for $p = Kn$, $n \in \mathbb{Z}$); therefore $g_z(p)$ is increasing on the entire real line. Taking into account that $g_z(0) = 0$ we obtain $g_z(p) \neq 0$ and therefore also $f_z(p) \neq 0$ for $p \neq 0$.

From equality (13) and Proposition 3.1 we obtain the following.

Corollary 3.1. *Let $\nu = (\lambda, t) \in N_2 \cap \{\alpha = 1,\ \beta = 0\}$. Then the equation $z_t = 0$ has no roots.*

In other words, the segments of non-inflectional elastics have non-zero algebraic area (see [8], § 3.2).

3.2. Roots of the equation $V = 0$ for $\nu \in N_2$. Let $\nu \in N_2$, $\alpha = 1$, $\beta = 0$. Then

$$V = xv + yw - \frac{1}{2}\, zr^2 = \frac{2 \operatorname{cn}\tau \operatorname{sn}\tau}{k^2 \Delta} f_V(p), \tag{14}$$

$$\begin{aligned} f_V(p) = \frac{4}{3}\big\{&3 \operatorname{dn} p\,(2\mathrm{E} - (2-k^2)p)^2 + \operatorname{cn} p\big[8\mathrm{E}^3 - 4\mathrm{E}(4+k^2) - 12\mathrm{E}^2(2-k^2)p \\ &+ 6\mathrm{E}(2-k^2)^2p^2 + p(16 - 4k^2 - 3k^4 - (2-k^2)^3p^2)\big] \operatorname{sn} p \\ &- 2 \operatorname{dn} p\,(-4k^2 + 3(2\mathrm{E} - (2-k^2)p)^2) \operatorname{sn}^2 p \\ &+ 12k^2 \operatorname{cn} p(2\mathrm{E} - (2-k^2)p) \operatorname{sn}^3 p - 8k^2 \operatorname{sn}^4 p \operatorname{dn} p\big\}. \end{aligned} \tag{15}$$

Proposition 3.2. *For any $k \in (0,1)$ the function $f_V(p)$ given by equality (15) has denumerably many zeros $p = p_n^V(k)$, $n \in \mathbb{Z}$. The roots p_n^V are odd and monotonic in n. The positive roots are located as follows:*

$$p_n^V \in (Kn, K + Kn), \qquad n \in \mathbb{N}.$$

Proof. Consider the function

$$g_V(p) := \frac{f_V(p)}{\operatorname{sn} p \operatorname{cn} p}, \qquad p \neq Kn, \quad n \in \mathbb{Z}. \tag{16}$$

We have

$$g_V'(p) = -\frac{k^2}{\operatorname{sn}^2 p \operatorname{cn}^2 p}(f_z(p))^2 < 0. \tag{17}$$

Consequently, the function $g_V(p)$ is decreasing on each interval $p \in (Kn, K + Kn)$, $n \in \mathbb{Z}$. We calculate the limits of $g_V(p)$ at the end-points of these intervals.

Let $n \in \mathbb{N}$. Then

$$f_V(2nK) = 8n\varphi_V^2(k) > 0, \qquad f_V((2n+1)K) = -4(2n+1)k'\varphi_V^2(k) < 0, \quad (18)$$

where $\varphi_V(k) = 2E(k) - (2-k^2)K(k) < 0$ by Lemma 2.3. Next,

$$\lim_{p \to 2Kn \pm 0} \operatorname{sn} p \operatorname{cn} p = \pm 0, \qquad \lim_{p \to K+2Kn \pm 0} \operatorname{sn} p \operatorname{cn} p = \mp 0.$$

Consequently,

$$\lim_{p \to Kn \pm 0} g_V(p) = \pm\infty, \qquad n \in \mathbb{N}.$$

To calculate the limit as $p \to 0$ we observe that

$$f_V(p) = -\frac{4}{45}\, k^8 p^6 + o(p^6), \quad \operatorname{sn} p \operatorname{cn} p = p + o(p), \qquad p \to 0,$$

and therefore

$$g_V(p) = -\frac{4}{45}\, k^8 p^5 + o(p^5) \to -0 \quad \text{as} \quad p \to +0.$$

Thus, the function $g_V(p)$ decreases from 0 to $-\infty$ on the interval $(0, K)$, and from $+\infty$ to $-\infty$ on the intervals $(Kn, K + Kn)$, $n \in \mathbb{N}$. Therefore the function $g_V(p)$ has a unique root $p = p_n^V$ on each interval $(Kn, K + Kn)$, $n \in \mathbb{N}$. The function $f_V(p)$ has the same zeros, since $f_V(Kn) \neq 0$ (see (18)).

The fact that the roots of the function $f_V(p)$ are odd follows from the fact that this function is even in p and from the equality $f_V(0) = 0$.

Proposition 3.2 describes the points where the centre of mass of a segment of a non-inflectional elastic crosses the perpendicular bisector of the chord (see [8], § 3.2).

Fig. 16 shows the graph of the function $p \mapsto f_V(p)$, and Figs. 17, 18 show the graphs of the functions $k \mapsto p_1^V$, $k \mapsto p_1^V/K$. On the ordinate axis in Fig. 18 we have marked the points $p/K \in \mathbb{Z}$ corresponding to a whole number of revolutions of the pendulum.

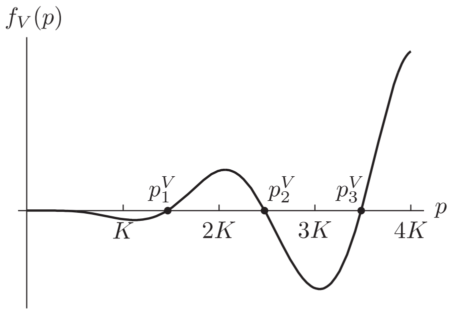

Figure 16. The graph of $p \mapsto f_V(p)$, $\lambda \in C_2$

We now describe the regularity properties of the curve $\{f_V = 0\}$ and of the functions $p = p_n^V(k)$ defining this curve.

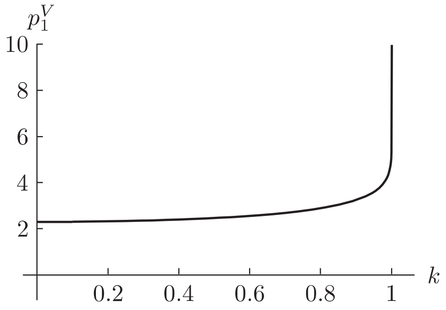

Figure 17. The graph of $k \mapsto p_1^V$, $\lambda \in C_2$

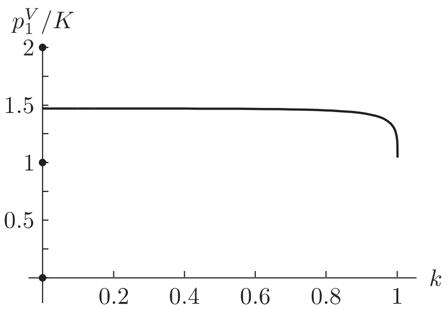

Figure 18. The graph of $k \mapsto p_1^V/K$, $\lambda \in C_2$

Lemma 3.1. *The curve* $\gamma_V = \{(p,k) \in (0,+\infty) \times (0,1) \mid f_V(p,k) = 0\}$ *is smooth and its tangent is nowhere parallel to the* p*-axis. The functions* $p = p_n^V(k)$, $k \in (0,1)$, $n \in \mathbb{N}$, *are smooth.*

Proof. By Proposition 3.2, on the curve γ_V we have $\operatorname{sn} p \operatorname{cn} p \neq 0$ and therefore $f_V(p) = \operatorname{sn} p \operatorname{cn} p\, g_V(p)$ (see (16)). Taking into account (17) we obtain

$$f'_V(p)\big|_{\gamma_V} = \operatorname{sn} p \operatorname{cn} p\, g'_V(p) \neq 0.$$

We now analyse the asymptotic behaviour of the functions $p = p_n^V(k)$ at the end-points of the interval $(0,1)$.

Proposition 3.3. *If* $k \to 1-0$, *then* $p_n^V(k) \to +\infty$ *for every* $n \in \mathbb{N}$.

Proof. This follows from the inclusion $p_n^V \in (Kn, K + Kn)$ (see Proposition 3.2) and the fact that $\lim_{k\to 1-0} K = +\infty$.

Proposition 3.4. *The function*

$$f_V^0(p) := \frac{1}{512}\left[(32p^2 - 1)\cos 2p - 8p \sin 2p + \cos 6p\right]$$

has denumerably many zeros $p = p_n^V(0)$, $n \in \mathbb{Z}$. *The roots* $p_n^V(0)$ *are odd and monotonic in* n. *The positive roots are located as follows*:

$$p_n^V(0) \in \left(\frac{\pi}{2}n, \frac{\pi}{2}(n+1)\right), \qquad n \in \mathbb{N}. \tag{19}$$

Extend the roots $p = p_n^V(k)$, $k \in (0,1)$ (*see Proposition* 3.2) *by the value* $p = p_n^V(0)$ *for* $k = 0$. *Then the function* $p = p_n^V(k)$ *is smooth on the interval* $k \in [0,1)$ *and* $(p_n^V)'_+(0) = 0$.

Proof. We observe that

$$f_V^0\left(\frac{\pi}{2}n\right) = \frac{(-1)^n}{64}\pi^2 n^2, \qquad n \in \mathbb{Z}.$$

Next, the function

$$g_V^0(p) := \frac{f_V^0(p)}{\sin p \, \cos p}$$

is decreasing on each interval $(\pi n/2, \pi(n+1)/2)$, $n \in \mathbb{Z}$, since

$$(g_V^0(p))' = -\frac{(\sin 4p - 4p)^2}{256 \sin^2 p \, \cos^2 p} < 0, \qquad p \neq \frac{\pi}{2}n.$$

We calculate the limits of this function at the end-points of the intervals:

$$\begin{aligned} n \neq 0 \quad &\Rightarrow \quad g_V^0(p) \to \pm\infty \quad \text{as} \;\; p \to \frac{\pi}{2}n \pm 0, \\ n = 0 \quad &\Rightarrow \quad g_V^0(p) \to \mp 0 \quad\;\; \text{as} \;\; p \to \pm 0. \end{aligned}$$

Hence the function $g_V^0(p)$ has a unique root on each interval $(\pi n/2, \pi(n+1)/2)$, $n \in \mathbb{N}$. The function $g_V^0(p)$ is negative on the interval $(0, \pi/2)$. All the zeros of the function $f_V^0(p)$ are exhausted by the roots $p_n^V(0)$. The assertions on the location and monotonicity of the roots follow from the preceding arguments; the fact that the roots are odd in n follows from the fact that the function $f_V^0(p)$ is even.

The fact that the extended function $p = p_n^V(k)$ is smooth at $k = 0$ follows from the Taylor expansion

$$f_V(p,k) = k^8 f_V^0(p) + O(k^{10}), \qquad k \to 0, \tag{20}$$

which can be obtained from the asymptotics of elliptic functions given in the Supplement and from the regularity of the roots $p = p_n^V(0)$ of the function $f_V^0(p)$. The equality $(p_n^V)'_+(0) = 0$ follows from the expansion (20) and the regularity of the positive roots of the function $f_V^0(p)$.

From equality (14) and Proposition 3.2 we obtain the following.

