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CONJUGATE AND CUT TIME IN THE SUB-RIEMANNIAN PROBLEM ON THE GROUP OF MOTIONS OF A PLANE *

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Abstract. The left-invariant sub-Riemannian problem on the group of motions (rototranslations) of a plane SE(2) is studied. Local and global optimality of extremal trajectories is characterized.

Lower and upper bounds on the first conjugate time are proved. The cut time is shown to be equal to the first Maxwell time corresponding to the group of discrete symmetries of the exponential mapping. Optimal synthesis on an open dense subset of the state space is described.

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Contents

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1. Introduction

This work is devoted to the study of the left-invariant sub-Riemannian problem on the group of motions of a plane. This problem can be stated as follows: given two unit vectors $v_0 = (\cos\theta_0, \sin\theta_0)$, $v_1 = (\cos\theta_1, \sin\theta_1)$ attached respectively at two given points (x_0, y_0), (x_1, y_1) in the plane, one should find an optimal motion in the plane that transfers the vector v_0 to the vector v_1, see Fig. 1. The vector can move forward or backward and rotate simultaneously. The required motion should be optimal in the sense of minimal length in the space (x, y, θ), where θ is the slope of the moving vector.

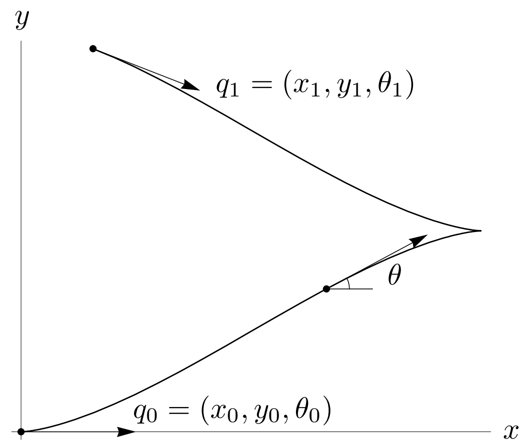

Figure 1. Problem statement

The corresponding optimal control problem reads as follows:

$$\dot{x} = u_1 \cos\theta, \quad \dot{y} = u_1 \sin\theta, \quad \dot{\theta} = u_2, \tag{1.1}$$

$$q = (x, y, \theta) \in M \cong \mathbb{R}^2_{x,y} \times S^1_\theta, \quad u = (u_1, u_2) \in \mathbb{R}^2, \tag{1.2}$$

$$q(0) = q_0 = (0, 0, 0), \qquad q(t_1) = q_1 = (x_1, y_1, \theta_1), \tag{1.3}$$

$$l = \int_0^{t_1} \sqrt{u_1^2 + u_2^2}\, dt \to \min, \tag{1.4}$$

or, equivalently,

$$J = \frac{1}{2}\int_0^{t_1} (u_1^2 + u_2^2)\, dt \to \min. \tag{1.5}$$

Problem (1.1)–(1.5) is a left-invariant sub-Riemannian problem on the group of motions of a plane $\mathrm{SE}(2) = \mathbb{R}^2 \ltimes \mathrm{SO}(2)$. This problem is closely connected to models of vision [6, 10, 11] and robotics [8]. On the other hand, this problem is important for understanding properties of the heat kernel of the diffusion equation on the group SE(2), see [4]. We expect that results of this work can be applied to these domains.

This is an immediate continuation of the previous work [9]. We use extensively the results obtained in that paper, and now we recall the most important of them.

The normal Hamiltonian system of Pontryagin Maximum Principle becomes triangular in appropriate coordinates on cotangent bundle T^*M, and its vertical subsystem is the equation of mathematical pendulum:

$$\dot\gamma = c, \quad \dot c = -\sin\gamma, \qquad (\gamma, c) \in C \cong (2S^1_\gamma) \times \mathbb{R}_c, \tag{1.6}$$

$$\dot x = \sin\frac{\gamma}{2}\cos\theta, \quad \dot y = \sin\frac{\gamma}{2}\sin\theta, \quad \dot\theta = -\cos\frac{\gamma}{2}. \tag{1.7}$$

The phase cylinder of pendulum (1.6) decomposes into invariant subsets according to values of the energy $E = c^2/2 - \cos\gamma$:

$$C = \bigcup_{i=1}^{5} C_i,$$

$$C_1 = \{\lambda \in C \mid E \in (-1,1)\}, \tag{1.8}$$

$$C_2 = \{\lambda \in C \mid E \in (1,+\infty)\}, \tag{1.9}$$

$$C_3 = \{\lambda \in C \mid E = 1, \ c \neq 0\}, \tag{1.10}$$

$$C_4 = \{\lambda \in C \mid E = -1\} = \{(\gamma, c) \in C \mid \gamma = 2\pi n, \ c = 0\}, \tag{1.11}$$

$$C_5 = \{\lambda \in C \mid E = 1, \ c = 0\} = \{(\gamma, c) \in C \mid \gamma = \pi + 2\pi n, \ c = 0\}. \tag{1.12}$$

In the subsets C_1, C_2, C_3 were introduced elliptic coordinates (φ, k) that rectify the flow of the pendulum: φ is the phase, and k a reparametrized energy of pendulum (1.6):

$$k = \sqrt{(E+1)/2} \text{ in } C_1 \cup C_3, \qquad k = \sqrt{2/(E+1)} \text{ in } C_2.$$

The Hamiltonian system (1.6), (1.7) was integrated in Jacobi's functions [22]. The equation of pendulum (1.6) has a discrete group of symmetries $G = \{\mathrm{Id}, \varepsilon^1, \dots, \varepsilon^7\} = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ generated by reflections in the axes of coordinates γ, c, and translations $(\gamma, c) \mapsto (\gamma + 2\pi, c)$. Action of the group G is naturally extended to extremal trajectories (x_t, y_t), this action modulo rotations is represented at Figs. 2–4.

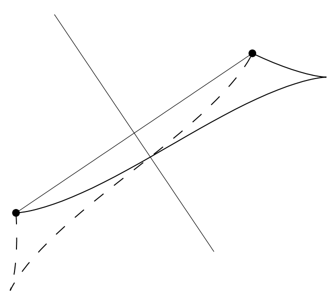

Figure 2. Action of ε^1, ε^2 on (x_t, y_t)

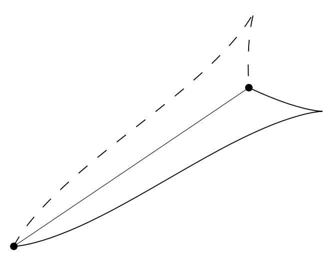

Figure 3. Action of ε^4, ε^7 on (x_t, y_t)

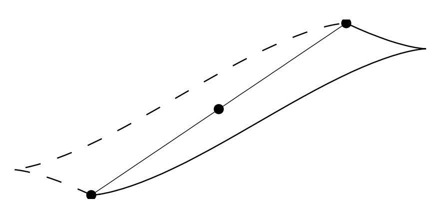

Figure 4. Action of ε^5, ε^6 on (x_t, y_t)

Reflections ε^i are symmetries of the exponential mapping $\mathrm{Exp} : N = C \times \mathbb{R}_+ \to M$, $\mathrm{Exp}(\lambda, t) = q_t$.

The main result of work [9] is an upper bound on cut time

$$t_{\mathrm{cut}} = \sup\{t_1 > 0 \mid q_s \text{ is optimal for } s \in [0, t_1]\}$$

along extremal trajectories q_s. It is based on the fact that a sub-Riemannian geodesic cannot be optimal after a Maxwell point, i.e., a point where two distinct geodesics of equal sub-Riemannian length meet one another.

A natural idea is to look for Maxwell points corresponding to discrete symmetries of the exponential mapping. For each extremal trajectory $q_s = \operatorname{Exp}(\lambda, s)$, we described Maxwell times $t^n_{\varepsilon^i}(\lambda)$, $i = 1, \dots, 7$, $n = 1, 2, \dots$, corresponding to discrete symmetries ε^i. The following upper bound is the main result of work [9]:

$$t_{\mathrm{cut}}(\lambda) \le \mathbf{t}(\lambda), \qquad \lambda \in C, \tag{1.13}$$

where $\mathbf{t}(\lambda) = \min(t^1_{\varepsilon^i}(\lambda))$ is the first Maxwell time corresponding to the group of symmetries G. We recall the explicit definition of the function $\mathbf{t}(\lambda)$ below in Eqs. (2.20)–(2.24).

In this work we continue the study of problem (1.1)–(1.5).

First we consider the local optimality of sub-Riemannian geodesics (Section 2). We show that extremal trajectories corresponding to oscillating pendulum (1.6) do not have conjugate points, thus they are locally optimal forever. In the case of rotating pendulum we prove that the first conjugate time is bounded from below and from above by the first Maxwell times $t^1_{\varepsilon^2}$ and $t^1_{\varepsilon^5}$ respectively. For critical values of energy of the pendulum, there are no conjugate points.

In Section 3 we study the global optimality of geodesics. We construct open dense subsets in the preimage and image of exponential mapping and prove that the exponential mapping transforms these domains diffeomorphically. As a consequence, we show that inequality (1.13) is in fact an equality. Moreover, we describe the optimal synthesis on the open dense subset of the state space.

In Section 4 we present plots of the sub-Riemannian caustic and spheres.

In the subsequent work [19] we complete our study of problem (1.1)–(1.5). There we describe explicitly the Maxwell strata and cut locus, and characterize the optimal synthesis in this problem. For some special terminal points q_1, we provide explicit optimal solutions.

2. Conjugate points

In this section we obtain bounds on conjugate time in the sub-Riemannian problem on SE(2), see Th. 2.5.

2.1. General facts

First we recall some known facts from the theory of conjugate points in optimal control problems. For details see, e.g., [2, 3, 20].

