Survey on Controllability of Invariant Systems on Solvable Lie Groups

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ABSTRACT. Known and new results on controllability of right-invariant systems on solvable Lie groups are presented and discussed. The main ideas and technique used are outlined, illustrating examples are given. Some open questions are suggested.

1. Introduction

Controllability properties of right-invariant systems on Lie groups have been a subject of active research in the mathematical control theory during the last 25 years. This question is motivated both by important applications (mechanics, geometry, bilinear systems) and by essential links with various branches of mathematics outside control theory (Lie groups and Lie algebras, differential geometry, Lie semigroups, dynamical systems).

The problem was stated and the basic properties of right-invariant systems and their reachable sets were established by R.W. Brockett [**8**], and V. Jurdjevic and H.J. Sussmann [**20**]. Controllability theory for various classes of Lie groups different from solvable ones was developed by V. Jurdjevic and H.J. Sussmann [**20**] (compact Lie groups), V. Jurdjevic and I. Kupka [**21, 22**], J.P. Gauthier and G. Bornard [**11**], J.P. Gauthier, I. Kupka and G. Sallet [**12**], R.El Assoudi and J.P. Gauthier [**2, 3**], R.El Assoudi, J.P. Gauthier, and I. Kupka [**4**], F. Silva Leite and P.E. Crouch [**35**], L.A.B. San Martin [**33**], L.A.B. San Martin and P.A. Tonelli [**34**] (semisimple Lie groups), B. Bonnard, V. Jurdjevic, I. Kupka and G. Sallet [**7**] (semidirect products of vector Lie groups with compact Lie groups), J. Hilgert [**14**] (reductive Lie groups).

The aim of this paper is to describe the state of the art in controllability on solvable Lie groups: to cover all results published, outline the main ideas and techniques used, and give illustrating examples. Notice that the case of solvable Lie groups is incompletely covered by surveys and textbooks on control theory and Lie semigroup theory (R.W. Brockett [**9**], G. Sallet [**30, 32**], A.A. Agrachev, S.V. Vakhrameev

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and R. V. Gamkrelidze [**1**], I. Kupka [**24**], J. Hilgert and K. H. Neeb [**15**], V. Jurdjevic [**23**]).

The structure of the paper is as follows. In Section 2 we introduce the general notation and definitions. In Section 3 we consider an important special class of solvable Lie groups — nilpotent ones. Subsection 3.1 is devoted to the controllability criterion for arbitrary right-invariant systems on a nilpotent Lie group by J. Hilgert, K. H. Hofmann and J. D. Lawson [**13**]. In Subsection 3.2 we present and discuss the controllability test for affine in control systems on simply connected nilpotent Lie groups due to V. Ayala Bravo [**6**]. In Section 4 we present controllability conditions for Lie groups with cocompact radical; this class of Lie groups includes all solvable Lie groups. This basic result is due to J. D. Lawson [**25**]. In Section 5 we consider a subclass of solvable Lie groups called completely solvable ones and a generalization of the controllability test of Subsection 3.2 from nilpotent to completely solvable Lie groups. Section 6 is devoted to single-input systems on Lie groups that differ from their derived subgroups; this class contains all solvable Lie groups. In Subsection 6.1 we give necessary and sufficient controllability conditions for such systems. These results yield the characterization of controllability on metabelian Lie groups presented in Subsection 6.2. In Subsection 6.3 we study right-invariant systems on a particular matrix metabelian Lie subgroup of the group of affine transformations of the n-space, and in Subsection 6.4 we treat bilinear systems in $\mathbb{R}^n$ that are projections of right-invariant systems on this matrix group. Finally, in Section 7 we classify single-input controllable systems on simply connected solvable Lie groups up to dimension 6, inclusive. The details on original sources are collected in Section 8; no references to authors are made immediately in the text. Some open questions related to the results presented are discussed in Section 9.

2. Definitions and basic facts

Let G be a Lie group and L its Lie algebra (i.e., the Lie algebra of right-invariant vector fields on G).

A *right-invariant control system* on G is an arbitrary subset $\Gamma \subset L$. An important particular class of systems is formed by systems *affine in control* $\Gamma = \{ A + \sum_{i=1}^m u_i B_i \mid u \in \mathbb{R} \}$, where $A, B_1, \dots, B_m$ are some elements of L.

The *attainable set* $\mathbb{A}$ of a system Γ is the subsemigroup of G generated by the set

$$\exp(\mathbb{R}_+\Gamma) = \{ \exp(tX) \mid X \in \Gamma,\, t \in \mathbb{R}_+ \}.$$

A system Γ is called *controllable* if $\mathbb{A} = G$.

The basic conditions necessary for controllability of a right-invariant system Γ on a Lie group G were given by V. Jurdjevic and H.J. Sussmann [**20**]: the Lie group G should be connected and the system Γ should satisfy the *rank controllability condition*, i.e., the Lie algebra $\mathrm{Lie}(\Gamma)$ generated by the system Γ must coincide with the whole Lie algebra L. That is why these conditions may be assumed without loss of generality. In particular, in the sequel all Lie groups are supposed connected.

The *derived series* of a Lie algebra L is constructed as follows:

$$L^{(1)} = [L, L],\ L^{(2)} = [L^{(1)}, L^{(1)}],\ \dots,\ L^{(i)} = [L^{(i-1)}, L^{(i-1)}],\ \dots, \quad i \in \mathbb{N}.$$

$L^{(1)}$ is the *derived subalgebra* of L, and the corresponding subgroup denoted by $G^{(1)}$ is the *derived subgroup* of G. The algebra L is called *solvable* if its derived series stabilizes at zero:

$$L \supset L^{(1)} \supset L^{(2)} \supset \ldots \supset L^{(N)} = \{0\}$$

for some $N \in \mathbb{N}$. Finally, a Lie group G is *solvable* if its Lie algebra L is solvable.

We denote topological closure and interior of a set M by $\operatorname{cl} M$ and $\operatorname{int} M$ respectively.

3. Nilpotent Lie groups

In this section we present controllability conditions for right-invariant systems on nilpotent Lie groups. Recall that a Lie algebra L is called *nilpotent* if its descending central series

$$L_{(1)} = [L, L],\ L_{(2)} = [L, L_{(1)}],\ \ldots,\ L_{(i)} = [L, L_{(i-1)}],\ \ldots, \quad i \in \mathbb{N},$$

stabilizes at zero:

$$L \supset L_{(1)} \supset L_{(2)} \supset \ldots \supset L_{(N)} = \{0\}$$

for some $N \in \mathbb{N}$. Any nilpotent Lie algebra is solvable since $L_{(i)} \supset L^{(i)}$, $i \in \mathbb{N}$. Another equivalent characterization of nilpotency of L is that all adjoint operators $\operatorname{ad} x$, $x \in L$, are nilpotent and thus have zero spectrum (this is important in Section 5).

3.1. Arbitrary systems. Controllability of a right-invariant system $\Gamma \subset L$ on a nilpotent Lie group G is completely characterized in terms of the wedge, i.e., the topologically closed convex cone $W(\Gamma) \subset L$ generated by Γ. It is standard that Γ and $W(\Gamma)$ are controllable or noncontrollable simultaneously.

THEOREM 3.1. *Let G be a nilpotent Lie group with Lie algebra L and let W be a wedge in L which generates L as a Lie algebra. Then W is controllable on G iff one of the following conditions holds:*

1. $\operatorname{int}_{W-W} W \cap L^{(1)} \neq \emptyset$,
2. $\operatorname{int} \operatorname{cl}(L^{(1)} + W) \cap \exp^{-1}(e) \neq \emptyset$.

REMARK. Here $\operatorname{int}_{W-W} W$ is the interior of the wedge W relative to the vector space $W - W$ generated by W, and e is the identity of the Lie group G.

The sufficiency in Theorem 3.1 is based on the description of maximal open subsemigroups S of nilpotent Lie groups in terms of their tangent objects

$$L(S) = \{\, x \in L \mid \exp(\mathbb{R}_+ x) \subset \operatorname{cl}(S) \,\}.$$

An open subsemigroup S of a Lie group G is proper, i.e., $S \neq G$ if and only if $e \notin S$. Hence the set of open subsemigroups in G is inductive, and any proper subsemigroup is contained in a *maximal* one.