Proposition 3.5. *Let* $\nu = (\lambda, t) \in N_2 \cap \{\alpha = 1, \; \beta = 0\}$. *Then*

$$V = 0 \quad \Leftrightarrow \quad \begin{cases} \operatorname{sn}\tau \; \operatorname{cn}\tau = 0, \quad \tau = \psi + \dfrac{t}{2k}, \\ \;\textit{or} \\ t = 2k\, p_n^V, \quad n \in \mathbb{N}, \end{cases}$$

where the roots p_n^V *are defined in Proposition* 3.2.

3.3. Roots of the equation $r^2 + \rho^2 = 0$ for $\nu \in N_2$.

Lemma 3.2. *Let $\nu \in N_2$. If $r_t^2 + \rho_t^2 = 0$, then $f_V(p) = 0$.*

Proof. Let $r_t^2 + \rho_t^2 = 0$. If $r_t^2 = x_t^2 + y_t^2 = 0$, then $V_t = x_t v_t + y_t w_t - z_t r_t^2/2 = 0$. We use the factorization (14).

a) If $\operatorname{cn}\tau \operatorname{sn}\tau \neq 0$, then we obtain from the factorization (14) that $f_V(p) = 0$.

b) Suppose that $\operatorname{sn}\tau = 0$. We have

$$x_t\big|_{\operatorname{sn}\tau=0} = \frac{2}{k}(2\mathrm{E} - (2-k^2)p) = 0 \quad \Rightarrow \quad \mathrm{E} = \frac{2-k^2}{2}p, \tag{21}$$

$$w_t\big|_{\operatorname{sn}\tau=0,\,\mathrm{E}=(2-k^2)p/2} = \frac{2}{3k}(8\operatorname{cn}p\operatorname{sn}p\operatorname{dn}p - k^2 p) = 0$$

$$\Rightarrow \quad p = \frac{8}{k^2}\operatorname{cn}p\operatorname{sn}p\operatorname{dn}p. \tag{22}$$

One can verify directly that

$$\mathrm{E} = \frac{2-k^2}{2}p, \quad p = \frac{8}{k^2}\operatorname{cn}p\operatorname{sn}p\operatorname{dn}p \quad \Rightarrow \quad f_V(p) = 0.$$

c) The case $\operatorname{cn}\tau = 0$ is considered in similar fashion. First we note that

$$x_t\big|_{\operatorname{cn}\tau=0} = \frac{2}{k}(2\mathrm{E} - (2-k^2)p) - \frac{4k}{\operatorname{dn}p}\operatorname{cn}p\operatorname{sn}p = 0$$

$$\Rightarrow \quad \mathrm{E} = \frac{2-k^2}{2}p + \frac{k^2}{\operatorname{dn}p}\operatorname{cn}p\operatorname{sn}p, \tag{23}$$

$$w_t\big|_{\operatorname{cn}\tau=0,\,\mathrm{E}=(2-k^2)p/2+(k^2/\operatorname{dn}p)\operatorname{cn}p\operatorname{sn}p} = -\frac{2}{3k}\left(p + 8(1-k^2)\frac{\operatorname{cn}p\operatorname{sn}p}{k^2\operatorname{dn}^3 p}\right) = 0$$

$$\Rightarrow \quad p = -8(1-k^2)\frac{\operatorname{cn}p\operatorname{sn}p}{k^2\operatorname{dn}^3 p}. \tag{24}$$

Then we verify directly that

$$\mathrm{E} = \frac{2-k^2}{2}p + \frac{k^2}{\operatorname{dn}p}\operatorname{cn}p\operatorname{sn}p, \quad p = -8(1-k^2)\frac{\operatorname{cn}p\operatorname{sn}p}{k^2\operatorname{dn}^3 p} \quad \Rightarrow \quad f_V(p) = 0.$$

We introduce notation for the functions that appeared in equalities (21)–(24):

$$f_1(p) = 2\mathrm{E} - (2-k^2)p, \qquad f_2(p) = 8\operatorname{cn}p\operatorname{sn}p\operatorname{dn}p - k^2 p,$$

$$f_3(p) = 2\mathrm{E} - (2-k^2)p - 2k^2\frac{\operatorname{cn}p\operatorname{sn}p}{\operatorname{dn}p}, \qquad f_4(p) = p + 8(1-k^2)\frac{\operatorname{cn}p\operatorname{sn}p}{k^2\operatorname{dn}^3 p}.$$

From the proof of Lemma 3.2 we obtain the following assertion.

Corollary 3.2. *Let $\nu \in N_2$. Then*

$$r_t^2 + \rho_t^2 = 0 \quad \Leftrightarrow \quad \left\{\begin{array}{l} \operatorname{sn}\tau = 0,\ f_1(p) = f_2(p) = 0 \\ \text{or} \\ \operatorname{cn}\tau = 0,\ f_3(p) = f_4(p) = 0. \end{array}\right.$$

We now analyse the structure of the set

$$S_{12} := \{(p,k) \in \mathbb{R} \times (0,1) \mid f_1(p,k) = f_2(p,k) = 0\}.$$

We set

$$g_1(u) = 2E(u) - (2-k^2)F(u), \qquad g_2(u) = 8\cos u\,\sin u\sqrt{1-k^2\sin^2 u} - k^2F(u),$$

so that $f_i(p) = g_i(\operatorname{am} p)$, $i = 1, 2$. We have the Taylor expansion

$$g_2(u,k) = \sum_{n=0}^{\infty} g_2^n k^{2n}, \qquad k \in [0,1),$$
$$g_2^0 = 4\sin 2u, \qquad g_2^1 = -(u + 2\sin^2 u\,\sin 2u),$$
$$g_2^n = \frac{(2n-3)!!}{2^{n-1}n!}\left(n\int_0^u \sin^{2n-2} t\,dt + 2\sin^{2n} u\,\sin 2u\right).$$

Lemma 3.3. *The following hold*:
1) $g_2^1(u) < 0$ *for* $u > 0$;
2) $g_2^{n+1}(u) < g_2^n(u)$ *for* $u > 0$;
3) $g_2^n(u) < 0$ *for* $u > 0$, $n \in \mathbb{N}$;
4) $\partial g_2/\partial k < 0$ *for* $u > 0$, $k \in (0,1)$.

Proof. Part 1) can be proved by standard analytical methods.

Part 2) can be proved by an elementary transformation of the difference $g_2^{n+1} - g_2^n$ using the identity

$$\int \sin^n t\,dt = \frac{\sin^{n+1} t\,\cos t}{n+1} + \frac{n+2}{n+1}\int \sin^{n+2} t\,dt, \qquad n \neq -1.$$

Part 3) follows immediately from parts 1) and 2).

Part 4) follows from part 3) in view of the expansion

$$\frac{\partial g_2}{\partial k} = 2\sum_{n=1}^{\infty} g_2^n k^{2n-1}, \qquad k \in [0,1).$$

Taking into account the explicit expression for $g_2^0(u)$ we obtain the following assertion from part 4) of the preceding lemma and from the implicit function theorem.

Corollary 3.3. *The curve* $\{g_2(u) = 0\}$ *is smooth and is contained in the domain*

$$\left\{u \in \left(\pi n, \frac{\pi}{2} + \pi n\right), \quad n = 0, 1, 2, \dots, \quad k \in (0,1)\right\}.$$

There exists a function $k = k_{g_2}(u)$ *that is smooth on the intervals* $u \in (\pi n, \pi/2 + \pi n)$ *and continuous at their end-points such that*

$$g_2(u,k) = 0 \quad \Leftrightarrow \quad k = k_{g_2}(u),$$

and $k_{g_2}(\pi n + 0) = k_{g_2}(\pi/2 + \pi n - 0) = 0$.

Lemma 3.4. *The set of points* $\{g_1 = g_2 = 0\}$ *is contained in the domain*

$$\left\{u \in \left(\pi n, \frac{\pi}{2} + \pi n\right),\ n \in \mathbb{N}\right\}.$$

Proof. From Corollary 3.3 we obtain the inclusion

$$\{g_2 = 0\} \subset \{u \in (\pi n, \pi/2 + \pi n),\ n = 0, 1, 2, \dots\}.$$

It remains to prove that $u \notin (0, \pi/2)$ on the set $\{g_1(u) = g_2(u) = 0\}$. This follows from Lemma 3.2 and Proposition 3.2: we obtain successively

$$\{f_1(p) = f_2(p) = 0\} \subset \{f_V(p) = 0\} \subset \{p \in (Kn, K + Kn),\ n \in \mathbb{N}\};$$

therefore $p \notin (0, K)$ for $f_1(p) = f_2(p) = 0$.

Lemma 3.5. *The set of points* $\{g_1 = g_2 = 0\}$ *is denumerable and each strip* $\{u \in (\pi n, \pi/2 + \pi n)\}$, $n \in \mathbb{N}$, *contains at least one point of this set.*

Remark. Computer calculations show that each strip $\{u \in (\pi n, \pi/2 + \pi n)\}$, $n \in \mathbb{N}$, contains exactly one point of the set $\{g_1 = g_2 = 0\}$ (see the graphs of the curves $\{g_1 = 0\}$ and $\{g_2 = 0\}$ in Fig. 19).

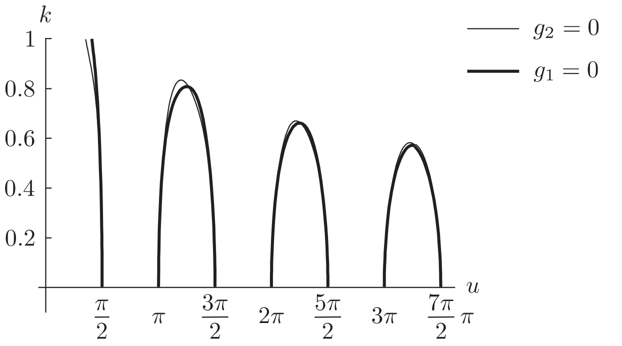

Figure 19. The curves $g_1 = 0$ and $g_2 = 0$; $\nu \in N_2$

The arcs of the elastics corresponding to the values of the parameters (p, τ, k) such that $\operatorname{sn}\tau = 0$, $f_1(p, k) = f_2(p, k) = 0$ satisfy the equalities $r_t^2 + \rho_t^2 = 0$, that is, $x_t = y_t = v_t = w_t = 0$. Taking into account the expression for the centre of mass of the segment of the elastic $c_x = (v - r^2 y/6)/z$, $c_y = (w + r^2 x/6)/z$ (see [1]) we obtain $x_t = y_t = c_x = c_y = 0$: this is a closed elastic bounding a domain with centre of mass at the initial point of the elastic (we called such elastics *remarkable*). Lemma 3.5 asserts that there exist denumerably many non-inflectional remarkable elastics. Note that the stratum MAX_0 does not intersect the sets N_i, $i = 1, 3, \dots, 7$; therefore no inflectional and critical elastics are remarkable. In Fig. 20 we depict the shortest remarkable elastic, which corresponds to the intersection point of the curves $f_3 = 0$ and $f_4 = 0$ in the domain $p \in (K, 2K)$. This is the shortest of all the closed smooth curves bounding a domain with centre of mass at the initial point of this curve.