Consider an optimal control problem of the form

$$\dot{q} = f(q, u), \qquad q \in M, \quad u \in U \subset \mathbb{R}^m, \tag{2.1}$$

$$q(0) = q_0, \qquad q(t_1) = q_1, \qquad t_1 \text{ fixed}, \tag{2.2}$$

$$J = \int_0^{t_1} \varphi(q(t), u(t))\, dt \to \min, \tag{2.3}$$

where M is a finite-dimensional analytic manifold, $f(q, u)$ and $\varphi(q, u)$ are respectively analytic in (q, u) families of vector fields and functions on M depending on the control parameter $u \in U$, and U an open subset of $\mathbb{R}^m$. Admissible controls are $u(\cdot) \in L_\infty([0, t_1], U)$, and admissible trajectories $q(\cdot)$ are Lipschitzian. Let

$$h_u(\lambda) = \langle \lambda, f(q, u) \rangle - \varphi(q, u), \qquad \lambda \in T^*M, \quad q = \pi(\lambda) \in M, \quad u \in U,$$

be the normal Hamiltonian of PMP for the problem (2.1)–(2.3). Fix a triple $(\widetilde{u}(t), \lambda_t, q(t))$ consisting of a normal extremal control $\widetilde{u}(t)$, the corresponding extremal λ_t, and the extremal trajectory $q(t)$ for the problem (2.1)–(2.3).

Let the following hypotheses hold:

(**H1**) *For all $\lambda \in T^*M$ and $u \in U$, the quadratic form $\dfrac{\partial^2 h_u}{\partial u^2}(\lambda)$ is negative definite.*

(**H2**) *For any $\lambda \in T^*M$, the function $u \mapsto h_u(\lambda)$, $u \in U$, has a maximum point $\bar{u}(\lambda) \in U$:*

$$h_{\bar{u}(\lambda)}(\lambda) = \max_{u \in U} h_u(\lambda), \qquad \lambda \in T^*M.$$

(**H3**) *The extremal control $\widetilde{u}(\cdot)$ is a corank one critical point of the endpoint mapping.*

(**H4**) *All trajectories of the Hamiltonian vector field $\vec{H}(\lambda)$, $\lambda \in T^*M$, are continued for $t \in [0, +\infty)$.*

An instant $t_* > 0$ is called a conjugate time (for the initial instant $t = 0$) along the extremal λ_t if the restriction of the second variation of the endpoint mapping to the kernel of its first variation is degenerate, see [3] for details. In this case the point $q(t_*) = \pi(\lambda_{t_*})$ is called conjugate for the initial point q_0 along the extremal trajectory $q(\cdot)$.

Under hypotheses (**H1**) –(**H4**) , we have the following:

1. Normal extremal trajectories lose their local optimality (both strong and weak) at the first conjugate point, see [3].
2. An instant $t > 0$ is a conjugate time iff the exponential mapping $\mathrm{Exp}_t = \pi \circ e^{t\vec{H}}$ is degenerate, see [2].
3. Along each normal extremal trajectory, conjugate times are isolated one from another, see [20].

We will apply the following statement for the proof of absence of conjugate points via homotopy.

Proposition 2.1 (Corollaries 2.2, 2.3 [18])**.** *Let $(u^s(t), \lambda_t^s)$, $t \in [0, +\infty)$, $s \in [0, 1]$, be a continuous in parameter s family of normal extremal pairs in the optimal control problem* (2.1)–(2.3) *satisfying hypotheses* (**H1**) –(**H4**) .

1. *Let $s \mapsto t_1^s$ be a continuous function, $s \in [0, 1]$, $t_1^s \in (0, +\infty)$. Assume that for any $s \in [0, 1]$ the instant $t = t_1^s$ is not a conjugate time along the extremal λ_t^s.*
   If the extremal trajectory $q^0(t) = \pi(\lambda_t^0)$, $t \in (0, t_1^0]$, does not contain conjugate points, then the extremal trajectory $q^1(t) = \pi(\lambda_t^1)$, $t \in (0, t_1^1]$, also does not contain conjugate points.
2. *Let for any $s \in [0, 1)$ and $T > 0$ the extremal λ_t^s has no conjugate points for $t \in (0, T]$. Then for any $T > 0$, the extremal λ_t^1 also has no conjugate points for $t \in (0, T]$.*

One easily checks that the sub-Riemannian problem (1.1)–(1.5) satisfies all hypotheses (**H1**) –(**H4**) , so the results cited in this subsection are applicable to this problem.

We denote the first conjugate time along an extremal trajectory $q(t) = \mathrm{Exp}(\lambda, t)$ as $t_1^{\mathrm{conj}}(\lambda)$.

2.2. Conjugate points for the case of oscillating pendulum

In this subsection we assume that $\lambda \in C_1$ and prove that the corresponding extremal trajectories do not contain conjugate points, see Th. 2.1.

Using the parametrization of extremal trajectories obtained in Subsec. 3.3 [9], we compute explicitly Jacobian of the exponential mapping:

$$J = \frac{\partial(x_t, y_t, \theta_t)}{\partial(t, \varphi, k)} = \frac{4}{k^3(1-k^2)(1-k^2 \operatorname{sn}^2 p \operatorname{sn}^2 \tau)} J_1,$$

$$p = t/2, \qquad \tau = \varphi + t/2, \tag{2.4}$$

$$J_1(\tau, p, k) = v_1 \operatorname{sn}^2 \tau + v_2 \operatorname{cn}^2 \tau, \tag{2.5}$$

$$v_1 = (1-k^2)(p - \mathrm{E}(p))(\mathrm{E}(p) - (1-k^2)p),$$

$$v_2 = (p - \mathrm{E}(p))(\mathrm{E}(p) - (1-k^2)p) + k^2 \operatorname{cn} p \operatorname{dn} p (2\mathrm{E}(p) + (k^2-2)p) \operatorname{sn} p$$
$$+ k^2((\mathrm{E}(p) - p)(\mathrm{E}(p) - (1-k^2)p) - k^2) \operatorname{sn}^2 p + k^4 \operatorname{sn}^4 p,$$

so that $\operatorname{sgn} J = \operatorname{sgn} J_1$.

2.2.1. *Preliminary lemmas*

Lemma 2.1. *For any $k \in (0,1)$ and $p > 0$ we have $v_1(p,k) > 0$.*

Proof. The statement follows from the relations

$$p - \mathrm{E}(p) = k^2 \int_0^p \mathrm{sn}^2\, t\, dt > 0, \qquad \mathrm{E}(p) - (1-k^2)p = k^2 \int_0^p \mathrm{cn}^2\, t\, dt > 0. \tag{2.6}$$

□

Lemma 2.2. *For any $k \in (0,1)$, $n \in \mathbb{N}$, and $\tau \in \mathbb{R}$ we have $J_1(\tau, 2Kn, k) > 0$.*

Proof. If $p = 2Kn$, $n \in \mathbb{N}$, then $v_2(p,k) = (p - \mathrm{E}(p))(\mathrm{E}(p) - (1-k^2)p) > 0$ by inequalities (2.6).
By virtue of Lemma 2.1 and decomposition (2.5), we obtain the inequality $J_1(\tau, 2Kn, k) > 0$. □

Lemma 2.3. $\forall\, p_1 > 0 \quad \exists\, \widehat{k} \in (0,1) \quad \forall\, k \in (0, \widehat{k}) \quad \forall p \in (0, p_1) \quad \forall \tau \in \mathbb{R}\ J_1(\tau, p, k) > 0$.

Proof. The statement of the lemma follows from the Taylor expansions:

$$J_1 = \frac{k^4}{16}(4p^2 - \sin^2 2p) + o(k^4), \qquad k \to 0, \tag{2.7}$$

$$J_1 = \frac{1}{3} k^4 p^4 + o(k^2 + p^2)^4, \qquad k^2 + p^2 \to 0. \tag{2.8}$$

By contradiction, if the statement is not verified, then there exists a converging sequence $(\tau_n, p_n, k_n) \to (\tau_0, p_0, 0)$ such that $J(\tau_n, p_n, k_n) \leq 0$ for all $n \in N$. If $p_0 > 0$, then a standard calculus argument yields contradiction with (2.7). And if $p_0 = 0$, then similarly one obtains a contradiction with (2.8). □

2.2.2. *Absence of conjugate points in C_1*

Theorem 2.1. *If $\lambda \in C_1$, then the extremal trajectory $q(t) = \mathrm{Exp}(\lambda, t)$, $t > 0$, does not contain conjugate points.*

Proof. We choose any $\widehat{\lambda} \in C_1$, $\widehat{t} > 0$, and prove that the extremal trajectory $\widehat{q}(t) = \mathrm{Exp}(\widehat{\lambda}, t)$ does not contain conjugate points for $t \in (0, \widehat{t}\,]$.

Find the elliptic coordinates $(\widehat{k}, \widehat{\varphi})$ corresponding to the covector $\widehat{\lambda} \in C_1$ according to Subsec. 3.2 [9], and let $\widehat{p} = \widehat{t}/2$, $\widehat{\tau} = \widehat{\varphi} + \widehat{p}$. Find $n \in \mathbb{N}$ such that $p_1 = 2K(\widehat{k})n > \widehat{p}$. Choose the following continuous curve in the plane (k,p):

$$\{(k^s, p^s) \mid s \in [0,1]\}, \qquad k^s = s\widehat{k}, \quad p^s = 2K(k^s)n,$$

with the endpoints $(k^0, p^0) = (0, \pi n)$ and $(k^1, p^1) = (\widehat{k}, 2K(\widehat{k})n)$.

Consider the following family of extremal trajectories:

$$\gamma^s = \{q^s(t) = \mathrm{Exp}(\varphi^s, k^s, t) \mid t \in [0, t^s]\}, \qquad s \in [0,1],$$
$$t^s = 2p^s, \qquad \varphi^s = \widehat{\tau} - p^s.$$

The endpoint $q^s(t^s)$ of each trajectory γ^s, $s \in [0,1]$, corresponds to the values of parameters $(\tau, p, k) = (\widehat{\tau}, 2K(k^s)n, k^s)$. Thus Lemma 2.2 implies that for any $s \in [0,1]$ the endpoint $q^s(t^s)$ is not a conjugate point.