THEOREM 3.2. *Let G be a nilpotent Lie group and let S be a maximal open proper subsemigroup of G. Then $L(S)$ is a halfspace bounded by a hyperplane algebra in $L(G)$.*

For necessity in Theorem 3.1, the Hahn-Banach theorem gives that $W + L^{(1)}$ is contained in a halfspace in L; then $\exp\left(\operatorname{int}\left(W + L^{(1)}\right)\right)$ is a proper open semigroup of G, which implies that $\exp(W)$ is contained in a proper subsemigroup of G.

The controllability test of Theorem 3.1 is essentially nilpotent. This result is false for the group $\mathrm{SL}(2, \mathbb{R})$; it also fails in the following solvable non-nilpotent example.

Example. Let G be the (unique) two-dimensional connected simply connected non-abelian Lie group, which is represented by the matrices

$$\left\{ \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \mid x > 0,\ y \in \mathbb{R} \right\}.$$

The Lie group G is solvable but not nilpotent. Its Lie algebra has the form

$$L = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in \mathbb{R} \right\}.$$

Consider the following wedge in L:

$$W = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a \in \mathbb{R},\ b \geq 0 \right\}.$$

Direct computations show that

$$\exp(\mathbb{R}_+ W) = \left\{ \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \mid x > 0,\ y \geq 0 \right\},$$

which is a proper subsemigroup of G, thus W is not controllable on G. On the other hand, it is easy to see that both conditions 1, 2 of Theorem 3.1 hold for the wedge W in this example.

3.2. Affine systems. For right-invariant systems affine in control

$$\Gamma = \left\{ A + \sum_{i=1}^{m} u_i B_i \mid u_i \in \mathbb{R} \right\} \subset L, \tag{3.1}$$

on simply connected nilpotent Lie groups there is a simple controllability criterion in terms of the Lie subalgebra L_0 of L generated by the non-drift vector fields:

$$L_0 = \mathrm{Lie}(B_1, \dots, B_m),$$

and the connected subgroup G_0 of G with Lie algebra L_0.

Theorem 3.3. *Let G be a simply connected nilpotent Lie group. Then system* (3.1) *is controllable on G iff $L_0 = L$.*

Sufficiency of the condition $L_0 = L$ for controllability of Γ is a well-known fact valid for arbitrary Lie groups G. So the essential part is necessity. Here the key role is played by the necessary controllability conditions in terms of the notion of a *symplectic vector*.

Consider the co-adjoint representation ρ^* of the group G in the dual space L^* of L. For any covector $\lambda \in L^*$, the co-adjoint orbit θ_λ of λ by the ρ^* action $\theta_\lambda = \rho^*_G(\lambda)$ is a smooth submanifold of L^* diffeomorphic to the homogeneous space G/E_λ, where E_λ is the isotropy subgroup of λ, $E_\lambda = \{ g \in G \mid \rho^*_g(\lambda) = \lambda \}$. Further, the system Γ can be projected from G onto the homogeneous space $G/E_\lambda \simeq \theta_\lambda$, and controllability of Γ on G obviously implies controllability of its projection Γ_λ on G/E_λ. This leads to necessary controllability conditions in terms of the co-adjoint representation.

Definition. $\lambda \in L^*$ is called a *symplectic vector* for $w \in L$ if the co-adjoint orbit θ_λ is not trivial and $\langle w, \beta \rangle > 0$ for all $\beta \in \theta_\lambda$.

(We denote by $\langle \cdot, \cdot \rangle$ the pairing of a vector and covector.)

THEOREM 3.4. *If there is a vector field $\xi \in L$ belonging to the centralizer of the subalgebra L_0 such that the nonzero vector field $[A, \xi]$ has a symplectic vector, then system* (3.1) *cannot be controllable on G.*

In fact, the existence of such vector field $\xi \in L$ yields that the function

$$f_\xi \ : \ \theta_\lambda \to \mathbb{R}, \qquad \beta \mapsto f_\xi(\beta) = -\langle \xi, \beta \rangle$$

is strictly increasing on the trajectories of projection of Γ onto the co-adjoint orbit θ_λ. Indeed, the solution of the Cauchy problem $\dot{g}(t) = A(g(t))$, $g(0) = g_0$ is given by $g(t) = \exp(tA)g_0$. Further, the function

$$h(t) = \operatorname{Ad}\left(g(t)^{-1}\right) = \operatorname{Ad}\left(g_0^{-1}\right) \circ \exp(-t \operatorname{ad} A)$$

has the derivative

$$\dot{h}(t) = \operatorname{Ad}\left(g_0^{-1}\right) \circ \exp(-t \operatorname{ad} A) \circ (-\operatorname{ad} A) = -\operatorname{Ad}\left(g(t)^{-1}\right) \circ \operatorname{ad} A.$$

Now for $\lambda \in L^*$ the co-adjoint action ρ^* of the element $g \in G$ is determined by

$$\rho_g^*(\lambda) = \operatorname{Ad}^*\left(g^{-1}\right)\lambda,$$

consequently, for any $\xi \in L$, $\lambda \in L^*$

$$\begin{aligned}
\frac{d}{dt} f_\xi(\rho^*_{g(t)}(\lambda)) &= -\frac{d}{dt}\langle \xi, \rho^*_{g(t)}(\lambda)\rangle = -\frac{d}{dt}\langle \xi, \operatorname{Ad}^*\left(g(t)^{-1}\right)\lambda\rangle \\
&= -\frac{d}{dt}\langle \operatorname{Ad}\left(g(t)^{-1}\right)\xi, \lambda\rangle = \langle \operatorname{Ad}\left(g(t)^{-1}\right) \circ \operatorname{ad} A(\xi), \lambda\rangle \\
&= \langle [A, \xi], \rho^*_{g(t)}(\lambda)\rangle.
\end{aligned}$$

Thus if λ is a symplectic vector for $[A, \xi]$, then f_ξ is increasing along co-adjoint orbits of trajectories of the field A. If in addition $\operatorname{ad} \xi$ vanishes on the subalgebra L_0, then the same holds for trajectories of the whole system Γ, which is impossible for a controllable system.

Another important fact for necessity in Theorem 3.3 is the following proposition related to *hypersurface* systems, i.e., affine systems (3.1) with L_0 a codimension one subalgebra of L.

THEOREM 3.5. *Let $\Gamma \subset L$ be an affine system* (3.1) *on G such that L_0 is a codimension one* ideal *of L.*

1. *If G_0 is closed in G, then Γ is controllable iff $A \notin L_0$ and $G/G_0 \simeq S^1$.*
2. *If G_0 is not closed in G, then Γ is controllable iff $A \notin L_0$.*

REMARK. In fact, the previous theorem holds without the assumption that L_0 is an ideal; this is important for a generalization of Theorem 3.3 to a subclass of solvable Lie groups including nilpotent ones (see Section 5 below).

Now we outline the scheme of proof of necessity in Theorem 3.3. Suppose the system Γ is controllable on the group G. Then the theory of symplectic vectors implies that the subalgebra L_0 is an ideal of L. The rank condition for Γ is satisfied: $\operatorname{Lie}(\Gamma) = \operatorname{Lie}(A, L_0) = L$, hence L_0 has codimension 0 or 1 in L. But the codimension one case is impossible since then Theorem 3.5 yields $G/G_0 \simeq S^1$, which contradicts simple connectedness of G. Thus $L_0 = L$, and necessity in Theorem 3.3 follows.

EXAMPLE. Let G be the Heisenberg group of dimension $2p+1$. It may be represented as a subgroup of $\mathrm{GL}(p+2,\mathbb{R})$ generated by the matrices

$$\mathrm{Id}+X_i,\ \mathrm{Id}+Y_i,\ Z,\quad i=1,\dots,p,$$

where

$$X_i = E_{1,i+1},\ Y_i = E_{i+1,p+2},\quad i=1,\dots,p.$$

(Id denotes the identity matrix, and E_{ij} stands for the square matrix with all zero entries except one unit in the i-th row and the j-th column.)