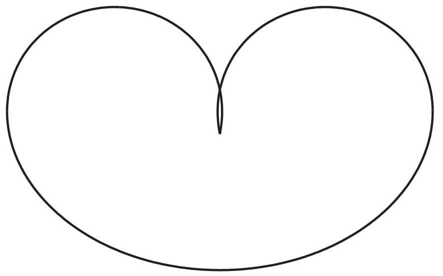

Figure 20. A shortest remarkable elastic

Proof of Lemma 3.5. It is sufficient to show that the function $\varphi(u) := g_1(u, k_{g_2}(u))$ changes sign on each interval $u \in (\pi n, \pi/2 + \pi n)$. To begin, we consider the first interval $u \in (\pi, 3\pi/2)$.

From the Taylor expansions

$$g_1(u) = 2(2E - (2-k^2)K) + k^2(u-\pi) + o(u-\pi),$$
$$g_2(u) = -2k^2K + (8-k^2)(u-\pi) + o(u-\pi)$$

we obtain the asymptotics of the curves near the point $(u,k) = (\pi, 0)$:

$$g_1 = 0\colon \quad u - \pi = \underbrace{-\frac{2}{k^2}(2E - (2-k^2)K)}_{>0} + o(1),$$

$$g_2 = 0\colon \quad u - \pi = \frac{2k^2K}{\underbrace{8-k^2}_{>0}} + o(1).$$

The inequality $(8-k^2)((2-k^2)K - 2E) > k^4K$, $k \in (0,1)$, which is proved in Lemma 3.6 below, and the expansion

$$g_1(u,k) = \underbrace{\sin u \cos u}_{>0}\, k^2 + o(k^2) \tag{25}$$

imply that $\varphi(u) < 0$ as $u \to \pi + 0$.

In a similar fashion we can determine the sign of $\varphi(u)$ as $u \to 3\pi/2 - 0$. We have

$$g_1(u) = 3(2E - (2-k^2)K) - \frac{k^2}{k'}\left(u - \frac{3\pi}{2}\right) + o\left(u - \frac{3\pi}{2}\right),$$
$$g_2(u) = -3k^2K - \frac{8-7k^2}{k'}\left(u - \frac{3\pi}{2}\right) + o\left(u - \frac{3\pi}{2}\right),$$

and therefore

$$g_1 = 0\colon \quad u - \frac{3\pi}{2} = \frac{3k'}{k^2}(2\mathrm{E} - (2-k^2)K) + o(1),$$
$$g_2 = 0\colon \quad u - \frac{3\pi}{2} = -\frac{3k'k^2K}{8-7k^2} + o(1).$$

From the asymptotics (25) and the inequality $(8-7k^2)(2E-(2-k^2)K) > -k^4K$, $k \in (0,1)$, which is proved in Lemma 3.6 below, it now follows that $\varphi(u) > 0$ as $u \to 3\pi/2 - 0$.

The function $\varphi(u)$ changes sign on each interval $(\pi n, \pi/2 + \pi n)$; one can prove this fact similarly using the asymptotics

$$g_1(u) = 2n(2E-(2-k^2)K) + k^2(u-\pi n) + o(u-\pi n),$$
$$g_2(u) = -2nk^2K + (8-k^2)(u-\pi n) + o(u-\pi n),$$
$$g_1(u) = (2n+1)(2E-(2-k^2)K) - \frac{k^2}{k'}\left(u - \frac{(2n+1)\pi}{2}\right) + o\left(u - \frac{(2n+1)\pi}{2}\right),$$
$$g_2(u) = -(2n+1)k^2K - \frac{8-7k^2}{k'}\left(u - \frac{(2n+1)\pi}{2}\right) + o\left(u - \frac{(2n+1)\pi}{2}\right).$$

We now prove the inequalities that we used above in the proof of Lemma 3.5.

Lemma 3.6. *The following hold*:
1) $(8-k^2)((2-k^2)K-2E) > k^4K, \quad k \in (0,1)$;
2) $(8-7k^2)(2E-(2-k^2)K) > -k^4K, \quad k \in (0,1)$.

Proof. 1) For the function $\alpha_1(k) := (8-k^2)((2-k^2)K-2E) - k^4K$ we have

$$\alpha_1(0) = 0, \qquad \alpha_1'(k) = \frac{6k}{1-k^2}\alpha_2(k),$$
$$\alpha_2(k) := (2-k^2)E - 2(1-k^2)K,$$
$$\alpha_2(0) = 0, \qquad \alpha_2'(k) = -3k(E-K) > 0,$$

and therefore $\alpha_1(k) > 0$, $k \in (0,1)$.

2) Similarly, define the function $\alpha_3(k) := (8-7k^2)(2E-(2-k^2)K) + k^4K$; then

$$\alpha_3(0) = 0, \qquad \alpha_3'(k) = -18k(2E-(2-k^2)K) > 0,$$

since $\varphi_V(k) = 2E(k) - (2-k^2)K(k) < 0$ (see Lemma 2.3). Therefore $\alpha_3(k) > 0$, $k \in (0,1)$.

Lemma 3.5 can be reformulated as follows.

Corollary 3.4. *The set of points $S_{12} = \{f_1 = f_2 = 0\}$ is denumerable and each domain $\{p \in (2Kn, K+2Kn)\}$ contains at least one point of this set.*

Remark. Computer calculations show that the set

$$S_{34} := \{(p,k) \mid f_3(p,k) = f_4(p,k) = 0\}$$

has the same structure as the set S_{12} examined above. The fact that the functions $f_3, f_4 \not\equiv \beta f_3$ are analytic implies that the set S_{34} is countable.

An important consequence of the analysis of the set S_{12} is that S_{12} is non-empty. Hence the stratum $\mathrm{MAX}_0 \cap N_2$ is non-empty (see Theorem 3.1). All the other strata $\mathrm{MAX}_0 \cap N_j$, $j \neq 2$, are empty. This means that only the geodesics that are projected to non-inflectional elastics ($\nu \in N_2$) connect the initial point q_0 with the fixed points of rotations $\{X_0 = 0\} = \{r^2 + \rho^2 = 0\}$. This is important for understanding the structure of optimal synthesis in the generalized Dido problem.

3.4. Complete description of the Maxwell strata in the domain N_2.

Theorem 3.1. *Let* $(\lambda, t) \in N_2$. *Then*

$$(\lambda, t) \in \mathrm{MAX}_0 \quad \Leftrightarrow \quad \begin{cases} f_1(p) = f_2(p) = 0, & \operatorname{sn} \tau = 0 \\ \ \textit{or} \\ f_3(p) = f_4(p) = 0, & \operatorname{cn} \tau = 0, \end{cases}$$

where $p = \sqrt{\alpha}\, t/(2k)$, $\tau = \sqrt{\alpha}\,(\psi + t/(2k))$. *The set* $\mathrm{MAX}_0 \cap N_2$ *is denumerable and is contained in the set* $\mathrm{MAX}_2 \cap N_2$.

Proof. The fact that the set $\mathrm{MAX}_0 \cap N_2$ is denumerable obviously follows from Theorem 5.13 in [8], Corollaries 3.2 and 3.4, and the fact that the set S_{34} is countable (see the remark after Corollary 3.4).

Let $\nu \in \mathrm{MAX}_0 \cap N_2$. Let us prove that $\nu \in \mathrm{MAX}_2$. We can assume that $\alpha = 1$, $\beta = 0$. By Theorem 5.1 in [8] we have $r_t^2 + \rho_t^2 = 0$, whence we obtain $f_V(p) = 0$ by Lemma 3.2. By Theorem 5.13 in [8] we have $\nu \in \mathrm{MAX}_2$ provided that $e^{\sigma \vec{h}_0}(\nu^2) \neq \nu$. But it is clear from Theorem 4.2 in [8] that this inequality holds for any $\sigma \neq 0$.

Theorem 3.2. *The following holds*: $\mathrm{MAX}_1 \cap N_2 = \mathrm{MAX}_3 \cap N_2 = \varnothing$.

Proof. This follows from Theorem 5.13 in [8] and Corollary 3.1.

Theorem 3.3. *Let* $(\lambda, t) \in N_2$. *Then*

$$(\lambda, t) \in \mathrm{MAX}_2 \quad \Leftrightarrow \quad \begin{cases} f_V(p) = 0, & \operatorname{sn} \tau \ \operatorname{cn} \tau \neq 0 \\ \ \textit{or} \\ (\lambda, t) \in \mathrm{MAX}_0, \end{cases}$$

where $p = \sqrt{\alpha}\, t/(2k)$, $\tau = \sqrt{\alpha}\,(\psi + t/(2k))$. *The positive roots* $p = p_n^V$, $n \in \mathbb{N}$, *of the function* $f_V(p)$ *are described in Proposition* 3.2.

Proof. This follows from Theorem 5.13 in [8] and the factorization (14).

We now describe the first Maxwell points along geodesics for $\lambda \in C_2$. By analogy with the domain C_1 we introduce notation for the first Maxwell time in the domain C_2:

$$p_1(k) = p_1^V(k), \qquad k \in (0,1), \quad \lambda \in C_2.$$

Theorem 3.4. *Let* $\lambda \in C_2$. *Then*

$$\operatorname{sn} \tau \ \operatorname{cn} \tau \neq 0 \quad \Rightarrow \quad t_1^{\mathrm{MAX}}(\lambda) = t_1^{\mathrm{MAX}_2}(\lambda) = \frac{2k}{\sqrt{\alpha}}\, p_1 = \frac{2k}{\sqrt{\alpha}}\, p_1^V,$$

$$\operatorname{sn} \tau = 0 \quad \Rightarrow \quad t_1^{\mathrm{MAX}}(\lambda) = t_1^{\mathrm{MAX}_0}(\lambda) = \min\left\{ \frac{2k}{\sqrt{\alpha}}\, p_n^V \ \middle| \ f_1(p_n^V) = f_2(p_n^V) = 0 \right\},$$

$$\operatorname{cn} \tau = 0 \quad \Rightarrow \quad t_1^{\mathrm{MAX}}(\lambda) = t_1^{\mathrm{MAX}_0}(\lambda) = \min\left\{ \frac{2k}{\sqrt{\alpha}}\, p_n^V \ \middle| \ f_3(p_n^V) = f_4(p_n^V) = 0 \right\}.$$

Proof. This follows from Theorems 3.1–3.3.

§ 4. Maxwell strata in N_3

We shall use the fact that the parametrisation of geodesics for $\lambda \in C_3$ is obtained from the formulae for geodesics for $\lambda \in C_1$, by passing to the limit $k \to 1-0$.

4.1. Roots of the equation $z = 0$ for $\nu \in N_3$.

Proposition 4.1. *Let $(\lambda, t) \in N_3$. Then the equation $z_t = 0$ has no roots.*

Proof. It is sufficient to consider the case $\alpha = 1$, $\beta = 0$. We pass to the limit $k \to 1-0$ in equality (2):

$$\begin{gathered} z_t = \frac{4}{\Delta \cosh \tau} f_z(p), \\ \Delta = 1 - \tanh^2 \tau \tanh^2 p > 0, \\ p = \frac{t}{2}, \quad \tau = \varphi + \frac{t}{2}, \\ f_z(p) = \frac{p - \tanh p}{\cosh p} > 0; \end{gathered}$$

therefore $z_t > 0$ for $t > 0$.