Further, Lemma 2.3 states that

$$\exists\, k_0 \in (0, \widehat{k}) \quad \forall \tau \in \mathbb{R} \quad \forall\, p \in (0, p_1) \qquad J(\tau, p, k) > 0. \tag{2.9}$$

Denote $s_0 = k_0/\widehat{k} \in (0,1)$, so that $k^{s_0} = k_0$. Condition (2.9) means that the extremal trajectory γ^{s_0} does not contain conjugate points for all $t \in [0, t^{s_0}]$.

Then Proposition 2.1 yields that for any $s \in [s_0, 1]$, the extremal trajectory $q^s(t)$ does not contain conjugate points for all $t \in [0, t^s]$. In particular, the trajectory $\widehat{q}(t) = q^1(t)$, $t \in (0, \widehat{t}]$, is free of conjugate points. □

So we proved that extremal trajectories $q(t) = \mathrm{Exp}(\lambda, t)$ with $\lambda \in C_1$ (i.e., corresponding to oscillating pendulum) are locally optimal at any segment $[0, t_1]$, $t_1 > 0$.

2.3. **Conjugate points for the case of rotating pendulum**

In this subsection we obtain bounds on conjugate points in the case $\lambda \in C_2$.

Using the formulas for extremal trajectories of Subsec. 3.3 [9], we get:

$$J = \frac{\partial(x_t, y_t, \theta_t)}{\partial(t, \varphi, k)} = -\frac{4k}{(1-k^2)(1-k^2 \operatorname{sn}^2 p \operatorname{sn}^2 \tau)} J_2,$$

$$p = t/(2k), \qquad \tau = \psi + t/(2k) = (2\varphi + t)/(2k), \tag{2.10}$$

$$J_2 = \alpha \operatorname{sn}^2 \tau + \beta \operatorname{cn}^2 \tau, \tag{2.11}$$

$$\alpha = (1-k^2) \operatorname{sn} p \, \alpha_1, \tag{2.12}$$

$$\alpha_1 = \operatorname{cn} p \operatorname{dn} p (p - 2\operatorname{E}(p)) + \operatorname{sn} p (\operatorname{dn}^2 p + \operatorname{E}(p)(p - \operatorname{E}(p))), \tag{2.13}$$

$$\beta = f_1(p)\beta_1, \qquad \beta_1 = \operatorname{cn} p \operatorname{E}(p) - \operatorname{dn} p \operatorname{sn} p, \tag{2.14}$$

where $f_1(p,k) = \operatorname{cn} p (\operatorname{E}(p) - p) - \operatorname{dn} p \operatorname{sn} p$, see Eq. (5.12) [9].

2.3.1. *Preliminary lemmas*

Recall that we denoted the first positive root of the function $f_1(p)$ by $p_1^1(k)$, see Lemma 5.3 [9].

Lemma 2.4. *If $k \in (0,1)$ and $p = p_1^1(k)$, then $\alpha(p,k) > 0$.*
If additionally $\operatorname{sn} \tau \neq 0$, then $J_2 > 0$ and $J < 0$.

Proof. In terms of the auxiliary function

$$\varphi(p,k) = \operatorname{sn} p \operatorname{dn} p - (2\operatorname{E}(p) - p) \operatorname{cn} p, \tag{2.15}$$

we have a decomposition

$$\alpha_1 = \operatorname{dn} p \, \varphi(p) + \operatorname{sn} p \operatorname{E}(p)(p - \operatorname{E}(p)). \tag{2.16}$$

Let $k \in (0,1)$ and $p = p_1^1(k)$. Then $f_1(p) = 0$, i.e., $\operatorname{sn} p \operatorname{dn} p = \operatorname{cn} p (\operatorname{E}(p) - p)$. Thus $\varphi(p) = \operatorname{cn} p (\operatorname{E}(p) - p) - (2\operatorname{E}(p) - p) \operatorname{cn} p = -\operatorname{E}(p) \operatorname{cn} p$. By virtue of Cor. 5.1 [9],we have $\operatorname{cn} p < 0$, so $\varphi(p) > 0$. Moreover, $\operatorname{sn} p > 0$. Then decomposition (2.16) yields $\alpha_1(p) > 0$, consequently, $\alpha(p) > 0$.

If additionally $\operatorname{sn} \tau \neq 0$, then it is obvious that $J_2 > 0$ and $J < 0$. □

Lemma 2.5. $\exists \, \widehat{k} \in (0,1) \quad \forall \, k \in (0, \widehat{k}) \quad \forall p \in (0, p_1^1] \qquad \alpha(p,k) > 0.$

Proof. The statement of this lemma follows by the argument used in the proof of Lemma 2.3 from the Taylor expansions

$$\alpha = \sin p (\sin p - p \cos p) + o(1), \qquad k \to 0,$$

$$\alpha = \frac{p^4}{3} + o(p^2 + k^2)^2, \qquad p^2 + k^2 \to 0.$$

□

Lemma 2.6. $\forall \, k \in (0,1) \quad \forall p \in (0, 2K] \qquad \beta_1(p,k) < 0.$

Proof. Since $(\beta_1(p)/\operatorname{cn} p)' = -(1-k^2) \operatorname{sn}^2 p / \operatorname{cn}^2 p$, the function $\beta_1(p)/\operatorname{cn} p$ decreases at the segments $p \in [0, K)$ and $p \in (K, 2K]$.

We have $\beta_1(0) = 0$, thus $\beta_1(p)/\operatorname{cn} p < 0$, so $\beta_1(p) < 0$ for $p \in (0, K)$.

Further, $\beta_1(K) = -\sqrt{1-k^2} < 0$.

Since $\beta_1(p)/\operatorname{cn} p \to +\infty$ as $p \to K+0$, and $\beta_1(2K)/\operatorname{cn}(2K) = \mathrm{E}(2K) > 0$, we have $\beta_1(p)/\operatorname{cn} p > 0$, so $\beta_1(p) < 0$ for $p \in (K, 2K]$. □

Lemma 2.7. *Let $k \in (0,1)$.*

(1) *Let $\operatorname{sn}\tau = 0$. Then $J_2(\tau,p,k) > 0$ for $p \in (0,p_1^1)$, and $J_2(\tau,p,k) = 0$ for $p = p_1^1$.*
(2) *Let $\operatorname{sn}\tau \neq 0$. Then $J_2(\tau,p,k) > 0$ for $p \in (0,p_1^1]$.*

Proof. If $p \in (0,p_1^1)$, then $f_1(p,k) < 0$ (Cor. 5.1 [9]), and $\beta_1(p,k) < 0$ (Lemma 2.6), thus $\beta(p,k) = f_1(p,k)\beta_1(p,k) > 0$.

(1) Let $\operatorname{sn}\tau = 0$. If $p \in (0,p_1^1)$, then $J_2(\tau,p,k) = \beta(p,k) > 0$. And if $p = p_1^1$, then $f_1(p,k) = 0$, thus $J_2(\tau,p,k) = \beta(p,k) = 0$.

(2) Let $\operatorname{sn}\tau \neq 0$.

(2.a) We prove that the function $\varphi(p)$ given by (2.15) satisfies the inequality

$$\varphi(p) > 0 \qquad \forall\, p \in (0,K].$$

First, $\varphi(p) = p^3/3 + o(p^3) > 0$ as $p \to +0$. Second,

$$(\varphi(p)/\operatorname{cn} p)' = \operatorname{dn}^2 p \operatorname{sn}^2 p / \operatorname{cn}^2 p > 0 \qquad \forall\, p \in (0,K).$$

Thus $\varphi(p) > 0$ for $p \in (0,K)$. And if $p = K$, then $\varphi(p) = \sqrt{1-k^2} > 0$.

(2.b) By virtue of the decomposition $\varphi(p) = -f_1(p) - \mathrm{E}(p)\operatorname{cn} p$, we get the inequality $\varphi(p) > 0$ for all $p \in (K, p_1^1]$. We proved that

$$\varphi(p) > 0 \qquad \forall\, p \in (0,p_1^1].$$

(2.c) In view of (2.16), we obtain that $\alpha_1(p) > 0$ for $p \in (0,p_1^1]$. Then Eq. (2.12) yields $\alpha(p) > 0$ for $p \in (0,p_1^1]$. Finally, Eq. (2.11) gives $J_2 > 0$ for $p \in (0,p_1^1]$. □

Lemma 2.8. $\forall\, z \in (0,1] \quad \exists\, \widehat{k} \in (0,1) \quad \forall\, k \in (0,\widehat{k}) \quad \forall\, p \in (0,p_1^1]$ *we have* $J_2(z,p,k) > 0$.

Proof. Fix any $z \in (0,1]$. By Lemma 2.5,

$$\exists\, \widehat{k} \in (0,1) \quad \forall\, k \in (0,\widehat{k}) \quad \forall\, p \in (0,p_1^1] \qquad \alpha(p,k) > 0.$$

But if $p \in (0,p_1^1]$, then $p \in (0,2K]$, thus $\beta_1(p,k) < 0$ by Lemma 2.6, so $\beta(p,k) > 0$ by Cor. 5.1 [9].

Then the inequalities $\alpha(p,k) > 0$, $\beta(p,k) > 0$ imply the required inequality $J_2(\tau,p,k) > 0$. □

2.3.2. *Conjugate points in C_2*

First we obtain a lower bound on the first conjugate time. It will play a crucial role in the subsequent analysis of the global structure of the exponential mapping in Section 3 and in the subsequent work [19].

Theorem 2.2. *If $\lambda \in C_2$, then $t_1^{\mathrm{conj}}(\lambda) \geq 2kp_1^1(k)$.*

Proof. Given any $\lambda \in C_2$, compute the corresponding elliptic coordinates (φ,k). If additionally we have $t > 0$, find the corresponding parameters $p = t/(2k)$, $\tau = \varphi/k + t/(2k)$ and denote $z = \operatorname{sn}^2\tau$.

We should prove that for any $\lambda \in C_2$ the interval $t \in (0, 2kp_1^1(k))$ does not contain conjugate times for the extremal trajectory $q(t) = \operatorname{Exp}(\lambda,t)$.