The Lie algebra L of G is spanned by the matrices

$$X_i,\ Y_i,\ Z,\quad i=1,\dots,p,$$

with the nonzero brackets

$$[X_i,Y_i]=Z,\quad i=1,\dots,p.$$

The Heisenberg group G is simply connected and nilpotent, hence Theorem 3.3 describes all controllable systems on G.

4. Lie groups with cocompact radical

Denote by $\mathrm{Rad}\, G$ the radical of a Lie group G, i.e., the maximal solvable normal subgroup of G. In this section we suppose that a Lie group G has cocompact radical, that is, the quotient group $K = G/\mathrm{Rad}\, G$ is compact. This wide class of Lie groups contains:

- solvable Lie groups ($K=\{e\}$),
- compact Lie groups,
- semidirect products of a vector space V with a compact Lie group ($V\subset \mathrm{Rad}\, G$).

The next theorem gives a Lie-algebraic description of controllability on such Lie groups, complete in the simply connected case.

THEOREM 4.1. *Suppose that $G/\mathrm{Rad}\, G$ is compact, and let $\Gamma\subset L$ be a system satisfying the rank condition $\mathrm{Lie}(\Gamma)=L$. If Γ is not contained in any half-space of L with boundary a subalgebra, then Γ is controllable on G. The converse holds if G is simply connected.*

This result is a consequence of the following classification of maximal subsemigroups of Lie groups with cocompact radical.

DEFINITION. A subsemigroup M of G is called a *maximal subsemigroup of G* if the only subsemigroups containing M are M and G, and M is not a subgroup.

THEOREM 4.2. *The maximal subsemigroups M with non-empty interior of a simply connected Lie group G with $G/\mathrm{Rad}\, G$ compact are in one-to-one correspondence with their tangent objects $L(M)$, and the latter are precisely the closed half-spaces with boundary a subalgebra. Further, M is the semigroup generated by $\exp(L(M))$.*

Theorem 4.1 follows from Theorem 4.2 since the attainable set of any noncontrollable right-invariant system $\Gamma\subset L$, $\mathrm{Lie}(\Gamma)=L$, is a proper subsemigroup of G contained in some maximal subsemigroup with non-empty interior.

5. Completely solvable Lie groups

In this section we suppose that Γ is a system affine in control as in (3.1).

DEFINITION. A solvable Lie algebra L is called *completely solvable* if all adjoint operators $\operatorname{ad} x$, $x \in L$, have real spectra. A Lie group is *completely solvable* if it has completely solvable Lie algebra.

(Such Lie algebras and Lie groups are also called *triangular over* $\mathbb{R}$ or *algebras*, resp. *groups, of type* (R) [**36**].)

The triangular group $\mathrm{T}(n)$ (see the example below) is completely solvable, as well as any of its subgroups. Nilpotent Lie groups are completely solvable since adjoint operators in nilpotent Lie algebras have zero spectrum. On the other hand, e.g., the group of motions of the plane $\mathrm{E}(2)$ is solvable but not completely solvable (the group $\mathrm{E}(2)$ and its simply connected covering $\widetilde{\mathrm{E}(2)}$ are treated in Section 6.3).

It turns out that the controllability criterion for systems affine in control on nilpotent Lie groups (Theorem 3.3) is valid for completely solvable Lie groups as well.

THEOREM 5.1. *Let G be a simply connected completely solvable Lie group. Then system* (3.1) *is controllable on G iff $L_0 = L$.*

Theorem 5.1 is based on the following general characterization of controllability for hypersurface systems.

THEOREM 5.2. *Let $\Gamma \subset L$ be an affine system* (3.1) *on G such that L_0 is a codimension one subalgebra of L.*

1. *If G_0 is closed in G, then Γ is controllable iff $A \notin L_0$ and $G/G_0 \simeq S^1$.*
2. *If G_0 is not closed in G, then Γ is controllable iff $A \notin L_0$.*

REMARK. Theorem 5.2 generalizes the analogous criterion of Theorem 3.5 with the additional assumption that L_0 is an ideal of L.

Theorem 5.2 implies the following *hypersurface principle* — a general necessary controllability condition for simply connected Lie groups.

THEOREM 5.3. *Let G be simply connected. Suppose that there exists a codimension one subalgebra l of L containing L_0. Then system* (3.1) *is not controllable.*

The sense of this proposition is that if the codimension one subalgebra $l \supset L_0$ exists, then attainable set of Γ lies "to one side" of the connected codimension one subgroup of G corresponding to l: by simple connectedness of G, this codimension one subgroup separates G into two disjoint parts.

Now we outline the proof of the controllability test of Theorem 5.1. Sufficiency of $L_0 = L$ for controllability is obvious, while the necessity follows from Theorem 5.3 and the observation that any proper subalgebra of a completely solvable Lie algebra L is contained in a codimension one subalgebra of L.

EXAMPLE. Let $G = \mathrm{T}(n)$ be the Lie group of all $n \times n$ upper triangular matrices with positive diagonal entries. $\mathrm{T}(n)$ is connected, simply connected and completely solvable. Its Lie algebra L consists of all $n \times n$ upper triangular matrices. By Theorem 5.1, an affine system Γ is controllable on G if and only if $L_0 = L$. The analogous result holds for any Lie subgroup of the triangular group $\mathrm{T}(n)$.

6. Lie groups differing from their derived subgroups

Lie groups G which satisfy the condition $G \neq G^{(1)}$ make up a wide class containing solvable Lie groups. On the other hand, no semisimple Lie groups belong to this class.

In this section we present controllability conditions for single-input systems

$$\Gamma = \{ A + uB \mid u \in \mathbb{R} \} = A + \mathbb{R}B \tag{6.1}$$

on such Lie groups.

Throughout this section we make the following assumptions:

1. $L \neq L^{(1)}$,
2. the adjoint operator $\operatorname{ad} B|_{L^{(1)}}$ has simple spectrum.

Notice that while the first assumption is obviously imposed by the class of Lie groups under consideration, the second one is added for simplicity of exposition and can be removed.

To formulate the results we need the following notation. Spectra of the adjoint operators in the first and second derived subalgebras $\operatorname{ad} B|_{L^{(1)}}$ and $\operatorname{ad} B|_{L^{(2)}}$ are denoted by $\operatorname{Sp}^{(1)}$ and $\operatorname{Sp}^{(2)}$ respectively. For any eigenvalue $a \in \operatorname{Sp}^{(1)}$ the eigenspace of $\operatorname{ad} B|_{L^{(1)}}$ corresponding to a is denoted by $L(a)$. In view of simplicity of the spectrum of $\operatorname{ad} B|_{L^{(1)}}$, there is the following decomposition of the derived subalgebra of L into a direct sum of one- and two-dimensional eigenspaces:

$$L^{(1)} = \sum\nolimits^{\oplus} \{ L(a) \mid a \in \operatorname{Sp}^{(1)}, \operatorname{Im} a \geq 0 \}.$$

If $B \notin L^{(1)}$ and $\dim L^{(1)} = \dim L - 1$, then we have

$$L = \mathbb{R}B \oplus L^{(1)} = \mathbb{R}B \oplus \sum\nolimits^{\oplus} \{ L(a) \mid a \in \operatorname{Sp}^{(1)}, \operatorname{Im} a \geq 0 \}$$

and the corresponding decomposition for a vector $A \in L$:

$$A = A_B + \sum \{ A(a) \in L(a) \mid a \in \operatorname{Sp}^{(1)}, \operatorname{Im} a \geq 0 \}.$$

(The signs $\oplus$ and $\sum^{\oplus}$ above denote direct sums of vector spaces.)

6.1. Controllability conditions. There are restrictive necessary controllability conditions for single-input systems on simply connected Lie groups $G \neq G^{(1)}$.

THEOREM 6.1. *Let a Lie group G be simply connected and $G \neq G^{(1)}$. If system* (6.1) *is controllable, then:*

1. $\dim L^{(1)} = \dim L - 1$,
2. $B \notin L^{(1)}$,
3. $\operatorname{Sp}^{(1)} \cap \mathbb{R} \subset \operatorname{Sp}^{(2)}$,
4. $A(a) \neq 0$ *for all* $a \in \operatorname{Sp}^{(1)} \setminus \operatorname{Sp}^{(2)}$.