4.2. Roots of the equation $V = 0$ for $\nu \in N_3$.

Proposition 4.2. *Let $(\lambda, t) \in N_3$. Then*

$$V_t = 0 \quad \Leftrightarrow \quad \tau = 0, \quad \tau = \sqrt{\alpha}\left(\varphi + \frac{t}{2}\right).$$

Proof. We pass to the limit $k \to 1-0$ in the factorization (5):

$$V = \frac{2 \tanh \tau}{\Delta \cosh \tau} f_V(p, 1), \tag{26}$$

$$g_V(p, 1) = \frac{\cosh p}{\tanh p} f_V(p, 1), \qquad p > 0,$$

$$(g_V(p, 1))' = -\frac{4 \cosh^2 p}{\tanh^2 p} (f_z(p, 1))^2 < 0, \qquad p > 0.$$

Therefore the function $g_V(p, 1)$ is decreasing for $p \in (0, +\infty)$. But $\lim_{p \to 0} g_V(p, 1) = 0$; therefore $g_V(p, 1) < 0$ and $f_V(p, 1) < 0$ for $p > 0$. The assertion now follows from the factorization (26).

4.3. Roots of the equation $r^2 + \rho^2 = 0$ for $\nu \in N_3$.

Proposition 4.3. *If $(\lambda, t) \in N_3$, then the equation $r_t^2 + \rho_t^2 = 0$ has no roots.*

Proof. Let $(\lambda, t) \in N_3$, $\alpha = 1$, $\beta = 0$, $r_t^2 + \rho_t^2 = 0$. From the factorization (4) we obtain

$$y_t = \frac{4 \tanh p \tanh \tau}{\Delta \cosh p \cosh \tau} = 0;$$

therefore $\tau = 0$. Then $x_t|_{\tau=0} = 2(2 \tanh p - p) = 0$. Consequently, $\tanh p = p/2$. Finally,

$$w_t\big|_{\tau=0,\, \tanh p = p/2} = \frac{2}{3}\left(8 \frac{\tanh p}{\cosh^2 p} - p\right) = 0.$$

But it is easy to prove that the system of equations

$$2\tanh p - p = 0, \qquad 8\frac{\tanh p}{\cosh^2 p} - p = 0$$

has no positive roots. The contradiction thus obtained completes the proof.

4.4. Complete description of the Maxwell strata in N_3.

Proposition 4.4. *The following holds*: $\mathrm{MAX}_i \cap N_3 = \varnothing$, $i = 0, 1, 2, 3$.

Proof. This follows from Theorem 5.13 in [8] and Propositions 4.1–4.3.

§ 5. Conjugate points

5.1. Limit points of the Maxwell set. A geodesic q_t is said to be *strictly normal* if it is the projection of at least one normal extremal λ_t but is not the projection of any abnormal extremal. In the generalized Dido problem the strictly normal geodesics are those corresponding to $\lambda \in C_i$, $i = 1, 2, 3, 6$. The geodesics corresponding to $\lambda \in C_i$, $i = 4, 5, 7$, are not strictly normal, since abnormal extremals are projected to them (see [1]).

A point q_t of a strictly normal geodesic $q_s = \mathrm{Exp}(\lambda, s)$, $s \in [0, t]$, is said to be *conjugate to the point* q_0 *along the geodesic* q_s if $\nu = (\lambda, t)$ is a critical point of the exponential map.

It is known that a strictly normal geodesic cannot be optimal after a conjugate point [10]. At the first conjugate point the geodesic ceases to be locally optimal, and at the first Maxwell point it ceases to be globally optimal. In this section we find conjugate points on the geodesics corresponding to $\lambda \in C_i$, $i = 1, 2, 6$, that contain no Maxwell points. These conjugate points are limits of pairs of the corresponding Maxwell points.

Proposition 5.1. *Suppose that* $\nu_n, \nu_n' \in N$, $\nu_n \neq \nu_n'$, $\mathrm{Exp}(\nu_n) = \mathrm{Exp}(\nu_n')$, $n \in \mathbb{N}$. *If both sequences* $\{\nu_n\}$, $\{\nu_n'\}$ *converge to some point* $\overline{\nu} = (\lambda, t)$ *and the geodesic* $q_s = \mathrm{Exp}(\lambda, s)$ *is strictly normal, then the end-point* $q_t = \mathrm{Exp}(\overline{\nu})$ *of this geodesic is a conjugate point.*

Proof. If ν is a regular point of the exponential map, then its restriction to a small neighbourhood of the point ν is a diffeomorphism. By the hypothesis of this proposition the exponential map is not bijective in any neighbourhood of the point $\overline{\nu}$. Consequently, $\overline{\nu}$ is a critical point of the map Exp and its image $q_t = \mathrm{Exp}(\lambda, t)$ is a conjugate point.

It is convenient to introduce the following set, which we call the *double closure of the Maxwell set*:

$$\mathrm{CMAX} = \bigl\{\overline{\nu} \in N \mid \exists\, \{\nu_n = (\lambda_n, t_n)\},\ \{\nu_n' = (\lambda_n', t_n)\} \subset N\colon\ \nu_n \neq \nu_n', \\ \mathrm{Exp}(\nu_n) = \mathrm{Exp}(\nu_n'),\ n \in \mathbb{N},\ \lim_{n\to\infty} \nu_n = \lim_{n\to\infty} \nu_n' = \overline{\nu}\bigr\}.$$

It is obvious that $\nu_n \in \mathrm{MAX}$; therefore $\mathrm{CMAX} \subset \mathrm{cl}(\mathrm{MAX})$.

Proposition 5.1 asserts that if $\nu = (\lambda, t) \in \mathrm{CMAX}$ and the geodesic $q_s = \mathrm{Exp}(\lambda, s)$ is strictly normal, then its end-point q_t is a conjugate point.

By analogy with the set CMAX we define the following subsets of it — the double closures of Maxwell strata:

$$\mathrm{CMAX}_i = \big\{\overline{\nu} \in N \mid \exists\, \{\nu_n\} \subset N,\ \sigma_n \in \mathbb{R}\colon\ \nu_n \neq \nu_n' = e^{\sigma_n \vec{h}_0} \circ \varepsilon^i(\nu_n), \\ \mathrm{Exp}(\nu_n) = \mathrm{Exp}(\nu_n'),\ n \in \mathbb{N},\ \lim_{n\to\infty} \nu_n = \lim_{n\to\infty} \nu_n' = \overline{\nu}\big\}.$$

Since $\nu_n \in \mathrm{MAX}_i$, we have the inclusion $\mathrm{CMAX}_i \subset \mathrm{cl}(\mathrm{MAX}_i)$.

5.2. Conjugate points in N_1.

Proposition 5.2. *Let $\nu = (\lambda, t) \in N_1$ be a point such that*

$$f_z(p) = 0, \quad \mathrm{cn}\,\tau = 0, \qquad p = \sqrt{\alpha}\,\frac{t}{2}\,, \quad \tau = \sqrt{\alpha}\Big(\varphi + \frac{t}{2}\Big).$$

Then $\nu \in \mathrm{CMAX}_1$.

Proof. The point $\nu = (k, p, \tau, \alpha, \beta)$ is the limit of the sequences

$$\nu_n^{\pm} = (k, p, \tau \pm 1/n, \alpha, \beta), \qquad n \in \mathbb{N},$$

and $\nu_n^- = \varepsilon^1(\nu_n^+) \neq \nu_n^+$, $\mathrm{Exp}(\nu_n^+) = \mathrm{Exp}(\nu_n^-)$ by Proposition 4.1 in [7] and Proposition 3.1 in [8].

Proposition 5.3. *Let $\nu = (\lambda, t) \in N_1$ be a point such that*

$$f_V(p) = 0, \quad \mathrm{sn}\,\tau = 0, \qquad p = \sqrt{\alpha}\,\frac{t}{2}\,, \quad \tau = \sqrt{\alpha}\Big(\varphi + \frac{t}{2}\Big).$$

Then $\nu \in \mathrm{CMAX}_2$.

Proof. It is easy to see that the double closures CMAX_i, as well as the strata MAX_i, are invariant under the rotations $\vec{h}_0$ and dilations Z; therefore we can set $\alpha = 1$, $\beta = 0$.

The point $\nu = (k,\ p,\ \tau,\ \alpha = 1,\ \beta = 0)$ is the limit of the sequences $\nu_n^{\pm} = (k, p, \tau \pm 1/n, 1, 0)$. According to Theorem 5.13 in [8] we have $\nu_n \in \mathrm{MAX}_2$ and

$$\nu_n' = e^{\sigma_n \vec{h}_0}(\nu_n^-) \neq \nu_n^+, \qquad \mathrm{Exp}(\nu_n') = \mathrm{Exp}(\nu_n),$$

where the rotation angles σ_n are determined by part 2) of Proposition 3.1 in [8]: $\sigma_n = 2\chi_n$ for $r_n > 0$, and $\sigma_n = 2\omega_n - \pi$ for $r_n = 0$, $\rho_n > 0$ (the case $r_n = \rho_n = 0$ is impossible by Lemma 2.5).

Suppose that $r > 0$ for the geodesic corresponding to ν. Then $\sigma_n \to \sigma = 2\chi$. From the factorization (4) we obtain $y = 0$; therefore $\chi \equiv 0 \pmod{\pi}$ and $\sigma = 0$.

Now suppose that $r = 0$ and $\rho > 0$. Then $\sigma_n \to \sigma = 2\omega - \pi$. From the explicit formulae for geodesics [1] we obtain $v = 0$ for $\lambda \in C_1$; therefore $\omega \equiv \pi/2 \pmod{\pi}$ and $\sigma = 0$.

Thus, in any case, $\sigma = 0$. Consequently,

$$\nu_n' = \nu_n^- \neq \nu_n^+, \quad \mathrm{Exp}(\nu_n^-) = \mathrm{Exp}(\nu_n^+), \qquad \nu_n^{\pm} \to \nu,$$

and therefore $\nu \in \mathrm{CMAX}_2$.

We define the following sets:

$$\mathrm{MAX}_{ij} = (\mathrm{MAX}_i \cup \mathrm{CMAX}_i) \cap N_j. \tag{27}$$

From Theorem 2.2 and Propositions 5.2, 5.3 we obtain the partitions of the sets MAX_{i1} into connected components:

$$\mathrm{MAX}_{11} = \left\{ (\lambda, t) \in N_1 \;\middle|\; f_z(p) = 0,\ p = \frac{t\sqrt{\alpha}}{2} \right\} = \bigcup_{n=1}^{\infty} \mathrm{MAX}_{11}^n,$$

$$\mathrm{MAX}_{11}^n = \left\{ (\lambda, t) \in N_1 \;\middle|\; t = \frac{2p_n^z}{\sqrt{\alpha}} \right\}, \tag{28}$$

$$\mathrm{MAX}_{21} = \left\{ (\lambda, t) \in N_1 \;\middle|\; f_V(p) = 0,\ p = \frac{t\sqrt{\alpha}}{2} \right\} = \bigcup_{n=1}^{\infty} \mathrm{MAX}_{21}^n,$$

$$\mathrm{MAX}_{21}^n = \left\{ (\lambda, t) \in N_1 \;\middle|\; t = \frac{2p_n^V}{\sqrt{\alpha}} \right\}. \tag{29}$$

5.3. Conjugate points in N_2.

Proposition 5.4. *The following holds*: $\mathrm{CMAX}_1 \cap N_2 = \varnothing$.

Proof. This follows from the equality $\mathrm{MAX}_1 \cap N_2 = \varnothing$.

Proposition 5.5. *Suppose that $\nu = (\lambda, t) \in N_2$ is a point such that*

$$f_V(p) = 0, \quad \operatorname{sn}\tau \operatorname{cn}\tau = 0, \qquad p = \sqrt{\alpha}\,\frac{t}{2k}, \quad \tau = \sqrt{\alpha}\left(\psi + \frac{t}{2k}\right).$$

Then $\nu \in \mathrm{CMAX}_2$.