Take any $\lambda^1 \in C_2$ and denote the corresponding elliptic coordinates (φ^1,k^1). For $t^1 = 2k^1p_1^1(k^1)$ we denote the corresponding parameters p^1, τ^1, z^1. In order to prove that the extremal trajectory $q^1(t) = \operatorname{Exp}(\lambda^1,t)$ does not have conjugate points at the interval $t \in (0,t^1)$, we show that

$$p \in (0,p^1) \quad \Rightarrow \quad J_2(z^1,p,k^1) > 0 \quad \Rightarrow \quad J(z^1,p,k^1) < 0.$$

(1) Assume first that $z^1 = \operatorname{sn}^2(\tau^1, k^1) \neq 0$, i.e., $z^1 \in (0,1]$. We prove that in this case

$$p \in (0, p^1] \quad \Rightarrow \quad J(z^1, p, k^1) < 0.$$

Consider the following continuous curve in the space (z, p, k):

$$\{(z^1, p^s, k^s) \mid s \in (0,1]\}, \qquad k^s = sk^1, \quad p^s = p_1^1(k^s).$$

The corresponding curve in the space (τ, p, k) is

$$\{(\tau^s, p^s, k^s) \mid s \in (0,1]\}, \qquad \tau^s = F(\operatorname{am}(\tau^1, k^1), k^s),$$

and in the space (t, φ, k) is

$$\{(t^s, \varphi^s, k^s) \mid s \in (0,1]\}, \qquad t^s = 2k^s p^s, \quad \varphi^s = (\tau^s - p^s)k^s.$$

Let $\lambda^s = (\varphi^s, k^s)$, $s \in (0,1]$, be the corresponding curve in C_2, and consider the continuous one-parameter family of extremal trajectories

$$q^s(t) = \operatorname{Exp}(\lambda^s, t), \qquad t \in [0, t^s], \quad s \in (0,1].$$

For any $s \in (0,1]$, if $t = t^s$, then by Lemma 2.4 we have $J_2(z^1, p_1^1(k^s), k^s) < 0$, i.e., the terminal instant $t = t^s$ is not a conjugate time along the extremal trajectory $q^s(t)$.

Further, by Lemma 2.8, for $z^1 \in (0,1]$

$$\exists\, \widehat{k} \in (0,1) \quad \forall\, k \in (0, \widehat{k}) \quad \forall\, p \in (0, p_1^1(k)] \qquad J_2(z^1, p, k) > 0.$$

Consequently, there exists $s_0 \in (0,1)$ such that the whole trajectory $q^{s_0}(t)$, $t \in (0, t^{s_0}]$, is free of conjugate points.

Then Propos. 2.1 implies that the trajectory $q^1(t)$, $t \in (0, t^1]$, also does not contain conjugate points.

We proved that if $z^1 \neq 0$, then the trajectory $q^1(t) = \operatorname{Exp}(\lambda^1, t)$, $t \in (0, t^1]$, does not have conjugate points.

(2) Now consider the case $z^1 = \operatorname{sn}^2(\tau^1, k^1) = 0$. Then Lemma 2.7 states that the terminal instant $t = 2k^1 p_1^1(k^1)$ is a conjugate point. We prove that all the less instants are not conjugate.

Since conjugate points are isolated one from another at each extremal trajectory, there exists $p < p_1^1(k^1)$ arbitrarily close to $p_1^1(k^1)$ such that the corresponding time $t = 2k^1 p$ is not conjugate.

Consider the continuous curve in the space (z, p, k):

$$\{(z_s, p, k^1) \mid s \in [0, \varepsilon)\}, \qquad z_s = sz^1.$$

By item (1) of this proof, there exists $\varepsilon > 0$ such that for any $s \in (0, \varepsilon)$ the extremal trajectory $q^s(t)$, $t \in (0, t^s]$, $t^s = 2k^1 p$, does not have conjugate points. By Propos. 2.1, for $s = 0$ the initial extremal trajectory $q^0(t)$, $t \in (0, t^0]$, also does not contain conjugate points. The endpoint $t^0 = 2k^1 p$ can be chosen arbitrarily close to $t^1 = 2k^2 p_1^1(k^1)$, so the initial extremal trajectory does not have conjugate points for $t \in (0, t^1)$. □

Now we obtain the final result on the first conjugate time in the domain C_2 — the following two-side bound.

Theorem 2.3. *If $\lambda \in C_2$, then*

$$2kp_1^1(k) \leq t_1^{\text{conj}}(\lambda) \leq 4kK(k). \tag{2.17}$$

Proof. We proved in Th. 2.2 that $2kp_1^1(k) \leq t_1^{\text{conj}}(\lambda)$; moreover, if $t \in (0, 2kp_1^1)$, then $J < 0$.

Let $t = 4kK$, then $p = 2K$, thus $\alpha = 0$, $f_1 = p - \mathrm{E}(p) > 0$, $\beta_1 = -\mathrm{E}(p) < 0$, so

$$J = -\frac{4k}{1-k^2} J_2 = \frac{4k}{1-k^2} \operatorname{cn}^2 \tau \, \mathrm{E}(p) \, (p - \mathrm{E}(p)) \geq 0.$$

It follows that for any $\lambda \in C_2$ the function $t \mapsto J$ has a root at the segment $t \in [2kp_1^1, 4kK]$. Consequently, also the first root $t_1^{\text{conj}} \in [2kp_1^1, 4kK]$. □

One can show that the bound (2.17) can be a little bit improved. The precise bound on the first conjugate time is

$$2kp_1^1(k) \leq t_1^{\text{conj}}(\lambda) \leq \gamma(k) = \min(4kK, 2kp_1^{\alpha_1}(k)), \tag{2.18}$$

where $p = p_1^{\alpha_1}(k)$ is the first positive root of the equation $\alpha_1(p) = 0$, and the function α_1 is given by Eq. (2.13). One can show that $\gamma(k) = 4kK$ for $k \in (0, k_0]$ and $\gamma(k) = 2kp_1^{\alpha_1}(k)$ for $[k_0, 1)$, where $k_0 \approx 0.909$ is the unique root of the equation $2E(k) - K(k) = 0$, see Proposition 11.5 [17]. Thus for $k \in (k_0, 1)$ the bound (2.17) is not exact and can be replaced by the following exact one:

$$2kp_1^1(k) \leq t_1^{\text{conj}}(\lambda) \leq 2kp_1^{\alpha_1}(k), \qquad k \in (k_0, 1). \tag{2.19}$$

The bound (2.17) is illustrated at Figs. 5, 6; and the bound (2.19) — at Fig. 7. The exact bounds (2.18) are plotted at Fig. 8.

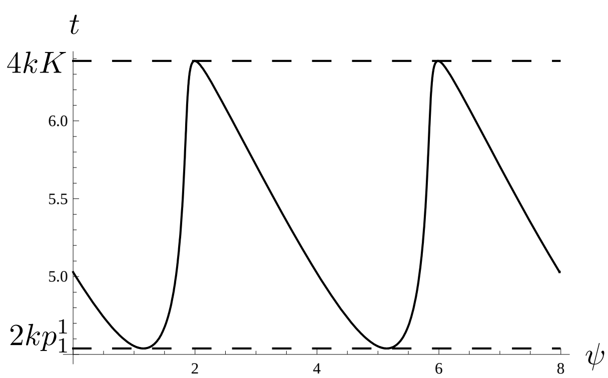

FIGURE 5. Plot of $t_1^{\text{conj}}(\psi, k)$, $k = 0.8 < k_0$

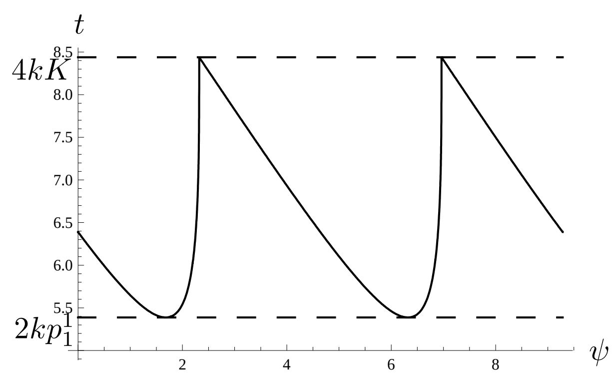

FIGURE 6. Plot of $t_1^{\text{conj}}(\psi, k)$, $k = k_0$

Proposition 2.2. *Let $\lambda \in C_2$ and $\tau = (2\varphi + 2kp_1^1)/(2k)$.*

(1) *If $\operatorname{sn} \tau = 0$, then $t_1^{\text{conj}}(\lambda) = 2kp_1^1$.*
(2) *If $\operatorname{sn} \tau \neq 0$, then $t_1^{\text{conj}}(\lambda) \in (2kp_1^1, 4kK]$.*

Proof. Notice first that by Th. 2.2, the interval $(0, 2kp_1^1)$ does not contain conjugate times. Then items (1), (2) of this proposition follow directly from the corresponding items of Lemma 2.7, and from Th. 2.3. □

2.4. **Conjugate points for the cases of critical energy of pendulum**

The subset $C_3 \cup C_4 \cup C_5$ of the cylinder C is the boundary of the domain C_1, see Eqs. (1.8)–(1.12) above, and Fig. 2 in [9]. Thus absence of conjugate points for the corresponding extremal trajectories follows by limit passage from C_1.