Notice that controllability of (at least one) single-input system on a simply connected Lie group $G \neq G^{(1)}$ implies restriction 1, $\dim G^{(1)} = \dim G - 1$, on the Lie group, not on the system. In fact, for solvable Lie algebras L restriction 3 of Theorem 6.1 depends on L but not on the system (provided that conditions 1 and 2 hold). This restrictive force of necessary conditions of Theorem 6.1 is crucial for classification of controllable systems on small-dimensional simply connected solvable Lie groups in Section 7.

Theorem 6.1 is based on the rank controllability condition and the necessary controllability condition of Theorem 5.3. The justification of the fact that these two results should be enough to give necessary controllability conditions on simply connected solvable Lie groups is provided by the general controllability test for simply connected Lie groups with cocompact radical of Theorem 4.1.

There are sufficient controllability conditions close to the previous necessary ones in the case of simply connected Lie groups.

THEOREM 6.2. *Suppose that the following conditions are satisfied for Lie algebra L of a Lie group G and system* (6.1)*:*

1. $\dim L^{(1)} = \dim L - 1$,
2. $B \notin L^{(1)}$,
3. $\mathrm{Sp}^{(1)} \cap \mathbb{R} \subset \mathrm{Sp}^{(2)}$,
4. $A(a) \neq 0$ *for all* $a \in \mathrm{Sp}^{(1)} \setminus \mathbb{R}$.
5. $\mathrm{Sp}^{(1)} \cap \mathbb{R} = \emptyset$ *or* $\mathrm{Sp}^{(1)} \subset \{\mathrm{Re}\, z > 0\}$ *or* $\mathrm{Sp}^{(1)} \subset \{\mathrm{Re}\, z < 0\}$.

Then system (6.1) *is controllable on G.*

This theorem is obtained with the help of the Lie saturation technique introduced by V. Jurdjevic and I. Kupka [**22**] (see also G. Sallet [**31**] and V. Jurdjevic [**23**]): a sequence of increasing lower bounds of the tangent cone to the attainability set at the identity is shown to stabilize at the whole Lie algebra L.

The above controllability conditions for Lie groups $G \neq G^{(1)}$ yield complete description of controllability for several particular classes of Lie groups: metabelian ones, some subgroups of the group of affine transformations of the n-space, and small-dimensional simply connected solvable Lie groups. These results are presented in the following subsections and in Section 7.

6.2. Metabelian groups. Lie algebras L having derived series of length 2:

$$L \supset L^{(1)} \supset L^{(2)} = \{0\},$$

are called *metabelian*. A Lie group with a metabelian Lie algebra is also called *metabelian*.

A metabelian Lie algebra is obviously solvable. Thus results of the previous subsection yield controllability conditions for metabelian Lie groups.

THEOREM 6.3. *Let G be a metabelian Lie group. Then the following conditions are sufficient for controllability of system* (6.1) *on G:*

1. $\dim L^{(1)} = \dim L - 1$,
2. $B \notin L^{(1)}$,
3. $\mathrm{Sp}^{(1)} \cap \mathbb{R} = \emptyset$,
4. $A(a) \neq 0$ *for any eigenvalue* $a \in \mathrm{Sp}^{(1)}$.

If the group G is simply connected, then conditions 1 – 4 *are also necessary for controllability of system* (6.1) *on G.*

6.3. Semidirect products. Let V be a real finite-dimensional vector space, $\dim V = n$, and M a nonzero linear operator in V. Consider the metabelian Lie algebra $L(M)$ which is the semidirect product of the abelian Lie algebra V with the one-dimensional Lie algebra $\mathbb{R}M$. This Lie algebra can be represented by $(n+1) \times (n+1)$ matrices:

$$L(M) = \left\{ \begin{pmatrix} Mt & b \\ 0 & 0 \end{pmatrix} \mid t \in \mathbb{R},\ b \in \mathbb{R}^n \right\} \subset \mathrm{gl}(n+1, \mathbb{R}). \tag{6.2}$$

The connected Lie subgroup of $\mathrm{GL}(n+1,\mathbb{R})$ corresponding to $L(M)$ is denoted by $G(M)$. It is the semidirect product of the vector Lie group $\mathbb{R}^n$ with the one-dimensional Lie group $G_1 = \{\exp(Mt) \mid t \in \mathbb{R}\}$. Elements of the group $G(M)$ are the matrices

$$\begin{pmatrix} \exp(Mt) & p \\ 0 & 1 \end{pmatrix}, \quad t \in \mathbb{R},\ p \in \mathbb{R}^n,$$

thus $G(M)$ may be viewed as a subgroup of the group of affine transformations of the n-space generated by the one-parameter group of automorphisms G_1 and all translations $p \in \mathbb{R}^n$. The group $G(M)$ is not simply connected iff the one-parameter subgroup G_1 is periodic, which obviously occurs iff

$$\left.\begin{array}{l} \text{the matrix } M \text{ is semisimple,} \\ \mathrm{Sp}(M) = ir \cdot (k_1, \dots, k_n) \quad \text{for some } r \in \mathbb{R},\ (k_1, \dots, k_n) \in \mathbb{Z}^n. \end{array}\right\} \tag{6.3}$$

If conditions (6.3) hold, then a controllability test on $G(M)$ is given by the result of B. Bonnard, V. Jurdjevic, I. Kupka and G. Sallet [**7**] for Lie groups that are semidirect products of vector spaces with compact Lie groups. Otherwise the controllability test for simply connected metabelian Lie groups (Theorem 6.3) implies the following.

THEOREM 6.4. *Suppose that conditions* (6.3) *are violated. System* (6.1) *is controllable on the group $G(M)$ if and only if the following conditions hold:*

1. $\mathrm{Sp}(M) \cap \mathbb{R} = \emptyset$,
2. $B \notin L^{(1)}$,
3. $\mathrm{span}(B, A, (\mathrm{ad}\, B)A, \dots, (\mathrm{ad}\, B)^{n-1}A) = L$.

REMARK. Suppose that conditions (6.3) hold, i.e., the group $G(M)$ is not simply connected. Then conditions 1–3 of the previous theorem are necessary and sufficient for controllability of system (6.1) on $\widetilde{G(M)}$ — the simply connected covering of $G(M)$. And for the group $G(M)$ itself conditions 1–3 are then sufficient for controllability.

EXAMPLE. Let $G = \mathrm{E}(2)$ be the Euclidean group of motions of the plane $\mathbb{R}^2$. It can be represented as the group of 3×3 matrices of the form

$$\begin{pmatrix} \cos t & -\sin t & s_1 \\ \sin t & \cos t & s_2 \\ 0 & 0 & 1 \end{pmatrix},$$

with the rotation matrix and translation vector respectively

$$\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \in \mathrm{SO}(2) \text{ for } t \in \mathbb{R}, \quad \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} \in \mathbb{R}^2.$$

The corresponding matrix Lie algebra L is spanned by the matrices

$$x = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and has form (6.2):

$$L = L(M), \quad M = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

It is solvable (in fact, metabelian):

$$L^{(1)} = \mathrm{span}(y, z) \supset L^{(2)} = \{0\},$$

but not completely solvable:

$$\mathrm{Sp}(\mathrm{ad}\, x) = \{\pm i,\ 0\}.$$

The Lie group $\mathrm{E}(2) = G(M)$ is connected but not simply connected, compare with (6.3).

Consider the system $\Gamma = A + \mathbb{R}B$ on $\widetilde{\mathrm{E}(2)}$ — the simply connected covering of $\mathrm{E}(2)$. A complete characterization of controllability of Γ on $\widetilde{\mathrm{E}(2)}$ is derived from the remark after Theorem 6.4:

Theorem 6.5. *System* (6.1) *is controllable on* $\widetilde{\mathrm{E}(2)}$ *if and only if the vectors* A, B *are linearly independent and* $B \notin \mathrm{span}(y, z)$.