Proof. The proof is similar to the proof of Proposition 5.3.

From Theorems 3.2, 3.3 and Propositions 5.4, 5.5 we obtain the following partition of the sets MAX_{i2} (see (27)) into connected components:

$$\mathrm{MAX}_{12} = \varnothing,$$

$$\mathrm{MAX}_{22} = \left\{ (\lambda, t) \in N_2 \;\middle|\; f_V(p) = 0,\ p = \frac{t\sqrt{\alpha}}{2k} \right\} = \bigcup_{n=1}^{\infty} \mathrm{MAX}_{22}^n,$$

$$\mathrm{MAX}_{22}^n = \left\{ (\lambda, t) \in N_2 \;\middle|\; t = \frac{2kp_n^V}{\sqrt{\alpha}} \right\}. \tag{30}$$

5.4. Conjugate points in N_3.

Proposition 5.6. *The following holds*:

$$\mathrm{CMAX}_1 \cap N_3 = \mathrm{CMAX}_2 \cap N_3 = \varnothing.$$

Proof. We shall prove the equality $\mathrm{CMAX}_1 \cap N_3 = \varnothing$; the proof of the other equality is similar, but simpler. Arguing by contradiction, suppose that there exists a point $\nu \in \mathrm{MAX}_2 \cap N_3$. Then one can find a sequence of points $\nu_n \in \mathrm{MAX}_2 \cap N_1$ or $\nu_n \in \mathrm{MAX}_2 \cap N_2$. Let $\nu_n \in \mathrm{MAX}_2 \cap N_1$; in the case $\nu_n \in \mathrm{MAX}_2 \cap N_2$ the proof is similar.

Using the symmetries $\vec{h}_0$, Z we can assume that $\nu_n = (k_n,\ p_n,\ \tau_n,\ \alpha = 1,\ \beta = 0)$, where $f_V(p_n) = 0$. By Proposition 2.2 we have $p_n \geqslant 2K(k_n)$. Since $\nu_n \to \nu \in N_3$, we obtain $k_n \to 1$; therefore $K(k_n) \to +\infty$, $p_n \to +\infty$, and $t_n \to +\infty$. This contradiction to the condition $\nu_n \to \nu$ completes the proof of the proposition.

5.5. Conjugate points in N_6.

Proposition 5.7. *The following holds*: $\mathrm{CMAX}_1 \cap N_6 = \varnothing$.

Proof. This follows from the equality $\mathrm{MAX}_1 \cap N_2 = \mathrm{MAX}_1 \cap N_6 = \varnothing$.

Proposition 5.8. *Let* $\overline{\nu} = (\overline{\theta},\ \overline{c} \neq 0,\ \overline{\alpha} = 0,\ \overline{t}\,) \in N_6$ *be a point such that*

$$\overline{t} = \frac{4}{|\overline{c}|}\overline{p}, \qquad f_V^0(\overline{p}) = 0,$$

where the function $f_V^0(p)$ *is defined in Proposition* 3.4. *Then* $\overline{\nu} \in \mathrm{CMAX}_2$.

Proof. Taking into account the symmetries $\vec{h}_0$ and Z we can set $\overline{\theta} = 0$, $\overline{c} = \pm 1$. We consider only the case $\overline{c} = 1$, since the case $\overline{c} = -1$ is quite similar. Thus, $\overline{\nu} = (\overline{\theta} = 0,\ \overline{c} = 1,\ \overline{\alpha} = 0,\ \overline{t}\,)$.

Let $\nu = (\theta, c, \alpha, \beta) \in \mathrm{MAX}_2 \cap N_2^+$, $\nu^2 = \varepsilon^2(\nu) = (\theta^2, c^2, \alpha, \beta^2, t)$, where by Theorem 3.3

$$t = \frac{2k}{\sqrt{\alpha}}, \quad p = p_n^V(k), \quad k \in (0,1), \quad \tau = \frac{\sqrt{\alpha}}{k}\left(\varphi + \frac{t}{2}\right) \neq Km, \quad m \in \mathbb{N},$$

$$\tau^2 = \frac{\sqrt{\alpha}}{k}\left(\varphi^2 + \frac{t}{2}\right), \qquad \beta^2 = -\beta.$$

Then by the definition of the Maxwell stratum we have

$$\widetilde{\nu} = e^{\sigma \vec{h}_0}(\nu^2) \neq \nu, \qquad \mathrm{Exp}(\nu) = \mathrm{Exp}(\widetilde{\nu}),$$

where by Proposition 3.1 in [8]

$$\sigma = 2\chi \quad \text{for} \quad r > 0, \qquad \sigma = 2\omega - \pi \quad \text{for} \quad r = 0,\ \rho > 0 \tag{31}$$

(we choose $\nu \in N_2$ so that $r^2 + \rho^2 \neq 0$).

Let $k \to 0$ and $\alpha \to \overline{\alpha} = 0$ so that $\sqrt{\alpha}/k = 1/2$. According to Proposition 3.4 we have $p = p_n^V(k) \to p_n^V(0) = \overline{p}$. Hence, $t = 2kp/\sqrt{\alpha} \to 4\overline{p} = \overline{t}$.

Let $\beta = \mathrm{const}$. We set $\overline{\varphi} = -\beta$. Obviously, we can choose $\varphi \to \overline{\varphi}$ so that

$$\tau = \frac{\sqrt{\alpha}}{k}\left(\varphi + \frac{t}{2}\right) \neq Km.$$

We claim that for this choice we obtain $\lim \nu = \lim \widetilde{\nu} = \overline{\nu}$; this will complete the proof of the proposition.

We use the expressions for elliptic coordinates in the domain C_2^+ in [7], § 4.1:

$$\sin\frac{\theta - \beta}{2} = \mathrm{sn}\,\frac{\sqrt{\alpha}}{k}, \qquad \cos\frac{\theta - \beta}{2} = \mathrm{cn}\,\frac{\sqrt{\alpha}}{k}, \tag{32}$$

$$\frac{c}{2} = \frac{\sqrt{\alpha}}{k}\,\mathrm{dn}\,\frac{\sqrt{\alpha}}{k}. \tag{33}$$

Then $\sqrt{\alpha}\,\varphi/k \to \overline{\varphi}/2$. Using (33) we obtain $c \to 1 = \overline{c}$; using (32), $(\theta-\beta)/2 \to \overline{\varphi}/2$. Therefore $\theta \to \overline{\varphi} + \beta = 0 = \overline{\theta}$. Thus,

$$\nu = (\theta, c, \alpha, \beta, t) \to \overline{\nu} = (\overline{\theta} = 0,\ \overline{c} = 1,\ \overline{\alpha} = 0,\ \overline{t}\,).$$

For the point $\nu^2 = (\theta^2, c^2, \alpha, \beta^2, t)$ we obtain $c^2 \to \overline{c}$ from equality (33). According to equality (6) in [7] we have $\theta^2 = -\theta_t$ and therefore $\theta^2 \to -\overline{\theta}_{\overline{t}}$, where $\overline{\theta}_t$ is the first component of the solution of the pendulum equation corresponding to $\overline{\nu}$:

$$\dot{\overline{\theta}}_t = \overline{c}_t, \quad \dot{\overline{c}}_t = 0, \quad \overline{\theta}_0 = 0, \quad \overline{c}_0 = 1 \quad \Rightarrow \quad \overline{\theta}_t = t, \quad \overline{c}_t = 1.$$

Consequently, $\theta^2 \to -\overline{\theta}_{\overline{t}} = -\overline{t}$. Thus,

$$\nu^2 = (\theta^2, c^2, \alpha, \beta^2, t) \to \overline{\nu}^{\,2} = (-\overline{t},\ \overline{c} = 1,\ \overline{\alpha} = 0,\ \overline{t}\,).$$

We now find the limit of the rotation angle σ; see (31). Consider the geodesic-circle corresponding to the covector $(\overline{\theta} = 0,\ \overline{c} = 1,\ \overline{\alpha} = 0) \in C_6$:

$$\overline{x}_t = \sin t, \qquad \overline{y}_t = 1 - \cos t. \tag{34}$$

By equality (19) we obtain $\overline{t} = 4\overline{p} = 4p_n^V(0) \in (2\pi n, 2\pi(n+1))$. Therefore $\overline{t} \neq 2\pi m$; consequently, $\overline{r}_{\overline{t}} = \sqrt{\overline{x}_{\overline{t}}^2 + \overline{y}_{\overline{t}}^2} > 0$. In other words, the first of equalities (31) holds, which implies $\sigma = 2\chi \to \overline{\sigma} = 2\overline{\chi}_{\overline{t}}$. From equalities (34) we obtain $\overline{\chi}_{\overline{t}} = \overline{t}/2$; consequently, $\overline{\sigma} = \overline{t}$. We can now complete the proof:

$$\widetilde{\nu} = e^{\sigma\vec{h}_0}(\nu^2) \to e^{\overline{\sigma}\vec{h}_0}(\overline{\nu}^{\,2}) = e^{\overline{t}\vec{h}_0}(-\overline{t},\ \overline{c} = 1,\ \overline{\alpha} = 0,\ \overline{t}\,) = (0, 1, 0, \overline{t}\,) = \overline{\nu};$$

therefore $\overline{\nu} \in \mathrm{CMAX}_2$.

§ 6. Cut time

Let $\lambda \in C$ and let $q_s = \operatorname{Exp}(\lambda, s)$ be the corresponding normal geodesic. The *cut time on the geodesic* q_s is defined to be the number

$$t_{\mathrm{cut}}(\lambda) = \sup\{t > 0 \mid q_s,\ s \in [0, t],\ \text{is optimal}\}.$$

The point q_t, $t = t_{\mathrm{cut}}(\lambda)$, is called the *cut point*. The cut time is the instant when a geodesic ceases to be optimal. All the geodesics in the generalized Dido problem are regular; therefore any small arc on one is optimal. In other words, $t_{\mathrm{cut}}(\lambda) > 0$ for all $\lambda \in C$.

6.1. Estimate of the cut time. In this subsection we bring together the results obtained in this paper and estimate the cut time from above. To do this, we define the following function on the initial cylinder C:

$$\mathbf{t}\colon C \to (0, +\infty], \qquad \lambda \mapsto \mathbf{t}(\lambda),$$

$$\lambda \in C_1 \quad \Rightarrow \quad \mathbf{t} = \frac{2}{\sqrt{\alpha}}\, p_1(k) = \min\left\{\frac{2}{\sqrt{\alpha}}\, p_1^z(k), \frac{2}{\sqrt{\alpha}}\, p_1^V(k)\right\},$$

$$\lambda \in C_2 \quad \Rightarrow \quad \mathbf{t} = \frac{2k}{\sqrt{\alpha}}\, p_1(k) = \frac{2k}{\sqrt{\alpha}}\, p_1^V(k),$$

$$\lambda \in C_6 \quad \Rightarrow \quad \mathbf{t} = \frac{4}{|c|} p_1^V(0),$$

$$\lambda \in C_i,\ i = 3, 4, 5, 7 \quad \Rightarrow \quad \mathbf{t} = +\infty.$$

Theorem 6.1. *For any $\lambda \in C$ we have the estimate $t_{\mathrm{cut}}(\lambda) \leqslant \mathbf{t}(\lambda)$.*

Proof. Let $\lambda \in C_1 \cup C_2 \cup C_3 \cup C_6$; then the geodesic $q_s = \operatorname{Exp}(\lambda, s)$ is strictly normal. According to Proposition 2.1 in [8] and Proposition 5.1, if $(\lambda, t) \in \mathrm{MAX} \cup \mathrm{CMAX}$, then the geodesic q_s is not optimal on any interval $s \in [0, t+\varepsilon]$, $\varepsilon > 0$, that is, $t_{\mathrm{cut}}(\lambda) \leqslant t$. To prove this theorem we will show that the pair $(\lambda, \mathbf{t}(\lambda))$ belongs to one of the sets MAX_i or CMAX_i for any covector $\lambda \in C$.