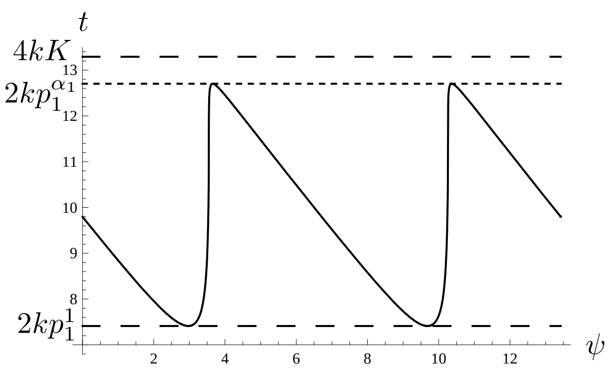

FIGURE 7. Plot of $t_1^{\mathrm{conj}}(\psi, k)$, $k = 0.99 > k_0$

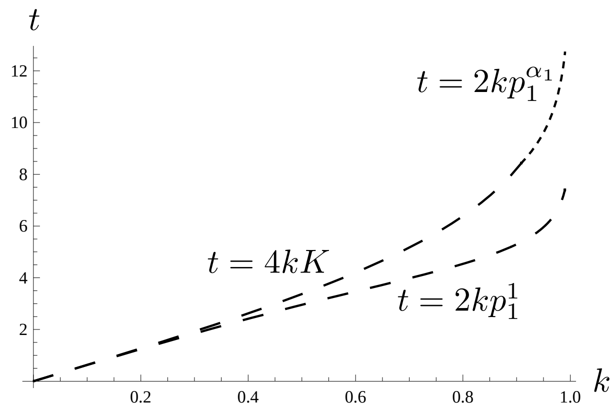

FIGURE 8. Bounds of $t_1^{\mathrm{conj}}($

Theorem 2.4. *If $\lambda \in C_3 \cup C_4 \cup C_5$, then the corresponding extremal trajectory $q(t) = \mathrm{Exp}(\lambda, t)$ does not have conjugate points for $t > 0$.*

Proof. For any $\lambda \in C_3 \cup C_4 \cup C_5$, there exists a continuous curve λ^s, $s \in [0, 1]$, such that $\lambda^s \in C_1$ for $s \in [0, 1)$ and $\lambda^1 = \lambda$. By Theorem 2.1, the trajectories $q^s(t) = \mathrm{Exp}(\lambda^s, t)$, $t > 0$, are free of conjugate points. Then Propos. 2.1 implies the same for the trajectory $q^1(t) = q(t)$. □

2.5. General bound of conjugate points

We collect the bounds on the first conjugate time obtained in the previous subsections. Moreover, we conclude that the first conjugate time t_1^{conj} admits the lower bound by the function $\mathbf{t} : C \to (0, +\infty]$ on the phase cylinder of pendulum $(2S^1_\gamma) \times \mathbb{R}_c = C = \sqcup_{i=1}^5 C_i$ introduced in [9]:

$$\lambda \in C_1 \quad \Rightarrow \quad \mathbf{t}(\lambda) = 2K(k), \tag{2.20}$$
$$\lambda \in C_2 \quad \Rightarrow \quad \mathbf{t}(\lambda) = 2kp_1^1(k), \tag{2.21}$$
$$\lambda \in C_3 \quad \Rightarrow \quad \mathbf{t}(\lambda) = +\infty, \tag{2.22}$$
$$\lambda \in C_4 \quad \Rightarrow \quad \mathbf{t}(\lambda) = \pi, \tag{2.23}$$
$$\lambda \in C_5 \quad \Rightarrow \quad \mathbf{t}(\lambda) = +\infty. \tag{2.24}$$

Theorem 2.5.
(1) *If $\lambda \in C_1 \cup C_3 \cup C_4 \cup C_5$, then $t_1^{\mathrm{conj}}(\lambda) = +\infty$.*
(2) *If $\lambda \in C_2$, then $t_1^{\mathrm{conj}}(\lambda) \in [2kp_1^1, 4kK]$.*
(3) *Consequently, $t_1^{\mathrm{conj}}(\lambda) \geq \mathbf{t}(\lambda)$ for all $\lambda \in C$.*

3. Exponential mapping of open stratas and cut time

In this section we show that there exist open dense domains $\widetilde{N} \subset N$, $\widetilde{M} \subset M$ transformed diffeomorphically by the exponential mapping. As a consequence, we prove that $t_{\mathrm{cut}}(\lambda) = \mathbf{t}(\lambda)$ for any $\lambda \in C$, and describe the optimal synthesis on $\widetilde{M}$.

3.1. Decompositions in preimage and image of exponential mapping

Denote $\widehat{M} = M \setminus \{q_0\}$. For any point $q \in \widehat{M}$ there exists an optimal trajectory $q(s) = \mathrm{Exp}(\lambda, s)$ such that $q(t) = q$, $(\lambda, t) \in N$. Thus the mapping $\mathrm{Exp} : N \to \widehat{M}$ is surjective. By Th. 5.4 [9], the optimal instant t

satisfies the inequality $t \le \mathbf{t}(\lambda)$. So the restriction

$$\begin{aligned}&\mathrm{Exp}\,:\,\widehat{N}\to\widehat{M},\\&\widehat{N}=\{(\lambda,t)\in N\mid t\le\mathbf{t}(\lambda)\},\end{aligned}$$

is surjective as well.

3.1.1. *Decomposition in* $\widehat{N}$

Now we select open dense subsets of $\widehat{N}$ such that restriction of Exp to these subsets will turn out to be a diffeomorphism. Let

$$\begin{aligned}&N_i=C_i\times\mathbb{R}_+,\qquad i=1,\dots,5,\\&\widetilde{N}=\{(\lambda,t)\in\cup_{i=1}^{3}N_i\mid t<\mathbf{t}(\lambda),\ \operatorname{sn}\tau\ \operatorname{cn}\tau\neq0\},\\&N'=\{(\lambda,t)\in\cup_{i=1}^{3}N_i\mid t=\mathbf{t}(\lambda)\text{ or }\operatorname{sn}\tau\ \operatorname{cn}\tau=0\}\cup\widehat{N}_4\cup N_5,\\&\widehat{N}_4=\widehat{N}\cap N_4.\end{aligned}\tag{3.1}$$

We have the obvious decomposition $\widehat{N}=\widetilde{N}\sqcup N'$ (we denote by $\sqcup$ the union of mutually non-intersecting sets).

There hold the following implications, see [9]:

$$\begin{aligned}&(\lambda,t)\in N_1\quad\Rightarrow\quad\mathbf{t}(\lambda)=2K,\ \tau\in\mathbb{R}/(4K\mathbb{Z}),\\&(\lambda,t)\in N_2\quad\Rightarrow\quad\mathbf{t}(\lambda)=2kp_1^1,\ \tau\in\mathbb{R}/(4K\mathbb{Z}),\\&(\lambda,t)\in N_3\quad\Rightarrow\quad\mathbf{t}(\lambda)=+\infty,\ \tau\in\mathbb{R}.\end{aligned}$$

Consequently, there holds the following decomposition:

$$\widetilde{N}=\sqcup_{i=1}^{8}D_i,$$

where the sets D_i, $i=1,\dots,8$, are defined by Table 1.

D_i	D_1	D_2	D_3	D_4	D_5	D_6	D_7	D_8
λ	C_1^0	C_1^0	C_1^0	C_1^0	C_1^1	C_1^1	C_1^1	C_1^1
τ	$(3K,4K)$	$(0,K)$	$(K,2K)$	$(2K,3K)$	$(-K,0)$	$(0,K)$	$(K,2K)$	$(2K,3K)$
p	$(0,K)$	$(0,K)$	$(0,K)$	$(0,K)$	$(0,K)$	$(0,K)$	$(0,K)$	$(0,K)$
λ	C_2^+	C_2^+	C_2^-	C_2^-	C_2^+	C_2^+	C_2^-	C_2^-
τ	$(-K,0)$	$(0,K)$	$(-K,0)$	$(0,K)$	$(K,2K)$	$(2K,3K)$	$(-3K,-2K)$	$(-2K,-K)$
p	$(0,p_1^1)$	$(0,p_1^1)$	$(0,p_1^1)$	$(0,p_1^1)$	$(0,p_1^1)$	$(0,p_1^1)$	$(0,p_1^1)$	$(0,p_1^1)$
λ	C_3^{0+}	C_3^{0+}	C_3^{0-}	C_3^{0-}	C_3^{1+}	C_3^{1+}	C_3^{1-}	C_3^{1-}
τ	$(-\infty,0)$	$(0,+\infty)$	$(-\infty,0)$	$(0,+\infty)$	$(-\infty,0)$	$(0,+\infty)$	$(-\infty,0)$	$(0,+\infty)$
p	$(0,+\infty)$	$(0,+\infty)$	$(0,+\infty)$	$(0,+\infty)$	$(0,+\infty)$	$(0,+\infty)$	$(0,+\infty)$	$(0,+\infty)$

TABLE 1. Definition of domains D_i

Table 1 should be read by columns. For example, the first column means that

$$
\begin{aligned}
&D_1 = (D_1 \cap N_1) \sqcup (D_1 \cap N_2) \sqcup (D_1 \cap N_3),\\
&D_1 \cap N_1 = \{(\tau, p, k) \in N_1 \mid \lambda \in C_1^0,\ \tau \in (3K, 4K),\ p \in (0, K),\ k \in (0,1)\},\\
&D_1 \cap N_2 = \{(\tau, p, k) \in N_2 \mid \lambda \in C_2^+,\ \tau \in (-K, 0),\ p \in (0, p_1^1),\ k \in (0,1)\},\\
&D_1 \cap N_3 = \{(\tau, p, k) \in N_3 \mid \lambda \in C_3^{0+},\ \tau \in (-\infty, 0),\ p \in (0, +\infty),\ k = 1\}.
\end{aligned}
$$

Projections of the sets D_i to the phase cylinder of the pendulum (γ, c) are shown at Fig. 9.