Compare the controllability conditions for $\widetilde{\mathrm{E}(2)}$ with the following conditions for $\mathrm{E}(2)$ (derived from Theorem 1 [**7**]):

Theorem 6.6. *System* (6.1) *is controllable on* $\mathrm{E}(2)$ *if and only if the vectors* A, B *are linearly independent and* $\{A, B\} \not\subset \mathrm{span}(y, z)$.

6.4. Bilinear systems. Global controllability conditions for bilinear systems of the form

$$\dot{x} = uAx + b, \qquad x \in \mathbb{R}^n, \quad u \in \mathbb{R}, \tag{Σ}$$

where A is a constant real $n \times n$ matrix and $b \in \mathbb{R}^n$, are obtained with the help of results for the matrix group of the previous subsection.

The system Σ may be rewritten as the following bilinear system in the hyperplane $\{x_{n+1} = 1\} \subset \mathbb{R}^{n+1}$:

$$\frac{d}{dt}\begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ 1 \end{pmatrix} + u\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ 1 \end{pmatrix}, \quad \begin{pmatrix} x \\ 1 \end{pmatrix} \in \mathbb{R}^{n+1} \tag{Σ'}$$

with the matrices

$$\overline{A} = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}, \ \overline{B} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \in L(A).$$

Moreover, Σ' is projection of the single-input right-invariant system $\overline{\Gamma} = \overline{A} + \mathbb{R}\overline{B} \subset L(A)$ on the matrix group $G(A)$ onto the hyperplane $\{x_{n+1} = 1\}$. Hence controllability of $\overline{\Gamma}$ on $G(A)$ implies global controllability of Σ' in the hyperplane $\{x_{n+1} = 1\}$ and thus global controllability of Σ in $\mathbb{R}^n$.

On the other hand, necessary controllability conditions for Σ can easily be obtained by studying codimension 1 and 2 invariant spaces of this system in $\mathbb{R}^n$.

This gives the following controllability test for Σ:

Theorem 6.7. *The system* Σ *is globally controllable on* $\mathbb{R}^n$ *if and only if the following conditions hold:*

1. *the matrix* A *has a purely complex spectrum,*
2. $\mathrm{span}(b, Ab, \dots, A^{n-1}b) = \mathbb{R}^n$.

7. Small-dimensional solvable Lie groups

Given a Lie algebra L, there is the "largest" connected Lie group G having Lie algebra L — the simply connected one. All other connected Lie groups with Lie algebra L are "smaller" than G in the sense that they are quotients G/C, where C is a discrete subgroup of center of G. A right-invariant system $\Gamma \subset L$ may thus be considered on any of these groups, and the simply connected group G is the hardest among them to control. Hence given a right-invariant system Γ on a Lie group (or a homogeneous space of a Lie group) H, it is natural first to study its controllability on the simply connected covering $\widetilde{H}$ of H. If Γ is controllable on $\widetilde{H}$, then it is obviously controllable on H (and on all its homogeneous spaces); in the opposite case one should use particular geometric properties of H (e.g., existence of periodic one-parameter subgroups) to verify controllability of Γ on H. It is obvious and remarkable that controllability conditions on a simply connected Lie group G should have a completely Lie-algebraic form: they are fully determined by the Lie algebra L and its subset Γ (see, e.g., Theorems 3.3, 5.1, 6.1, 6.3, 6.4).

This motivates the following definition.

Definition. A system $\Gamma \subset L$ is called *controllable* if it is controllable on the (unique) connected simply connected Lie group with Lie algebra L.

And the next definition makes sense at least for solvable Lie algebras in small dimensions.

Definition. A Lie algebra L is called *controllable* if there are $A, B \in L$ such that the system $\Gamma = A + \mathbb{R}B$ is controllable.

Indeed, it turns out that controllability conditions on solvable Lie groups (Sections 4 and 6) imply that for solvable small-dimensional Lie algebras L:

- existence of a controllable single-input system $\Gamma \subset L$, i.e., controllability of L, is a strong restriction on L;
- if L is controllable, then almost all pairs $(A, B) \in L \times L$ give rise to controllable systems $\Gamma = A + \mathbb{R}B$;
- controllability of a system $\Gamma \subset L$ depends primarily on L but not on Γ.

Moreover, these results yield a complete description of controllability in small-dimensional solvable Lie algebras presented in the following subsections.

Up to dimension 6 inclusive we describe all solvable Lie algebras L that are controllable, and give controllability tests for single-input systems $\Gamma = A + \mathbb{R}B \subset L$ (the only gap in this picture is the class $L_{6,IV}$ of six-dimensional Lie algebras not completely studied).

The general "bird's-eye view" of controllable small-dimensional solvable Lie algebras is as follows:

$\dim L = 1$: the (unique) Lie algebra is controllable;
$\dim L = 2$: the two Lie algebras are noncontrollable;
$\dim L = 3$: there is one family of controllable Lie algebras $L_3(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$;
$\dim L = 4$: there is one family of controllable Lie algebras $L_4(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$;
$\dim L = 5$: there are two families of controllable Lie algebras:

1. $L_{5,I}(\lambda, \mu)$, $\lambda, \mu \in \mathbb{C} \setminus \mathbb{R}$, $\lambda \neq \mu, \bar{\mu}$,
2. $L_{5,II}(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$;

$\dim L = 6$: there are five families of controllable Lie algebras:

1. $L_{6,I}(\lambda, \mu)$, $\lambda, \mu \in \mathbb{C} \setminus \mathbb{R}$, $\lambda \neq \mu, \bar{\mu}$,
2. $L_{6,II}(\lambda, \mu, k)$, $\lambda, \mu \in \mathbb{C} \setminus \mathbb{R}$, $\operatorname{Re}\lambda = \operatorname{Re}\mu$, $\lambda \neq \mu, \bar{\mu}$, $k \in \mathbb{R} \setminus \{0\}$,
3. $L_{6,III}(\lambda, k, l)$, $\lambda \in \mathbb{C} \setminus (\mathbb{R} \cup i\mathbb{R})$, $k, l \in \mathbb{R}$, $k^2 + l^2 \neq 0$,
4. $L_{6,IV}(\lambda, k, l)$, $\lambda \in \mathbb{C} \setminus (\mathbb{R} \cup i\mathbb{R})$, $k, l \in \mathbb{R}$, $k^2 + l^2 \neq 0$,
5. $L_{6,V}(\lambda, k, l)$, $\lambda \in \mathbb{C} \setminus (\mathbb{R} \cup i\mathbb{R})$, $k, l \in \mathbb{R}$, $k^2 + l^2 \neq 0$,

and one exceptional class $L_{6,IV}(bi)$, $b \in \mathbb{R} \setminus \{0\}$, containing both controllable and noncontrollable Lie algebras.

All controllable Lie algebras L are presented by a scheme in the complex plane $\mathbb{C}$ containing eigenvalues of the adjoint operator $\operatorname{ad} B|_{L^{(1)}}$, $B \in L \setminus L^{(1)}$, and arrows between these eigenvalues describing Lie brackets between eigenvectors of the operator $\operatorname{ad} B|_{L^{(1)}}$ (these schemes are given at the very end of this section). Notice that for solvable Lie algebras L with codimension one subalgebras $L^{(1)}$ (and only such solvable Lie algebras may be controllable, see condition 1 of Theorem 6.1), spectra of all adjoint operators $\operatorname{ad} B|_{L^{(1)}}$, $B \in L \setminus L^{(1)}$, are homothetic with respect to the origin $0 \in \mathbb{C}$, and the homothety equivalence class of spectra of $\operatorname{ad} B|_{L^{(1)}}$, $B \in L \setminus L^{(1)}$, is determined not by $B \in L \setminus L^{(1)}$ but by L itself (in fact, by the isomorphism class of L).

Now we present the classification of controllability in small-dimensional solvable Lie algebras. These results are obtained by virtue of controllability conditions of Sections 4 and 6.

7.1. One-dimensional Lie algebras. The unique one-dimensional Lie algebra is abelian and isomorphic to $\mathbb{R}$.

THEOREM 7.1. *The one-dimensional Lie algebra $\mathbb{R}$ is controllable.*
A system $\Gamma = A + \mathbb{R}B \subset \mathbb{R}$ is controllable iff $B \neq 0$.