Let $\lambda \in C_1$. If $k \in (0, k_1) \cup (k_0, 1)$, then

$$\begin{aligned}
&\operatorname{cn}\tau \neq 0 \quad \Rightarrow \quad \mathbf{t}(\lambda) = \frac{2}{\sqrt{\alpha}}\, p_1^z = t_1^{\mathrm{MAX}_1}(\lambda) &&\text{(Theorem 2.3)},\\
&\operatorname{cn}\tau = 0 \quad \Rightarrow \quad (\lambda, \mathbf{t}(\lambda)) = \left(\lambda, \frac{2}{\sqrt{\alpha}}\, p_1^z\right) \in \mathrm{CMAX}_1 &&\text{(Proposition 5.2)}.
\end{aligned}$$

If $k \in (k_1, k_0)$, then

$$\begin{aligned}
&\operatorname{sn}\tau \neq 0 \quad \Rightarrow \quad \mathbf{t}(\lambda) = \frac{2}{\sqrt{\alpha}}\, p_1^V = t_1^{\mathrm{MAX}_2}(\lambda) &&\text{(Theorem 2.3)},\\
&\operatorname{sn}\tau = 0 \quad \Rightarrow \quad (\lambda, \mathbf{t}(\lambda)) = \left(\lambda, \frac{2}{\sqrt{\alpha}}\, p_1^V\right) \in \mathrm{CMAX}_2 &&\text{(Proposition 5.3)}.
\end{aligned}$$

Finally, if $k = k_1$, k_0, then

$$\mathbf{t}(\lambda) = \frac{2}{\sqrt{\alpha}}\, p_1^z = t_1^{\mathrm{MAX}_3}(\lambda) \qquad \text{(Theorem 2.3)}.$$

Let $\lambda \in C_2$; then

$$\begin{aligned}
&\operatorname{cn}\tau \,\operatorname{sn}\tau \neq 0 \quad \Rightarrow \quad \mathbf{t}(\lambda) = \frac{2k}{\sqrt{\alpha}}\, p_1^V = t_1^{\mathrm{MAX}_2}(\lambda) &&\text{(Theorem 3.4)},\\
&\operatorname{cn}\tau \,\operatorname{sn}\tau = 0 \quad \Rightarrow \quad (\lambda, \mathbf{t}(\lambda)) = \left(\lambda, \frac{2k}{\sqrt{\alpha}}\, p_1^V\right) \in \mathrm{CMAX}_2 &&\text{(Proposition 5.5)}.
\end{aligned}$$

Let $\lambda \in C_6$; then

$$(\lambda, \mathbf{t}(\lambda)) = \left(\lambda, \frac{4p_1^V(0)}{|c|}\right) \in \mathrm{CMAX}_2 \qquad \text{(Proposition 5.8)}.$$

If $\lambda \in C_3$, then there is nothing to prove, since $\mathbf{t}(\lambda) = +\infty$; note that in this case $\mathrm{MAX}_i = \mathrm{CMAX}_i = \varnothing$ (Propositions 4.4, 5.6).

If $\lambda \in C_4 \cup C_5 \cup C_6$, then there is also nothing to prove; in this case the geodesics q_s are optimal on the entire ray $s \in [0, +\infty)$ and $t_{\mathrm{cut}}(\lambda) = \mathbf{t}(\lambda) = +\infty$.

Computer calculations corroborate the following.

Conjecture. *For any $\lambda \in C$ we have the equality $t_{\mathrm{cut}}(\lambda) = \mathbf{t}(\lambda)$.*

We aim to prove this conjecture in a subsequent paper. At present it has only been proved for some of the geodesics.

6.2. Properties of the function $\mathbf{t}$. To obtain a global representation of the function $\mathbf{t}$ it is convenient to define the following function on the cylinder C:

$$\kappa = \sqrt{\alpha^2 + \frac{(E+\alpha)^2}{4}}\,, \tag{35}$$

and to extend the functions introduced earlier:

$$k = \begin{cases} 0, & \lambda \in C_4 \cup C_6; \\ 1, & \lambda \in C_3 \cup C_5, \end{cases}$$

$$p_1(k) = \begin{cases} p_1^V(0), & \lambda \in C_6; \\ +\infty, & \lambda \in C_3 \cup C_4 \cup C_5 \cup C_7. \end{cases}$$

Theorem 6.2. *The function $\mathbf{t}\colon C \to (0,+\infty]$ is invariant under the flow of the generalized pendulum and under rotations, it is also homogeneous of order 1 under dilations. We have the global representation*

$$\mathbf{t}(\lambda) = \frac{2\sqrt[4]{1+k^4}}{\sqrt{\kappa}}\, p_1(k), \qquad \lambda \in C. \tag{36}$$

The function $\mathbf{t}$ is continuous on $C \setminus C_4$ and discontinuous on C_4; but after extension by continuity the function $\mathbf{t}$ becomes smooth on C_4. The function $\mathbf{t}$ is smooth in the domains $C_1 \setminus \{k = k_1, k_0\}$ and $C_2 \cup C_6$. The function $\mathbf{t}$ ceases to be smooth on the surfaces $C_1 \cap \{k = k_1\}$ and $C_1 \cap \{k = k_0\}$.

Proof. The invariance of the function $\mathbf{t}$ under the flow of the generalized pendulum $\ddot\theta = -\alpha \sin(\theta - \beta)$ (the vertical part of the Hamiltonian system of the Pontryagin maximum principle) follows from the fact that, for $\lambda \in C_1 \cup C_2$, the value $\mathbf{t}(\lambda)$ depends only on the invariants of the generalized pendulum k, α, and for $\lambda \in C_6$, the value $\mathbf{t}(\lambda)$ depends only on the coordinate c, which is constant for the generalized pendulum on C_6. It is easy to see that the function $\mathbf{t}$ is invariant under rotations and is homogeneous of order 1 under dilations: $\vec{h}_0 \mathbf{t} = 0$, $Z\mathbf{t} = \mathbf{t}$; this follows from the invariance of the Maxwell strata under the flows of these fields.

We now prove formula (36). Both the function $\mathbf{t}$ and the right-hand side of this formula are invariant under the flow of the generalized pendulum and under rotations; therefore it is sufficient to prove equality (36) in the quadrant

$$K_+ = \{\theta = \beta = 0,\ c \geqslant 0,\ \alpha \geqslant 0\} \subset C.$$

In this quadrant we introduce the generalized polar coordinates (κ, η), $\kappa \geqslant 0$, $\eta \in [0, \pi/2]$ as follows:

$$\begin{aligned} c^2 &= 4\kappa \cos\eta, & \alpha &= \kappa \sin\eta, \\ \kappa^2 &= \alpha^2 + \frac{c^4}{16}, & \tan\eta &= \frac{4\alpha}{c^2}\,. \end{aligned} \tag{37}$$

It is obvious that in the quadrant K_+ the function κ is given by equality (35):

$$\theta = \beta = 0 \quad\Rightarrow\quad E = \frac{c^2}{2} - \alpha \quad\Rightarrow\quad \alpha^2 + \frac{c^4}{16} = \alpha^2 + \frac{(E+\alpha)^2}{4}\,.$$

If $\lambda \in C_1 \cap K_+$, then $k = |c|/(2\sqrt{\alpha}) = \sqrt{\cot\eta}$ and therefore $\eta = \operatorname{arccot} k^2$ and

$$\sin\eta = \frac{1}{\sqrt{1+k^4}}\,, \qquad \cos\eta = \frac{k^2}{\sqrt{1+k^4}}\,.$$

Consequently,

$$c^2 = \frac{4\kappa k^2}{\sqrt{1+k^4}}\,, \qquad \alpha = \frac{\kappa}{\sqrt{1+k^4}}\,,$$

and we finally obtain

$$\mathbf{t}\big|_{C_1\cap K_+} = \frac{2}{\sqrt{\alpha}}\,p_1(k) = \frac{2\sqrt[4]{1+k^4}}{\sqrt{\kappa}}\,p_1(k).$$

If $\lambda \in C_2 \cap K_+$, then $k = 2\sqrt{\alpha}/|c| = \sqrt{\tan\eta}$ and therefore $\eta = \arctan k^2$ and

$$\sin\eta = \frac{k^2}{\sqrt{1+k^4}}\,, \qquad \cos\eta = \frac{1}{\sqrt{1+k^4}}\,.$$

Thus,

$$c^2 = \frac{4\kappa}{\sqrt{1+k^4}}\,, \qquad \alpha = \frac{\kappa k^2}{\sqrt{1+k^4}}\,,$$

whence

$$\mathbf{t}\big|_{C_2\cap K_+} = \frac{2k}{\sqrt{\alpha}}\,p_1(k) = \frac{2\sqrt[4]{1+k^4}}{\sqrt{\kappa}}\,p_1(k).$$

If $\lambda \in C_6 \cap K_+$, then $k = 0$, $\kappa = c^2/4$, $p_1(0) = p_1^V(0)$, and therefore

$$\mathbf{t}\big|_{C_6\cap K_+} = \frac{4}{|c|}\,p_1^V(0) = \frac{2\sqrt[4]{1+k^4}}{\sqrt{\kappa}}\,p_1(k).$$

Finally, for $\lambda \in C_i \cap K_+$, $i = 3,4,5,7$, the validity of formula (36) follows from the equalities $\mathbf{t}(\lambda) = +\infty$ and $p_1(1) = +\infty$.

By the properties of the invariance under the flow of the generalized pendulum and under rotations the global representation (36) is proved on the entire cylinder C.

The continuity of the function $\mathbf{t}$ in the domains C_1, C_2 follows from the continuity of the function $p_1(k)$ (see Corollary 2.5 and Lemma 3.1). The continuity of $\mathbf{t}$ on C_6 follows from the continuity of the root $p_1^V(k)$, $\lambda \in C_2$, at the point $k = 0$. The continuity $\mathbf{t}$ on C_7 follows from the fact that $\kappa \to 0$ as $\lambda \to \overline{\lambda} \in C_7$, and the function $\sqrt[4]{1+k^4}\,p_1(k)$ is isolated from zero from below, and therefore $\mathbf{t}(\lambda) \to +\infty = \mathbf{t}(\overline{\lambda})$. The continuity of $\mathbf{t}$ on C_i, $i = 3,5$, can be proved in similar fashion.

On the set C_4 the function $\mathbf{t}$ is discontinuous:

$$\overline{\lambda} \in C_4 \quad\Rightarrow\quad \lim_{\lambda\to\overline{\lambda}} \mathbf{t}(\lambda) = \frac{2p_1^z(+0)}{\sqrt{\kappa(\overline{\lambda})}} < +\infty = \mathbf{t}(\overline{\lambda}).$$

But if we redefine $\mathbf{t}|_{C_4} = 2p_1^z(0)/\sqrt{\alpha}$ by continuity, then the function $\mathbf{t}$ becomes smooth on C_4. This follows from the fact that the function $p_1(k)$ redefined by continuity has zero right derivative at the point $k = 0$ (see Corollary 2.5).