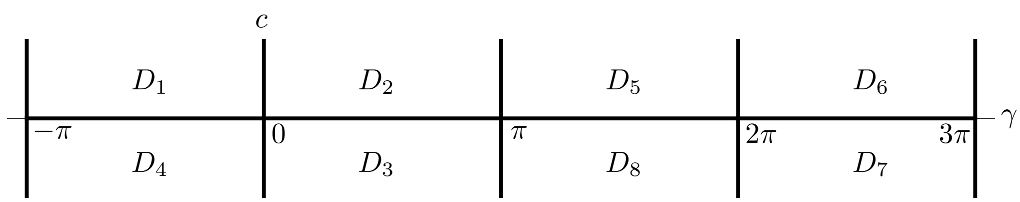

FIGURE 9. Projections of domains D_i to the phase cylinder of pendulum $(2S^1_\gamma) \times \mathbb{R}^1_c$

Lemma 3.1. *Each set D_i, $i = 1, \dots, 8$, is homeomorphic to $\mathbb{R}^3$.*

Proof. We prove the statement only for the set D_2 since all other sets D_i can be defined in the coordinates (τ, p, k) by the same inequalities as D_2 by a shift of origin in elliptic coordinate φ. Taking into account Table 1 and Eqs. (2.4), (2.10), we get:

$$
\begin{aligned}
&D_2 = (D_2 \cap N_1) \sqcup (D_2 \cap N_2) \sqcup (D_2 \cap N_3),\\
&D_2 \cap N_1 = \{(\lambda, t) \in N_1 \mid \lambda \in C_1^0,\ k \in (0,1),\ t \in (0, 2K),\ \varphi \in (-t, -t + 2K)\},\\
&D_2 \cap N_2 = \{(\lambda, t) \in N_1 \mid \lambda \in C_2^+,\ k \in (0,1),\ t \in (0, 2kp_1^1),\ \varphi \in (-t, -t + 2kK)\},\\
&D_2 \cap N_3 = \{(\lambda, t) \in N_1 \mid \lambda \in C_3^{0+},\ k = 1,\ t \in (0, +\infty),\ \varphi \in (-t, +\infty)\}.
\end{aligned}
$$

As shown in [17], one can choose regular system of coordinates (k_1, φ, t) on the set D_2, where

$$
k_1 = k \text{ for } \lambda \in C_1; \quad k_1 = 1/k \text{ for } \lambda \in C_2; \quad k_1 = 1 \text{ for } \lambda \in C_3.
$$

In this system of coordinates

$$
D_2 = \{\nu = (k_1, \varphi, t) \mid k_1 \in (0, +\infty),\ t \in (0, t_1(k_1),\ \varphi \in (-t, -t + t_2(k_2))\}, \tag{3.2}
$$

where $t_1(k_1) = 2K(k_1)$ for $k_1 \in (0,1)$, $t_1(k_1) = +\infty$ for $k_1 = 1$, $t_1(k_1) = (2/k_1)p_1^1(1/k_1)$ for $k_1 \in (1, +\infty)$; and $t_2(k_1) = 2K(k_1)$ for $k_1 \in (0,1)$, $t_2(k_1) = +\infty$ for $k_1 = 1$, $t_2(k_1) = (2/k_1)K(1/k_1)$ for $k_1 \in (1, +\infty)$. The both functions $t_i : (0, +\infty) \to (0, +\infty]$, $i = 1, 2$, are continuous. Thus representation (3.2) implies that the domain D_2 is homeomorphic to $\mathbb{R}^3$. □

Consequently, all domains D_i are open, connected, and simply connected. These domains are schematically represented in the left-hand side of Fig. 10.

3.1.2. *Decomposition in* $\widehat{M}$

The state space of the problem admits a decomposition of the form

$$M = \widetilde{M} \sqcup M',$$
$$\widetilde{M} = \{q \in M \mid R_1(q)R_2(q)\sin\theta \neq 0\},$$
$$M' = \{q \in M \mid R_1(q)R_2(q)\sin\theta = 0\},$$

where

$$R_1 = y\cos\frac{\theta}{2} - x\sin\frac{\theta}{2}, \qquad R_2 = x\cos\frac{\theta}{2} + y\sin\frac{\theta}{2}.$$

Further,

$$\widetilde{M} = \sqcup_{i=1}^{8} M_i,$$

where each of the sets M_i is characterized by constant signs of the functions $\sin\theta$, R_1, R_2 described in Table 2.

M_i	M_1	M_2	M_3	M_4	M_5	M_6	M_7	M_8
$\mathrm{sgn}(\sin\theta)$	$-$	$-$	$-$	$-$	$+$	$+$	$+$	$+$
$\mathrm{sgn}(R_1)$	$+$	$+$	$-$	$-$	$-$	$-$	$+$	$+$
$\mathrm{sgn}(R_2)$	$+$	$-$	$-$	$+$	$+$	$-$	$-$	$+$

Table 2. Definition of domains M_i

For example, $M_1 = \{q \in M \mid \sin\theta < 0,\ R_1 > 0,\ R_2 > 0\}$. The numeration of the sets M_i is chosen so that it correspond to numeration of the sets N_i (we prove below in Th. 3.1 that each mapping $\mathrm{Exp} : N_i \to M_i$ is a diffeomorphism). It is obvious that all the sets M_i are diffeomorphic to $\mathbb{R}^3$.

All the domains M_i are contained in the set $\{q \in M \mid \theta \neq 0\}$. At this set θ is a single-valued function, and we choose the branch $\theta \in (0, 2\pi)$. Thus in the sequel we assume that $\theta \in (0, 2\pi)$ on the sets M_i. Then R_1, R_2 become single-valued functions, and the last two rows of Table 2 reflect the signs of these single-valued functions R_i on the sets M_i.

The boundary M' of the domain $\widetilde{M}$ decomposes into four mutually orthogonal surfaces: two planes $\{\theta = 0\}$, $\{\theta = \pi\}$ and two Moebius strips $\{R_1 = 0\}$, $\{R_2 = 0\}$, see the right-hand side of Fig. 10, and Fig. 7 [9].

3.2. Diffeomorphic properties of exponential mapping

Lemma 3.2. *The restriction* $\mathrm{Exp}|_{\widetilde{N}}$ *is nondegenerate.*

Proof. If $\nu = (\lambda, t) \in \widetilde{N}$, then $t < \mathbf{t}(\lambda)$. By Th. 2.5, $t_1^{\mathrm{conj}}(\lambda) \geq \mathbf{t}(\lambda)$, thus $t < t_1^{\mathrm{conj}}(\lambda) = \inf\{s > 0 \mid \mathrm{Exp}(\lambda, s)$ is degenerate $\}$. Consequently, the exponential mapping is nondegenerate at the point (λ, t). □

Lemma 3.3. *For any* $i = 1, \dots, 8$*, we have* $\mathrm{Exp}(D_i) \subset M_i$*.*

Proof. We prove only the inclusion $\mathrm{Exp}(D_1) \subset M_1$ since the rest inclusions are proved similarly.

Let $(\lambda, t) \in D_1 \cap N_1 = \{(\lambda, t) \in N_1 \mid \lambda \in C_1^0,\ \tau \in (3K, 4K),\ p \in (0, K),\ k \in (0, 1)\}$, see Table 1. Since $\lambda \in C_1^0$, then $s_1 = \mathrm{sgn}(\gamma_t/2) = 1$. Moreover, we have $\mathrm{cn}\, p\, \mathrm{sn}\, p\, \mathrm{dn}\, \tau > 0$, thus $\sin\theta_t < 0$ by virtue of Eq. (5.2) [9]. So $\theta_t/2 \in (\pi/2, \pi)$. Consequently, $\cos(\theta_t/2) < 0$, $\sin\theta_t/2 > 0$ on $D_1 \cap N_1$, thus $s_3 = -1$, $s_4 = 1$ in Eq. (5.3)–(5.6) [9]. Then we get $R_1 > 0$ from Eq. (5.5) [9], and $R_2 > 0$ from Eq. (5.6) [9] and Lemma 5.2 [9]. We proved that if $\nu \in D_1 \cap N_1$, then $\sin\theta_t < 0$, $R_1 > 0$, $R_2 > 0$, i.e., $\mathrm{Exp}(\nu) \in M_1$, see Table 2. That is, $\mathrm{Exp}(D_1 \cap N_1) \subset M_1$.

A similar argument works in the case $(\lambda, t) \in D_1 \cap N_2 = \{(\lambda, t) \in N_2 \mid \lambda \in C_2^+,\ \tau \in (-K, 0),\ p \in (0, p_1^1),\ k \in (0, 1)\}$. We have $s_2 = \operatorname{sgn} c_t = 1$; $\sin\theta_t < 0$ by Eq. (5.7) [9]; $R_1 > 0$ and $R_2 > 0$ by Eq. (5.10) and (5.11) [9]. Thus $\operatorname{Exp}(D_1 \cap N_2) \subset M_1$.

It follows from the definition of N_3 and N_2 (see Table 1) that $D_1 \cap N_3 \subset \operatorname{cl}(D_2 \cap N_3)$, thus $\operatorname{Exp}(D_1 \cap N_3) \subset \operatorname{cl}(M_1)$. So

$$\operatorname{Exp}(D_1) \subset \operatorname{cl}(M_1). \tag{3.3}$$

Let $\nu = (\lambda, t) \in D_1 \cap N_3$, then $q_t = \operatorname{Exp}(\nu) \in \operatorname{cl}(M_1)$. In order to prove that $q_t \in M_1$, assume by contradiction that $q_t \in \partial M_1$. By Lemma 3.2, the exponential mapping is a local diffeomorphism near ν. Thus there exist a neighborhood $U \subset D_1$ of the point ν and a neighborhood $V \subset M$ of the point q_t such that $\operatorname{Exp} : U \to V$ is a diffeomorphism. Since $q_t \in \partial M_1$, there exists a point $\widetilde{q} \in V \setminus \operatorname{cl}(M_1)$. Then $\widetilde{\nu} = \operatorname{Exp}^{-1}(\widetilde{q}) \in U \subset D_1$, but $\widetilde{q} = \operatorname{Exp}(\widetilde{\nu}) \notin \operatorname{cl}(M_1)$, which contradicts to (3.3). Thus $q_t \in M_1$. So $\operatorname{Exp}(D_1 \cap N_3) \subset M_1$, and the required inclusion $\operatorname{Exp}(D_1) \subset M_1$ follows. □

Lemma 3.4. *For any $i = 1, \dots, 8$, the mapping $\operatorname{Exp} : D_i \to M_i$ is proper.*

Proof. Similarly to Lemma 3.1, we can consider only the case $i = 2$. Let $K \subset M_2$ be a compact, we show that $S = \operatorname{Exp}^{-1}(K) \subset D_2$ is a compact as well, i.e., S is bounded and closed.