7.2. Two-dimensional Lie algebras. There are two nonisomorphic two-dimensional Lie algebras: abelian $\mathbb{R}^2$, and solvable non-abelian $S_2 = \operatorname{span}(x, y)$, $[x, y] = y$.

THEOREM 7.2. *Both two-dimensional Lie algebras $\mathbb{R}^2$ and S_2 are not controllable.*

7.3. Three-dimensional Lie algebras.

CONSTRUCTION. The Lie algebra $L_3(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, Fig. 1.

$$L_3(\lambda) = \operatorname{span}(x, y, z),$$
$$\operatorname{ad} x|_{\operatorname{span}(y,z)} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \quad \lambda = a + bi.$$

The Lie algebra $L_3(\lambda)$ is schematically represented in Fig. 1 by the eigenvalues $\lambda, \bar{\lambda} \in \mathbb{C}$ and realifications of the eigenvectors $y, z \in L_3(\lambda)$ of the adjoint operator $\operatorname{ad} x|_{\operatorname{span}(y,z)}$.

THEOREM 7.3. *A three-dimensional solvable Lie algebra is controllable iff it is isomorphic to $L_3(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$.*

THEOREM 7.4. *Let $L = L_3(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, and let $A, B \in L$. The system $\Gamma = A + \mathbb{R}B \subset L$ is controllable iff the following conditions are satisfied:*
1. *$B \notin L^{(1)}$,*
2. *the vectors A and B are linearly independent.*

7.4. Four-dimensional Lie algebras.

CONSTRUCTION. The Lie algebra $L_4(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, Fig. 2.

$$L_4(\lambda) = \operatorname{span}(x, y, z, w),$$
$$\operatorname{ad} x|_{\operatorname{span}(y,z,w)} = \begin{pmatrix} a & -b & 0 \\ b & a & 0 \\ 0 & 0 & 2a \end{pmatrix}, \quad \lambda = a + bi,$$
$$[y, z] = w.$$

The arrows in the schematic representation of the Lie algebra $L_4(\lambda)$ in Fig. 2 mean that Lie bracket of the vectors y and z gives the vector w.

THEOREM 7.5. *A four-dimensional solvable Lie algebra is controllable iff it is isomorphic to* $L_4(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

THEOREM 7.6. *Let* $L = L_4(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, *and let* $A, B \in L$. *The system* $\Gamma = A + \mathbb{R}B \subset L$ *is controllable iff the following conditions are satisfied:*

1. $B \notin L^{(1)}$,
2. $A(\lambda) \neq 0$.

7.5. Five-dimensional Lie algebras.

CONSTRUCTION. The Lie algebra $L_{5,I}(\lambda, \mu)$, $\lambda, \mu \in \mathbb{C} \setminus \mathbb{R}$, Fig. 3.

$$L_{5,I}(\lambda, \mu) = \operatorname{span}(x, y, z, u, v),$$
$$\operatorname{ad} x|_{\operatorname{span}(y,z,u,v)} = \begin{pmatrix} a & -b & 0 & 0 \\ b & a & 0 & 0 \\ 0 & 0 & c & -d \\ 0 & 0 & d & c \end{pmatrix}, \quad \lambda = a + bi, \ \mu = c + di.$$

CONSTRUCTION. The Lie algebra $L_{5,II}(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, Fig. 4.

$$L_{5,II}(\lambda) = \operatorname{span}(x, y, z, u, v),$$
$$\operatorname{ad} x|_{\operatorname{span}(y,z,u,v)} = \begin{pmatrix} a & -b & 0 & 0 \\ b & a & 0 & 0 \\ 1 & 0 & a & -b \\ 0 & 1 & b & a \end{pmatrix}, \quad \lambda = a + bi.$$

The circles around the eigenvalues λ, $\bar{\lambda}$ in Fig. 4 mean that they have double algebraic multiplicity. (Notice that according to the previous matrix their geometric multiplicity is single.)

THEOREM 7.7. *A five-dimensional solvable Lie algebra is controllable iff it is isomorphic to* $L_{5,I}(\lambda, \mu)$, $\lambda, \mu \in \mathbb{C} \setminus \mathbb{R}$, $\lambda \neq \mu, \bar{\mu}$, *or* $L_{5,II}(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

THEOREM 7.8. *Let* $L = L_{5,I}(\lambda, \mu)$, $\lambda, \mu \in \mathbb{C} \setminus \mathbb{R}$, $\lambda \neq \mu, \bar{\mu}$, *and let* $A, B \in L$. *The system* $\Gamma = A + \mathbb{R}B \subset L$ *is controllable iff the following conditions are satisfied:*

1. $B \notin L^{(1)}$,
2. $A(\lambda) \neq 0$ *and* $A(\mu) \neq 0$.

THEOREM 7.9. *Let* $L = L_{5,II}(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, *and let* $A, B \in L$. *The system* $\Gamma = A + \mathbb{R}B \subset L$ *is controllable iff the following conditions are satisfied:*

1. $B \notin L^{(1)}$,
2. $\operatorname{top}(A, \lambda) \neq 0$.

REMARK. The notation $\operatorname{top}(A, \lambda) \neq 0$ in Theorem 7.9 (and in Theorem 7.15 below) means that the vector A has a nonzero component corresponding to the higher order root space of the operator $\operatorname{ad} B|_{L^{(1)}}$ corresponding to its eigenvalue λ.

7.6. Six-dimensional Lie algebras.

CONSTRUCTION. The Lie algebra $L_{6,I}(\lambda,\mu)$, $\lambda,\mu\in\mathbb{C}\setminus\mathbb{R}$, Fig. 5.

$$L_{6,I}(\lambda,\mu)=\operatorname{span}(x,y,z,u,v,w),$$
$$\operatorname{ad}x|_{\operatorname{span}(y,z,u,v,w)}=\begin{pmatrix} a & -b & 0 & 0 & 0\\ b & a & 0 & 0 & 0\\ 0 & 0 & c & -d & 0\\ 0 & 0 & d & c & 0\\ 0 & 0 & 0 & 0 & 2a\end{pmatrix},\quad \lambda=a+bi,\ \mu=c+di,$$
$$[y,z]=w.$$

CONSTRUCTION. The Lie algebra $L_{6,II}(\lambda,\mu,k)$, $\lambda,\mu\in\mathbb{C}\setminus\mathbb{R}$, $\operatorname{Re}\lambda=\operatorname{Re}\mu$, $k\in\mathbb{R}\setminus\{0\}$, Fig. 6.

$$L_{6,II}(\lambda,\mu,k)=\operatorname{span}(x,y,z,u,v,w),$$
$$\operatorname{ad}x|_{\operatorname{span}(y,z,u,v,w)}=\begin{pmatrix} a & -b & 0 & 0 & 0\\ b & a & 0 & 0 & 0\\ 0 & 0 & a & -d & 0\\ 0 & 0 & d & a & 0\\ 0 & 0 & 0 & 0 & 2a\end{pmatrix},\quad \lambda=a+bi,\ \mu=a+di,$$
$$[y,z]=w,\quad [u,v]=kw.$$

CONSTRUCTION. The Lie algebra $L_{6,III}(\lambda,k,l)$, $\lambda\in\mathbb{C}\setminus(\mathbb{R}\cup i\,\mathbb{R})$, $k,l\in\mathbb{R}$, $k^2+l^2\neq 0$, Fig. 7.

$$L_{6,III}(\lambda,k,l)=\operatorname{span}(x,y,z,u,v,w),$$
$$\operatorname{ad}x|_{\operatorname{span}(y,z,u,v,w)}=\begin{pmatrix} a & -b & 0 & 0 & 0\\ b & a & 0 & 0 & 0\\ 0 & 0 & 3a & -b & 0\\ 0 & 0 & b & 3a & 0\\ 0 & 0 & 0 & 0 & 2a\end{pmatrix},\quad \lambda=a+bi,$$
$$[w,y]=ku+lv,\quad [w,z]=-lu+kz.$$