The smoothness of $\mathbf{t}$ in the domains $C_1 \cap \{k \neq k_1, k_0\}$, C_2 and the loss of smoothness on the surfaces $C_1 \cap \{k = k_1\}$, $C_1 \cap \{k = k_0\}$ follow from the corresponding assertions on the smoothness of the function $p_1(k)$ (see Corollary 2.5 and Lemma 3.1).

The smoothness of $\mathbf{t}$ at the points of C_6 follows from the fact that $p_1^V(k) = O(k^2)$, $k \to 0$, $\lambda \in C_2$ (see Proposition 3.4).

All the peculiarities of the behaviour of the function $\mathbf{t}$ mentioned in Theorem 6.2 can be clearly seen in Fig. 21.

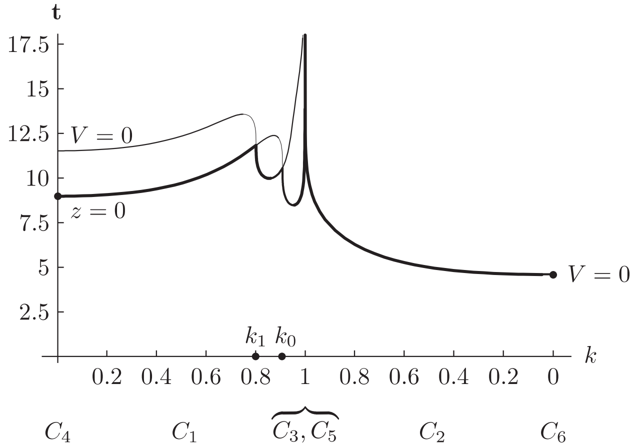

Figure 21. The graph of $k \mapsto \mathbf{t}|_{\varkappa=1} = 2\sqrt[4]{1+k^4}\, p_1(k)$

6.3. Maxwell strata in the image of the exponential map. We define the *Maxwell set* and *Maxwell strata in the image of the exponential map* as follows:

$$\mathrm{Max} = \mathrm{Exp}(\mathrm{MAX}), \qquad \mathrm{Max}_i = \mathrm{Exp}(\mathrm{MAX}_i).$$

From Theorems 2.1, 3.1, 3.3 and Lemma 3.2 we obtain the inclusions

$$\mathrm{Max}_1 \subset \{z = 0\}, \qquad \mathrm{Max}_2 \subset \{V = 0\},$$
$$\mathrm{Max}_3 \subset \{z = V = 0\}, \qquad \mathrm{Max}_0 \subset \{r = \rho = 0\}.$$

We define the first components of the Maxwell strata in the image of the exponential map to be the images of the corresponding first components $\mathrm{MAX}_{ij}^1 \subset N$ (see (28)–(30)):

$$\mathrm{Max}_{ij}^1 = \mathrm{Exp}(\mathrm{MAX}_{ij}^1) \subset M;$$

we also define their projections onto the quotient space by rotations and dilations $M'' = \pi_1''(M)$, $\pi_1''\colon q \mapsto e^{\mathbb{R}X_0} \circ e^{\mathbb{R}Y}(q)$:

$$(\mathrm{Max}_{ij}^1)'' = \pi_1''(\mathrm{Max}_{ij}^1) \subset M''.$$

In Figs. 22–25 we have depicted the first components $(\mathrm{Max}_{11}^1)''$, $(\mathrm{Max}_{21}^1)''$, $(\mathrm{Max}_{22}^1)''$. To do this, in the chart $\{\rho > 0\}''$ of the manifold M'', we have taken

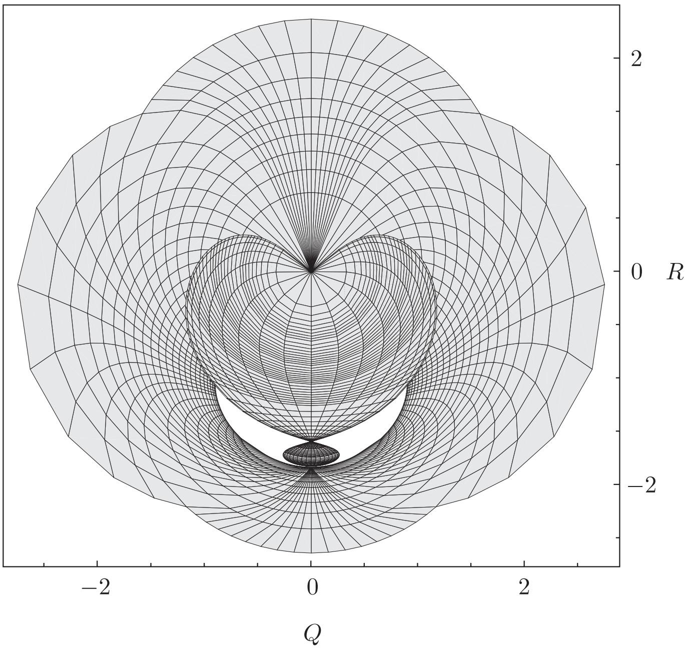

Figure 22. The set $(\mathrm{Max}_{11}^1)''$

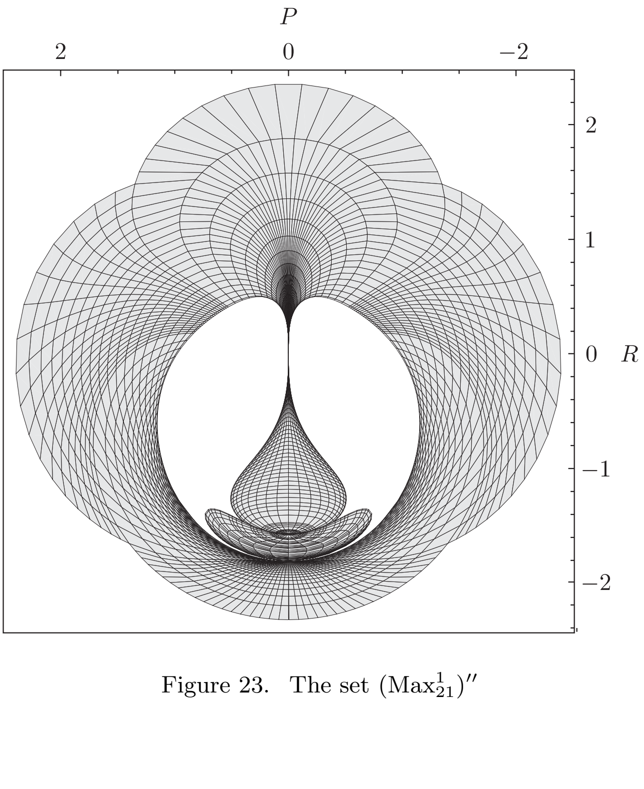

Figure 23. The set $(\mathrm{Max}_{21}^1)''$

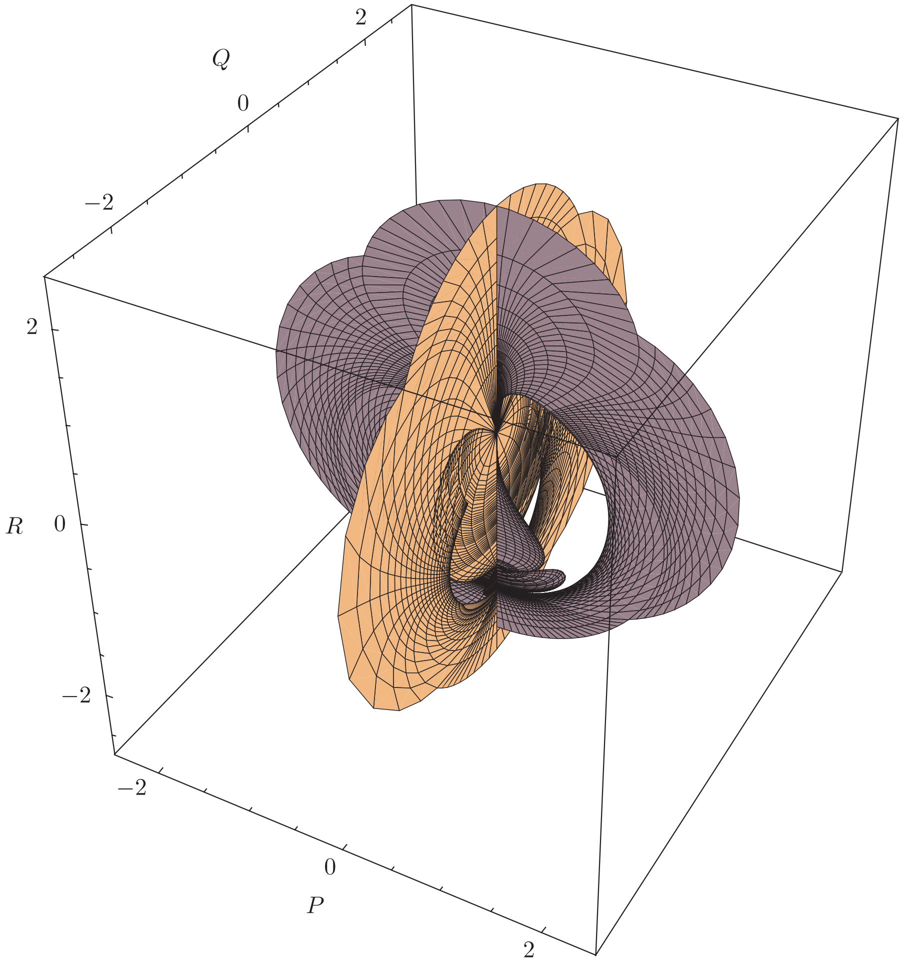

Figure 24. The set $(\mathrm{Max}_{11}^1)'' \cup (\mathrm{Max}_{21}^1)''$

the following coordinates rectifying the hypersurfaces $\{z = 0\}''$, $\{V = 0\}''$:

$$P_2 = P'(Q'^2 + R'^2) = \frac{zr^2}{2\rho^{4/3}},$$

$$Q_2 = Q' - P'(Q'^2 + R'^2) = \frac{xv + yw - zr^2/2}{\rho^{4/3}},$$

$$R_2 = R' = \frac{-yv + xw}{\rho^{4/3}},$$

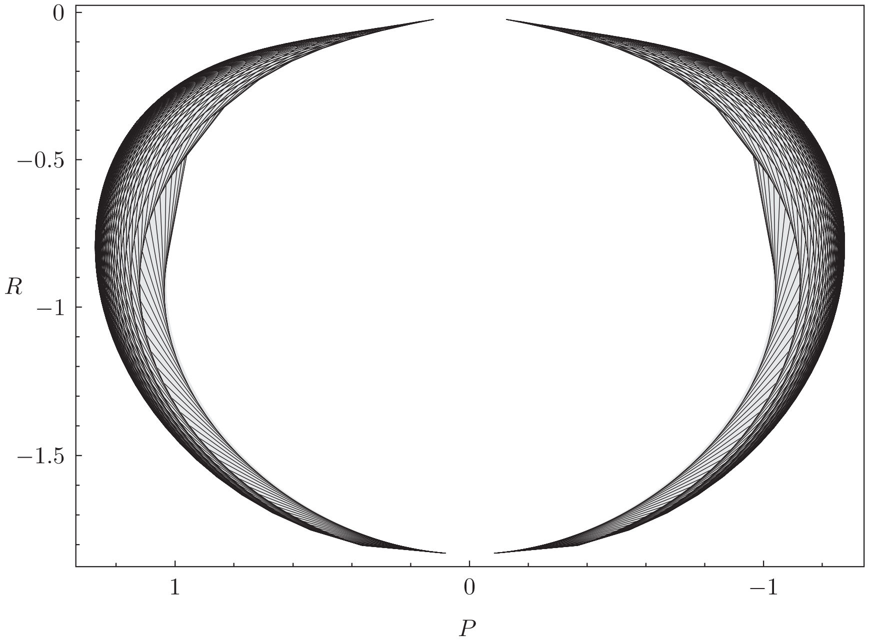

Figure 25. The set $(\mathrm{Max}^1_{22})''$

while the coordinates

$$P' = \frac{z}{2\rho^{2/3}}, \qquad Q' = \frac{xv + yw}{\rho^{4/3}}, \qquad R' = \frac{-yv + xw}{\rho^{4/3}}$$

in the chart $\{\rho > 0\}''$ of the manifold $M'' \cong S^3$ were introduced in [6].