There exists $\varepsilon > 0$ such that

$$|\sin\theta| \geq \varepsilon, \qquad \varepsilon \leq |R_1|, |R_2| \leq 1/\varepsilon \qquad \text{for all } q \in K.$$

(1) We show that S is bounded. By contradiction, let $\nu_n = (k_n, \varphi_n, t_n) \to \infty$ for some sequence $\{\nu_n\} \subset S$. Then there exists a sequence $\{\nu_n\} \subset S \cap N_i$ for some $i = 1, 2, 3$ with $\nu_n \to \infty$.

Let $S \cap N_1 \ni \nu_n = (k_n, \varphi_n, t_n) \to \infty$. Then $t_n = 2p_n \in (0, K(k_n))$, $\tau_n = (\varphi_n + t_n)/2 \in (0, K(k_n))$. If k_n is separated from 1, then p_n, τ_n are bounded, thus t_n, φ_n are bounded, a contradiction. Thus $k_n \to 1$ for a subsequence (we will assume that this holds for the initial sequence).

If $(\gamma_n, c_n) \to (\pm\pi, 0)$, then $(\theta_t, y_t) \to 0$, thus $R_1 \to 0$, a contradiction. Thus the sequence (γ_n, c_n) is separated from the point $(\pm\pi, 0)$.

Then there exists a sequence such that $k_n \to 1$ and $\varphi_n \to \varphi \in (-\infty, +\infty)$, thus $t_n \to +\infty$, $p_n \to +\infty$, $\tau_n \to +\infty$. Then $(p_n - \operatorname{E}(p_n))/(k_n\sqrt{\Delta}) \to \infty$, $f_2(p_n, k_n)/(k_n\sqrt{\Delta}) \to \infty$.

If $\operatorname{cn}(\tau_n)$ is separated from zero, then $R_1 \to \infty$ (see Eq. (5.5) [9]). And if $\operatorname{cn}(\tau_n)$ is not separated from zero, then there exists a sequence such that $\operatorname{cn}(\tau_n) \to 0$, thus $\operatorname{sn}(\tau_n)$ is separated from zero, then $R_2 \to \infty$ (see Eq. (5.6) [9]).

So the hypothesis $S \cap C_1 \ni \nu_n = (k_n, \varphi_n, t_n) \to \infty$ leads to a contradiction.

Similarly the hypotheses $C \cap C_i \ni \nu_n \to \infty$, $i = 2, 3$, lead to a contradiction.

Thus the set $S = \operatorname{Exp}^{-1}(K)$ is bounded.

(2) We show that S is closed. Let $\{\nu_n\} \subset S$, we have to prove that there exists a subsequence ν_{n_k} converging in D_2. By contradiction, let $\nu_n \to \infty$ or $\nu_n \to \nu \in \partial D_2$.

Consider the case $\nu_n = (\tau_n, p_n, k_n) \in S \cap N_1$.

If $k_n \to 0$, then $(x, y) \to 0$, thus $R_1, R_2 \to 0$, a contradiction.

Let $k_n \to 1$. If $(\gamma_n, c_n) \to (\pm\pi, 0)$, then $(\theta, y) \to (0, 0)$, thus $R_1 \to 0$, a contradiction.

If $(\gamma_n, c_n) \to (\gamma, c) \neq (\pm\pi, 0)$, then $\nu \in N_3$, a contradiction.

Thus $k_n \to k \in (0, 1)$. Then $\tau_n \to \tau \in [3K(k), 4K(k)]$. If $\tau = 3K$, then $R_1 \to 0$, and if $\tau = 4K$, then $R_2 \to 0$, a contradiction. Thus $\tau_n \to \tau \in (3K(k), 4K(k))$.

Further, $p_n \to p \in [0, K(k)]$. If $p = 0$, then $t = 0$ and $R_1, R_2 \to 0$. If $p = K$, then $R_1 \to 0$. Thus $p_n \to p \in (0, K)$.

So $(\tau_n, p_n, k_n) \to (\tau, p, k) \in N_1$, a contradiction.

We proved that any sequence $\nu_n \in S \cap N_1$ contains a subsequence converging in D_2. Similarly one proves the same for a sequence $\nu_n \in S \cap N_i$, $i = 2, 3$.

Thus any sequence $\nu_n \in S$ contains a subsequence converging in D_2, thus converging in S. So the set $S = \operatorname{Exp}^{-1}(K)$ is closed. □

Theorem 3.1. *For any* $i = 1, \dots, 8$, *we have* $\mathrm{Exp}(D_i) \subset M_i$, *and the mapping* $\mathrm{Exp} : D_i \to M_i$ *is a diffeomorphism.*

Proof. The inclusion $\mathrm{Exp}(D_i) \subset M_i$ was proved in Lemma 3.3. The mapping $\mathrm{Exp} : D_i \to M_i$ is smooth, nondegenerate (Lemma 3.2), and proper (Lemma 3.4), thus it is a covering. Since M_i is simply connected, the mapping $\mathrm{Exp} : D_i \to M_i$ is a diffeomorphism. □

Lemma 3.5. $\mathrm{Exp}(N_4) = \{q \in M \mid R_1 = R_2 = 0\} = \{q \in M \mid x = y = 0\}$, $\mathrm{Exp}(N_5) = \{q \in M \mid R_1 = 0,\ R_2 \neq 0,\ \theta = 0\} = \{q \in M \mid x \neq 0,\ y = 0,\ \theta = 0\}$.

Proof. Follows immediately from the corresponding formulas for extremal trajectories of Subsec. 3.3 [9]. □

Lemma 3.6. $\mathrm{Exp}(N') \subset M'$.

Proof. Follows from formulas (5.2)–(5.11) [9]. □

Theorem 3.1 implies the following statement.

Corollary 3.1. *The mapping* $\mathrm{Exp} : \widetilde{N} \to \widetilde{M}$ *is a diffeomorphism.*

In view of Lemma 3.6, for any $q \in \widetilde{M}$ there exists a unique $\nu = (\lambda, t) = \mathrm{Exp}^{-1}(q) \in \widetilde{N}$, $\lambda = \lambda(q)$, $t = t(q)$.

The diffeomorphism $\mathrm{Exp} : \widetilde{N} = \cup_{i=1}^8 D_i \to \widetilde{M} = \cup_{i=1}^8 M_i$ is schematically shown at Fig. 10. At this figure in the left, each set N_i, $i = 1, \dots, 5$, projects down to the corresponding subset C_i in the phase cylinder of the pendulum, see (1.8)–(1.12) and Fig. 2 [9]. Each domain D_i, $i = 1, \dots, 8$, is mapped to one of the domains M_i, $i = 1, \dots, 8$, cut out in the solid torus by the discs $\{\theta = 0\}$, $\{\theta = \pi\}$, and the Moebius strips $\{R_1 = 0\}$, $\{R_2 = 0\}$. Boundaries of the domains D_i are mapped to these discs and Moebius strips in a more complicated way to be described in detail in the forthcoming work [19].

3.3. Cut time

Theorem 3.2. *For any* $q_1 \in \widetilde{M}$, *let* $(\lambda_1, t_1) = \mathrm{Exp}^{-1}(q_1) \in \widetilde{N}$. *Then the extremal trajectory* $q(s) = \mathrm{Exp}(\lambda_1, s)$ *is optimal with* $q(t_1) = q_1$.

Thus optimal synthesis on the domain $\widetilde{M}$ *is given by*

$$u_i(q) = h_i(\lambda), \quad i = 1, 2, \qquad (\lambda, t) = \mathrm{Exp}^{-1}(q) \in \widetilde{N}, \qquad q \in \widetilde{M}.$$

Proof. Let $q_1 \in \widetilde{M}$. There exists $\nu_1 = (\lambda_1, t_1) \in \widehat{N} = \widetilde{N} \sqcup N'$ such that the trajectory $q(s) = \mathrm{Exp}(\lambda_1, s)$ is optimal and $q(t_1) = \mathrm{Exp}(\nu_1) = q_1$. By Lemmas 3.5 and 3.6, we have $\nu_1 \in \widetilde{N}$. By Cor. 3.1, there exists a unique $\nu_1 \in \widetilde{N}$ such that $\mathrm{Exp}(\nu_1) = q_1$. So $q(s) = \mathrm{Exp}(\lambda_1, s)$ is a unique optimal trajectory coming to q_1. □

In work [9] we proved the inequality $t_{\mathrm{cut}}(\lambda) \leq \mathbf{t}(\lambda)$. Now we prove the corresponding equality.

Theorem 3.3. *For any* $\lambda \in C$ *we have* $t_{\mathrm{cut}}(\lambda) = \mathbf{t}(\lambda)$.

Proof. We proved the inequality $t_{\mathrm{cut}}(\lambda) \leq \mathbf{t}(\lambda)$ in Th. 5.4 [9].

(1) Consider first the generic case $\lambda_1 \in \cup_{i=1}^3 C_i$. There exists $t_1 \in (0, \mathbf{t}(\lambda_1))$ and arbitrarily close to $\mathbf{t}(\lambda_1)$ such that $\operatorname{sn} \tau_1 \operatorname{cn} \tau_1 \neq 0$. Then $\nu_1 = (\lambda_1, t_1) \in \widetilde{N}$, thus $q_1 = \mathrm{Exp}(\nu_1) \in \widetilde{M}$. By Th. 3.2, the trajectory $q(s) = \mathrm{Exp}(\lambda_1, s)$, $s \in [0, t_1]$, is optimal, thus $t_1 \leq t_{\mathrm{cut}}(\lambda_1)$.

So there exists $t_1 \in (0, \mathbf{t}(\lambda_1))$ arbitrarily close to $\mathbf{t}(\lambda_1)$ such that $t_1 \leq t_{\mathrm{cut}}(\lambda_1)$. Consequently, $\mathbf{t}(\lambda_1) \leq t_{\mathrm{cut}}(\lambda_1)$.

We proved that $t_{\mathrm{cut}}(\lambda_1) = \mathbf{t}(\lambda_1)$ for any $\lambda_1 \in \cup_{i=1}^3 C_i$.