CONSTRUCTION. The Lie algebra $L_{6,IV}(\lambda,k,l)$, $\lambda\in\mathbb{C}\setminus(\mathbb{R}\cup i\,\mathbb{R})$, $k,l\in\mathbb{R}$, $k^2+l^2\neq 0$, Fig. 8.

$$L_{6,IV}(\lambda,k,l)=\operatorname{span}(x,y,z,u,v,w),$$
$$\operatorname{ad}x|_{\operatorname{span}(y,z,u,v,w)}=\begin{pmatrix} a & -b & 0 & 0 & 0\\ b & a & 0 & 0 & 0\\ 0 & 0 & -a & -b & 0\\ 0 & 0 & b & -a & 0\\ 0 & 0 & 0 & 0 & 0\end{pmatrix},\quad \lambda=a+bi,$$
$$[y,v]=-[z,u]=kw,\quad [y,u]=[z,v]=lw.$$

Construction. The Lie algebra $L_{6,V}(\lambda, k, l)$, $\lambda \in \mathbb{C} \setminus (\mathbb{R} \cup i\,\mathbb{R})$, $k, l \in \mathbb{R}$, $k^2 + l^2 \neq 0$, Fig. 9.

$$L_{6,V}(\lambda, k, l) = \operatorname{span}(x, y, z, u, v, w),$$

$$\operatorname{ad} x|_{\operatorname{span}(y,z,u,v,w)} = \begin{pmatrix} a & -b & 0 & 0 & 0 \\ b & a & 0 & 0 & 0 \\ 1 & 0 & a & -b & 0 \\ 0 & 1 & b & a & 0 \\ 0 & 0 & 0 & 0 & 2a \end{pmatrix}, \quad \lambda = a + bi,$$

$$[y, z] = kw, \quad [y, u] = [z, v] = lw.$$

Construction. The class of Lie algebras $L_{6,VI}(bi)$, $b \in \mathbb{R} \setminus \{0\}$, Fig. 10.

A Lie algebra L belongs to the class $L_{6,VI}(bi)$ if:

$L = \operatorname{span}(x, y, z, u, v, w)$,
$L^{(1)} = \operatorname{span}(y, z, u, v, w)$,
$\operatorname{Sp}(\operatorname{ad} x|_{L^{(1)}}) = \{\pm bi, 0\}$,
both eigenvalues $\pm bi$ have double algebraic multiplicity,
$w \in L^{(2)}$.

The class $L_{6,VI}$ contains a lot of nonisomorphic Lie algebras in which multiplication can not be described in detail as in Lie algebras $L_{6,I}$–$L_{6,V}$.

Theorem 7.10. *Let a six-dimensional solvable Lie algebra L not belong to the class $L_{6,VI}(bi)$, $b \in \mathbb{R} \setminus \{0\}$. Then L is controllable iff it is isomorphic to one of the following Lie algebras:*

1. $L_{6,I}(\lambda, \mu)$, $\lambda, \mu \in \mathbb{C} \setminus \mathbb{R}$, $\lambda \neq \mu, \bar{\mu}$;
2. $L_{6,II}(\lambda, \mu, k)$, $\lambda, \mu \in \mathbb{C} \setminus \mathbb{R}$, $\operatorname{Re} \lambda = \operatorname{Re} \mu$, $\lambda \neq \mu, \bar{\mu}$, $k \in \mathbb{R} \setminus \{0\}$;
3. $L_{6,III}(\lambda, k, l)$, $\lambda \in \mathbb{C} \setminus (\mathbb{R} \cup i\,\mathbb{R})$, $k, l \in \mathbb{R}$, $k^2 + l^2 \neq 0$;
4. $L_{6,IV}(\lambda, k, l)$, $\lambda \in \mathbb{C} \setminus (\mathbb{R} \cup i\,\mathbb{R})$, $k, l \in \mathbb{R}$, $k^2 + l^2 \neq 0$;
5. $L_{6,V}(\lambda, k, l)$, $\lambda \in \mathbb{C} \setminus (\mathbb{R} \cup i\,\mathbb{R})$, $k, l \in \mathbb{R}$, $k^2 + l^2 \neq 0$.

Theorem 7.11. *Let $L = L_{6,I}(\lambda, \mu)$, $\lambda, \mu \in \mathbb{C} \setminus \mathbb{R}$, $\lambda \neq \mu, \bar{\mu}$, and let $A, B \in L$. The system $\Gamma = A + \mathbb{R} B \subset L$ is controllable iff the following conditions are satisfied:*

1. $B \notin L^{(1)}$,
2. $A(\lambda) \neq 0$ *and* $A(\mu) \neq 0$.

Theorem 7.12. *Let $L = L_{6,II}(\lambda, \mu, k)$, $\lambda, \mu \in \mathbb{C} \setminus \mathbb{R}$, $\operatorname{Re} \lambda = \operatorname{Re} \mu$, $\lambda \neq \mu, \bar{\mu}$, $k \in \mathbb{R} \setminus \{0\}$, and let $A, B \in L$. The system $\Gamma = A + \mathbb{R} B \subset L$ is controllable iff the following conditions are satisfied:*

1. $B \notin L^{(1)}$,
2. $A(\lambda) \neq 0$ *and* $A(\mu) \neq 0$.

Theorem 7.13. *Let $L = L_{6,III}(\lambda, k, l)$, $\lambda \in \mathbb{C} \setminus (\mathbb{R} \cup i\,\mathbb{R})$, $k, l \in \mathbb{R}$, $k^2 + l^2 \neq 0$, and let $A, B \in L$. The system $\Gamma = A + \mathbb{R} B \subset L$ is controllable iff the following conditions are satisfied:*

1. $B \notin L^{(1)}$,
2. $A(\lambda) \neq 0$.

Theorem 7.14. *Let $L = L_{6,IV}(\lambda, k, l)$, $\lambda \in \mathbb{C} \setminus (\mathbb{R} \cup i\,\mathbb{R})$, $k, l \in \mathbb{R}$, $k^2 + l^2 \neq 0$, and let $A, B \in L$. The system $\Gamma = A + \mathbb{R} B \subset L$ is controllable iff the following conditions are satisfied:*

1. $B \notin L^{(1)}$,
2. $A(\lambda) \neq 0$ *and* $A(-\lambda) \neq 0$.

THEOREM 7.15. *Let* $L = L_{6,V}(\lambda, k, l)$, $\lambda \in \mathbb{C} \setminus (\mathbb{R} \cup i\mathbb{R})$, $k, l \in \mathbb{R}$, $k^2 + l^2 \neq 0$, *and let* $A, B \in L$. *The system* $\Gamma = A + \mathbb{R}B \subset L$ *is controllable iff the following conditions are satisfied:*

1. $B \notin L^{(1)}$,
2. $\mathrm{top}(A, \lambda) \neq 0$.

REMARK. The class $L_{6,VI}(bi)$, $b \in \mathbb{R} \setminus \{0\}$, contains both controllable and noncontrollable Lie algebras.

Controllable solvable Lie algebras up to dimension 6:

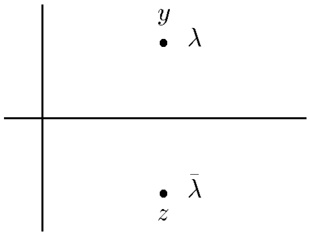

Fig. 1. $L_3(\lambda)$.

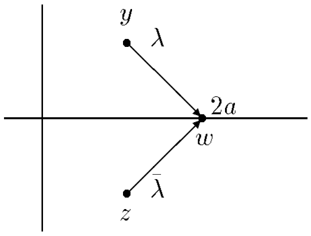

Fig. 2. $L_4(\lambda)$, $\mathrm{Re}\,\lambda = a$.

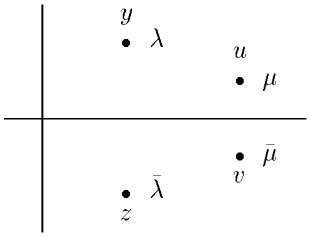

Fig. 3. $L_{5,I}(\lambda, \mu)$.