In the sketches of each of the sets $(\mathrm{Max}^1_{11})''$ and $(\mathrm{Max}^1_{21})''$ (Figs. 22 and 23) one can clearly see the three subdomains corresponding to different intervals of values of the parameter k. There is a domain that has non-compact intersection with the chart $\{\rho > 0\}''$: this is the 'exterior' domain extending to infinity ($k \in (k_0, 1)$). There are also two compact 'interior' domains: the upper one ($k \in (k_1, k_0)$), and the lower one ($k \in (0, k_1)$). The interior points of all the three domains are Maxwell points, and the boundary edges are conjugate points. The interior points of the set $(\mathrm{Max}^1_{22})''$ in Fig. 25 are also Maxwell points; the boundary edges consist of conjugate points, as well as of limit points of the Maxwell set.

Supplement: derivatives and asymptotics of the Jacobi elliptic functions

In this supplement for the Jacobi elliptic functions $\mathrm{sn}(u,k)$, $\mathrm{cn}(u,k)$, $\mathrm{dn}(u,k)$, $\mathrm{E}(u,k)$ we give the partial derivatives with respect to the parameter k, as well as the Taylor expansions as $k \to 0$, which we used in this paper. The expansions were obtained by the method of indeterminate coefficients from the formulae for partial derivatives. A detailed exposition of the theory of elliptic functions can be found in the books [11], [12].

Derivatives of the Jacobi elliptic functions with respect to the parameter:

$$\begin{aligned}
\frac{\partial \operatorname{sn} u}{\partial k} &= \frac{1}{k}\, u \operatorname{cn} u \operatorname{dn} u + \frac{k}{1-k^2} \operatorname{sn} u \operatorname{cn}^2 u - \frac{1}{k(1-k^2)} \operatorname{E}(u) \operatorname{cn} u \operatorname{dn} u,\\
\frac{\partial \operatorname{cn} u}{\partial k} &= -\frac{1}{k}\, u \operatorname{sn} u \operatorname{dn} u - \frac{k}{1-k^2} \operatorname{sn}^2 u \operatorname{cn} u + \frac{1}{k(1-k^2)} \operatorname{E}(u) \operatorname{sn} u \operatorname{dn} u,\\
\frac{\partial \operatorname{dn} u}{\partial k} &= -\frac{k}{1-k^2} \operatorname{sn}^2 u \operatorname{dn} u - ku \operatorname{sn} u \operatorname{cn} u + \frac{k}{1-k^2} \operatorname{E}(u) \operatorname{sn} u \operatorname{cn} u,\\
\frac{\partial \operatorname{E}(u)}{\partial k} &= \frac{k}{1-k^2} \operatorname{sn} u \operatorname{cn} u \operatorname{dn} u - ku \operatorname{sn}^2 u - \frac{k}{1-k^2} \operatorname{E}(u) \operatorname{cn}^2 u.
\end{aligned}$$

Asymptotics of the Jacobi elliptic functions:

$$\begin{aligned}
\operatorname{sn} u &= s_0(u) + k^2 s_2(u) + k^4 s_4(u) + k^6 s_6(u) + k^8 s_8(u) + O(k^{10}), \qquad k \to 0,\\
s_0(u) &= \sin u,\\
s_2(u) &= \frac{1}{8} \cos u(\sin 2u - 2u),\\
s_4(u) &= \frac{1}{128}\big((8 - 4u^2 + 9\cos 2u + \cos 4u)\sin u - 6u(2\cos u + \cos 3u)\big),\\
s_6(u) &= \frac{1}{3072}\big(8u(u^2-21)\cos u - 3u(39\cos 3u + 5\cos 5u + 22u\sin u + 18u\sin 3u)\\
&\quad + 3\cos^2 u(53\sin u + 14\sin 3u + \sin 5u)\big),\\
s_8(u) &= \frac{1}{49152}\big\{u[(128u^2 - 1845)\cos u + 18(-83 + 12u^2)\cos 3u\\
&\quad - 21(15\cos 5u + \cos 7u) - 2u(951 - 4u^2 + 1122\cos 2u + 150\cos 4u)\sin u]\\
&\quad + 3\cos^2 u[553\sin u + 185\sin 3u + 22\sin 5u + \sin 7u]\big\};
\end{aligned}$$

$$\begin{aligned}
\operatorname{cn} u &= c_0(u) + k^2 c_2(u) + k^4 c_4(u) + k^6 c_6(u) + k^8 c_8(u) + O(k^{10}), \qquad k \to 0,\\
c_0(u) &= \cos u,\\
c_2(u) &= \frac{1}{8} \sin u(2u - \sin 2u),\\
c_4(u) &= \frac{1}{256}\big(-(9 + 8u^2)\cos u + 8\cos 3u + \cos 5u + 16u\sin u + 12u\sin 3u\big),\\
c_6(u) &= \frac{1}{12288}\big[-27(11 + 8u^2)\cos u + 6(41 - 36u^2)\cos 3u + 48\cos 5u\\
&\quad + 3\cos 7u + 8u(111 - 4u^2 + 132\cos 2u + 15\cos 4u)\sin u\big],\\
c_8(u) &= \frac{1}{196608}\big\{[-3594 - 2256u^2 + 32u^4]\cos u\\
&\quad + 3[943\cos 3u + 230\cos 5u + 24\cos 7u + \cos 9u]\\
&\quad + 4u[-2u(486\cos 3u + 75\cos 5u + 56u\sin u + 108u\sin 3u)\\
&\quad + 3(281\sin u + 498\sin 3u + 7(15\sin 5u + \sin 7u))]\big\};
\end{aligned}$$

$$\operatorname{dn} u = d_0(u) + k^2 d_2(u) + k^4 d_4(u) + k^6 d_6(u) + k^8 d_8(u) + O(k^{10}), \qquad k \to 0,$$

$$
\begin{aligned}
d_0(u) &= 1,\\
d_2(u) &= -\frac{1}{2}\sin^2 u,\\
d_4(u) &= -\frac{1}{32}\sin u(5\sin u+\sin 3u-8u\cos u),\\
d_6(u) &= \frac{1}{1024}\bigl(-44+(31-32u^2)\cos 2u+12\cos 4u+\cos 6u\\
&\qquad +72u\sin 2u+16u\sin 4u\bigr),\\
d_8(u) &= \frac{1}{49152}\bigl[-1407+(900-1344u^2)\cos 2u+(444-384u^2)\cos 4u\\
&\qquad +60\cos 6u+3\cos 8u+16u(147-16u^2+102\cos 2u+9\cos 4u)\sin 2u\bigr];\\
\mathrm{E}(u) &= \mathrm{E}_0(u)+k^2\mathrm{E}_2(u)+k^4\mathrm{E}_4(u)+k^6\mathrm{E}_6(u)+k^8\mathrm{E}_8(u)+O(k^{10}), \qquad k\to 0,\\
\mathrm{E}_0(u) &= u,\\
\mathrm{E}_2(u) &= \frac{1}{4}(\sin 2u-2u),\\
\mathrm{E}_4(u) &= \frac{1}{64}(-4u-8u\cos 2u+4\sin 2u+\sin 4u),\\
\mathrm{E}_6(u) &= \frac{1}{1024}\bigl(-32u+33\sin 2u-8u(9\cos 2u+2\cos 4u+4u\sin 2u)\\
&\qquad +12\sin 4u+\sin 6u\bigr),\\
\mathrm{E}_8(u) &= \frac{1}{49152}\bigl[8u(-291+32u^2)\cos 2u+3(-4(82u+68u\cos 4u+6u\cos 6u\\
&\qquad +(-85+112u^2)\sin 2u+(-37+32u^2)\sin 4u-5\sin 6u)+\sin 8u)\bigr].
\end{aligned}
$$

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Bibliography

[1] Yu. L. Sachkov, "Exponential map in the generalized Dido problem", *Mat. Sb.* **194**:9 (2003), 63–90; English transl. in *Sb. Math.* **194** (2003).

[2] A. A. Agrachev and A. V. Sarychev, "Filtrations of a Lie algebra of vector fields and the nilpotent approximation of controllable systems", *Dokl. Akad. Nauk SSSR* **295**:4 (1987), 777–781; English transl. in *Soviet Math.–Dokl.* **36** (1988).

[3] A. Bellaïche, "The tangent space in sub-Riemannian geometry", *Sub-Riemannian geometry*, Birkhäuser, Basel 1996, pp. 1–78.

[4] A. A. Agrachev and Yu. L. Sachkov, "An intrinsic approach to the control of rolling bodies", *Proc. 38th IEEE Conference on Decision and Control* (*Phoenix*, *Arizona* 1999), vol. 1, IEEE, Piscataway, NJ 1999, pp. 431–435.

[5] Yu. L. Sachkov, "Symmetries of flat rank two distributions and sub-Riemannian structures", *Trans. Amer. Math. Soc.* **356** (2004), 457–494.

[6] Yu. L. Sachkov and E. F. Sachkova, "The geometric meaning of invariants and the global structure of the quotient space in the generalized Dido problem", *Program systems*: *theory and applications*, vol. 2, Fizmatlit, Moscow 2004, pp. 387–407. (Russian)

[7] Yu. L. Sachkov, "Discrete symmetries in the generalized Dido problem", *Mat. Sb.* **197**:2 (2006), 95–116; English transl. in *Sb. Math.* **197** (2006).

[8] Yu. L. Sachkov, "The Maxwell set in the generalized Dido problem", *Mat. Sb.* **197**:4 (2006), 123–150; English transl. in *Sb. Math.* **197** (2006).

[9] S. Wolfram, *Mathematica: a system for doing mathematics by computer*, Addison-Wesley, Reading, MA 1992.

[10] A. A. Agrachev and Yu. L. Sachkov, *Geometric control theory*, Fizmatlit, Moscow 2004; English transl., *Control theory from the geometric viewpoint*, Springer-Verlag, Berlin 2004.

[11] E. T. Whittaker and G. N. Watson, *A course of modern analysis. An introduction to the general theory of infinite processes and of analytic functions; with an account of the principal transcendental functions*, Cambridge University Press, Cambridge 1996.

[12] D. F. Lawden, *Elliptic functions and applications*, Springer-Verlag, New York 1989.

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