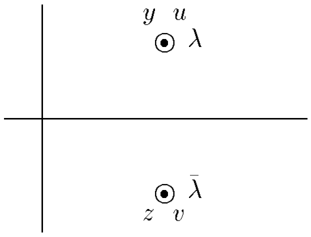

Fig. 4. $L_{5,II}(\lambda)$.

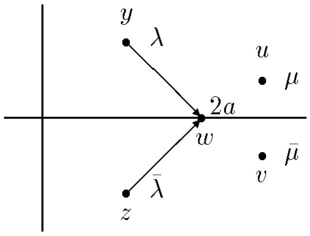

Fig. 5. $L_{6,I}(\lambda, \mu)$, $\mathrm{Re}\,\lambda = a$.

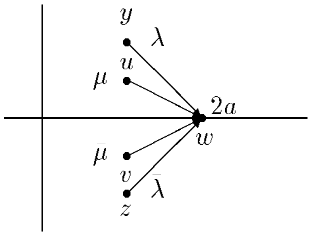

Fig. 6. $L_{6,II}(\lambda, \mu, k)$, $\mathrm{Re}\,\lambda = \mathrm{Re}\,\mu = a$.

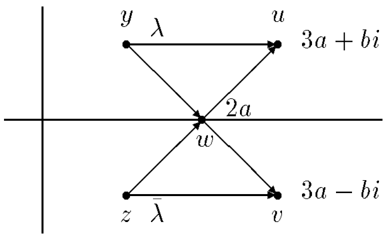

Fig. 7. $L_{6,III}(\lambda, k, l)$, $\lambda = a + bi$.

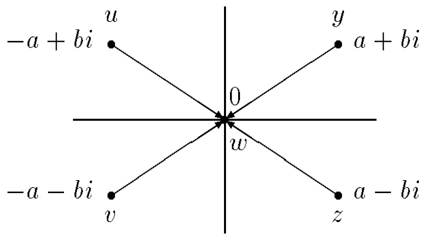

Fig. 8. $L_{6,IV}(\lambda, k, l)$, $\lambda = a + bi$.

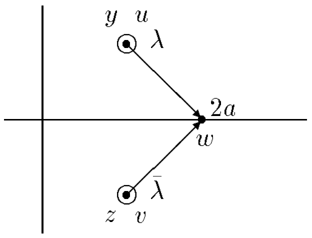

Fig. 9. $L_{6,V}(\lambda, \mu)$, $\operatorname{Re}\lambda = a$.

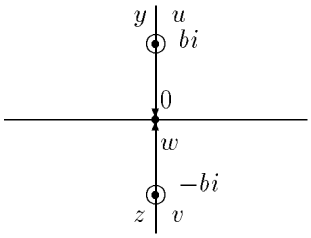

Fig. 10. $L_{6,VI}(bi)$.

8. Bibliographical notes

The results presented in this paper were obtained by the following authors:
Subsection 3.1: J. Hilgert, K. H. Hofmann, and J. D. Lawson [**13**].
Subsection 3.2: V. Ayala Bravo [**6**].
Section 4: J. D. Lawson [**25**].
Sections 5–7: Yu. L. Sachkov [**26, 27, 28**].

9. Questions and suggestions

In this section we present and discuss several challenging open questions related to the results considered above.

9.1. The hypersurface principle. The hypersurface principle given by Theorem 5.3, Section 5, is a necessary controllability condition for an arbitrary simply connected Lie group. If a simply connected Lie group has cocompact radical, then this principle is also sufficient for controllability (Theorem 4.1, Section 4). Is it possible to extend the class of simply connected Lie groups with cocompact radical so that the hypersurface principle remain a criterion of controllability?

9.2. Lie algebras hard to control. For any Lie group G and any system affine in control $\Gamma = \{\, A + \sum_{i=1}^{m} u_i B_i \mid u_i \in \mathbb{R} \,\}$ on G controllability of the homogeneous part $\Gamma_0 = \{\, \sum_{i=1}^{m} u_i B_i \mid u_i \in \mathbb{R} \,\}$ is sufficient for controllability of Γ on G. We call a Lie algebra L *hard to control* if any affine in control system $\Gamma \subset L$ and its homogeneous part Γ_0 are simultaneously controllable or noncontrollable (on the connected simply connected Lie group G corresponding to L). In Lie algebras L

hard to control the drift term A in an affine system $\Gamma \subset L$ does not help in control, which is not the case for general Lie algebras.

There is the expanding chain of classes of Lie algebras hard to control:

$$\text{abelian} \subset \text{nilpotent} \subset \text{completely solvable}. \qquad (*)$$

The abelian case is obvious, the nilpotent one is Theorem 3.3, Subsection 3.2, and the completely solvable one is Theorem 5.1, Section 5.

On the other hand, the Lie algebra of the group $\mathrm{E}(2)$ of motions of the plane is solvable, not completely solvable and not hard to control (see the example in Subsection 6.3).

Corollary 3.3 [**26**] states that all Lie algebras satisfying the following property:

$$\left.\begin{array}{l}\text{any subalgebra } l \subset L,\ l \neq L, \text{ is contained}\\ \text{in a codimension one subalgebra of } L\end{array}\right\} \qquad (**)$$

are included in the set of Lie algebras hard to control. The author does not know, whether this inclusion is strict. (By Lemma 4.2 [**26**], completely solvable Lie algebras satisfy property (**).)

Are there any Lie algebras hard to control not contained in chain (*)? If yes, can this chain be continued by any reasonable class of Lie algebras?

The theory of K. H. Hofmann on hyperplane subalgebras of Lie algebras [**16, 17, 19**] may be important for this question.

9.3. Small-dimensional groups. A complete and visual classification of controllable systems might be obtained for small-dimensional groups with the help of the known results for the following classes of groups: compact [**20**], semi-simple [**21**], [**22**], [**11**], reductive [**14**], nilpotent [**13**], [**6**], and solvable [**25**], [**26**], [**27**], [**28**]. An attempt in this direction was made in [**29**].

9.4. Solvable not simply connected Lie groups. The results of K. H. Hofmann on compact elements in solvable Lie algebras [**18**] might be applied in order to understand controllability for solvable Lie groups without the assumption of simple connectedness essential in Sections 5–7.

9.5. General groups. On the basis of the results listed in Subsection 9.3, controllability theory for general Lie groups can be started synthesizing the "semi-simple" and "solvable" theory via Levi decomposition (I. Kupka [**24**]).

9.6. Nilpotent and solvable manifolds. Controllability of projections of right-invariant systems onto nilpotent and solvable manifolds can be studied via application of the theory of flows on these manifolds [**5**]. This may be important for studying local controllability of nonlinear systems via nilpotent approximations (P. E. Crouch and C. I. Byrnes [**10**]).

9.7. Codimension one and two subalgebras. The solution of the controllability problem for completely solvable Lie groups (see Section 5) is based upon the following fact: any proper subalgebra of a real *completely solvable* Lie algebra is contained in a codimension one subalgebra. On the other hand, any proper subalgebra of a real *solvable* Lie algebra is included in some subalgebra of codimension one or two.

This suggests the following approach to controllability on solvable Lie groups. Project a system along the connected subgroup corresponding to the indicated codimension one or two subalgebra. Then: 1) if this group is closed and normal, we obtain a right-invariant system on a one- or two-dimensional Lie group (such systems are transparent); 2) if this subgroup is closed, we obtain a nonlinear system on a one- or two-dimensional smooth manifold (such systems are tractable by the nonlinear controllability theory); 3) and if this subgroup is not closed, then try to apply and develop the theory of control systems on foliations.

The progress in this direction may be useful for the controllability theory on general Lie groups (Subsection 9.5) as well.

9.8. Rank condition and hypersurface principle. The customary procedure to show noncontrollability is either to show the violation of the rank controllability principle [**20**] or to construct a (not necessarily smooth) hypersurface in the state space of a system intersected by all trajectories of the system in one direction only. For right-invariant systems on simply connected Lie groups with cocompact radical such hypersurface can always be found among codimension one subgroups (Section 4).

Does every full-rank noncontrollable right-invariant system have such codimension one subgroup? A positive answer will give a new method of obtaining sufficient controllability conditions, and a negative one will give an example of a complex obstacle to controllability.

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