(2) If $\lambda \in C_4$, then the extremal trajectory $(x, y, \theta) = (0, 0, \pm t)$ is a Riemannian geodesic for the restriction of the sub-Riemannian problem on SE(2) to the circle $\{(0, 0, \theta) \mid \theta \in S^1\}$. It is optimal up to the antipodal point, thus $t_{\mathrm{cut}}(\lambda) = \pi = \mathbf{t}(\lambda)$.

(3) In the case $\lambda \in C_5$ the extremal trajectory is a line $(x, y, \theta) = (\pm t, 0, 0)$, thus it is optimal forever: $t_{\mathrm{cut}}(\lambda) = +\infty = \mathbf{t}(\lambda)$. □

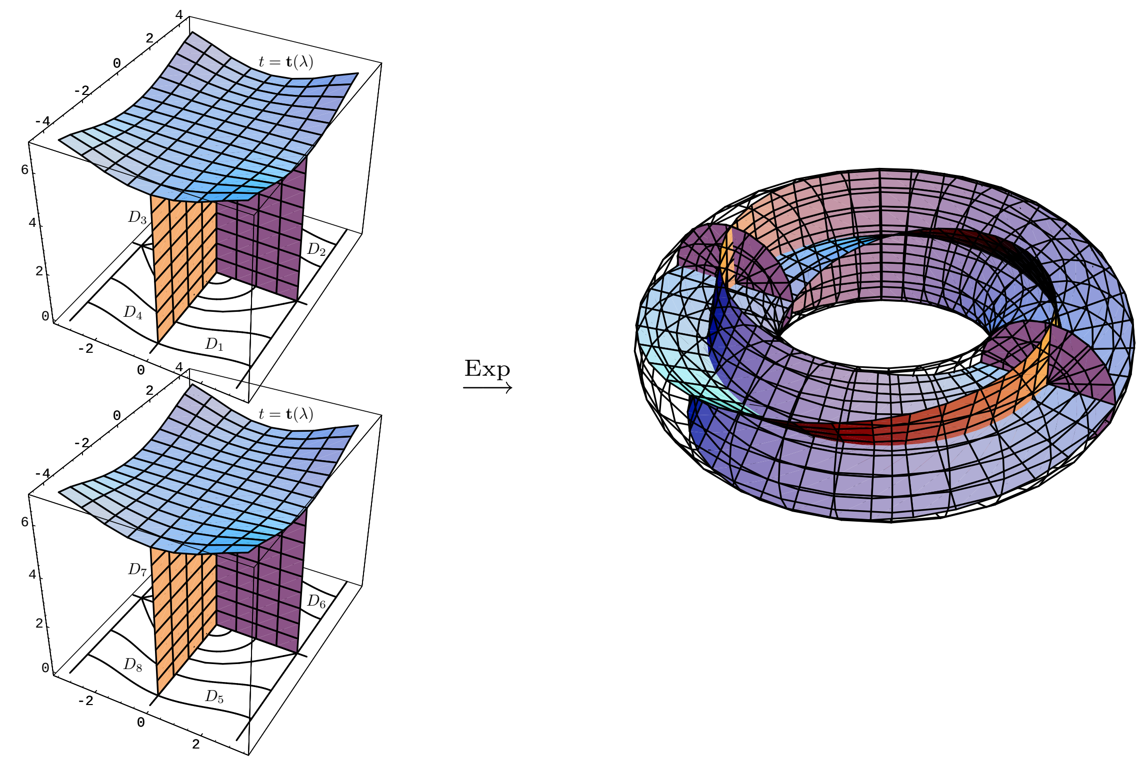

FIGURE 10. Global structure of exponential mapping

4. Plots of sub-Riemannian caustic and spheres

Here we collect 3-dimensional plots of some essential objects in the sub-Riemannian problem on SE(2).

Figure 11 shows the sub-Riemannian caustic

$$\{\mathrm{Exp}(\lambda, t) \mid \lambda \in C,\ t = t_1^{\mathrm{conj}}(\lambda)\}$$

in the rectifying coordinates (R_1, R_2, θ).

Figures 12–20 present sub-Riemannian spheres

$$S_R = \{q \in M \mid d(q_0, q) = R\} = \{\mathrm{Exp}(\lambda, R) \mid \lambda \in C,\ t_{\mathrm{cut}}(\lambda) \geq R\}$$

of different radii R, where

$$d(q_0, q_1) = \inf\{l(q(\cdot)) \mid q(\cdot) \text{ trajectory of (1.1)}, q(0) = q_0,\ q(t_1) = q_1\}$$

is the sub-Riemannian distance — the cost function in the sub-Riemannian problem (1.1)–(1.5).

In this problem sub-Riemannian spheres can be of three different topological classes. If $R \in (0, \pi)$, then S_R is homeomorphic to the standard 2-dimensional Euclidean sphere S^2, see Fig. 14. For $R = \pi$ the sphere S_R is homeomorphic to the sphere S^2 with its north pole N and south pole S identified: $S_\pi \cong S^2/\{N = S\}$, see Fig. 17. And if $R > \pi$, then S_R is homeomorphic to the 2-dimensional torus, see Fig. 20.

Figure 12 shows the sphere $S_{\pi/2}$ in rectifying coordinates (R_1, R_2, θ). Figure 13 represents the same sphere with a cut-out opening the singularities of the sphere: the sphere intersects itself at the local components

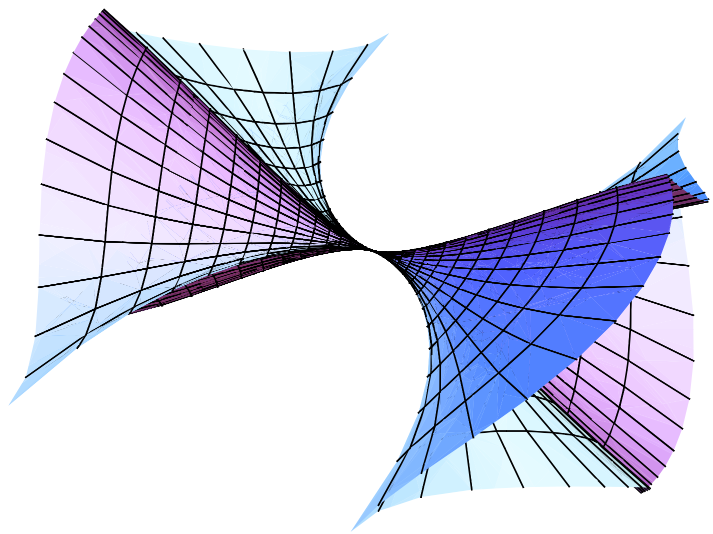

Figure 11. Sub-Riemannian caustic

of the cut locus $\mathrm{Cut}_{\mathrm{loc}}$ described in [19]. Figure 14 shows embedding of the same sphere to the solid torus. Sub-Riemannian spheres of small radii resemble the well-known apple-shaped sub-Riemannian sphere in the Heisenberg group [21]. Although, there is a major difference: the sphere in the Heisenberg group has a one-parameter family of symmetries (rotations), but the sphere in SE(2) has only a discrete group of symmetries $G = \{\mathrm{Id}, \varepsilon^1, \dots, \varepsilon^7\}$ (reflections).

Figures 15–17 represent similarly the sub-Riemannian sphere of the critical radius π. In addition to self-intersections at the local component $\mathrm{Cut}_{\mathrm{loc}}$, the sphere S_π has one self-intersection point at the global component $\mathrm{Cut}_{\mathrm{glob}}$.

Figures 18–20 show similar images of the sphere of radius $3\pi/2$. For $R > \pi$ the sphere S_R has two topological segments of self-intersection points at $\mathrm{Cut}^{\pm}_{\mathrm{loc}}$ respectively, and a topological circle of self-intersection points at $\mathrm{Cut}_{\mathrm{glob}}$.

Figures 21, 21 shows intersections of the spheres $S_{\pi/2}$, S_π, $S_{3\pi/2}$ with the half-spaces $\theta < 0$ and $R_2 > 0$ respectively.

Figure 23 shows self-intersections of the wavefront

$$W_R = \{\mathrm{Exp}(\lambda, R) \mid \lambda \in C\}$$

for $R = \pi$.

5. Conclusion

The solution to the sub-Riemannian problem on SE(2) obtained in the previous paper [9], this one, and the subsequent paper [19] is based on a detailed study of the action of the discrete group of symmetries of the exponential mapping. This techniques was already partially developed in the study of related optimal control problems (nilpotent sub-Riemannian problem with the growth vector (2,3,5) [13–16] and Euler's elastic problem [17, 18]). The sub-Riemannian problem on SE(2) is the first problem in this series, where a complete solution was obtained (local and global optimality, cut time and cut locus, optimal synthesis). We believe that our approach based on the study of symmetries will provide such complete results in other symmetric

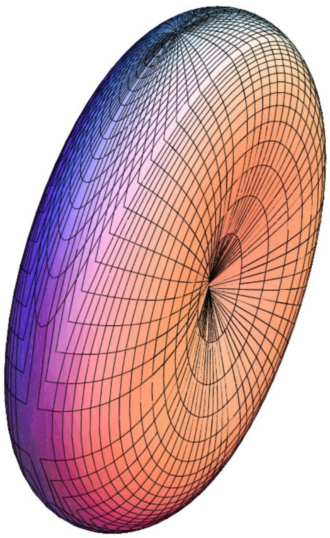

FIGURE 12. Sub-Riemannian sphere $S_{\pi/2}$

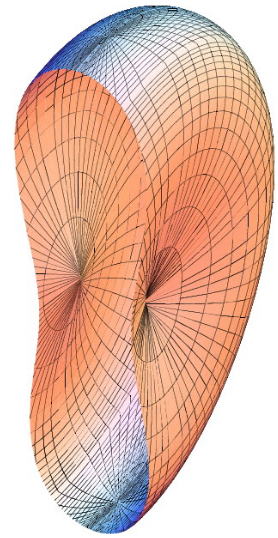

FIGURE 13. Sub-Riemannian sphere $S_{\pi/2}$ with cut-out

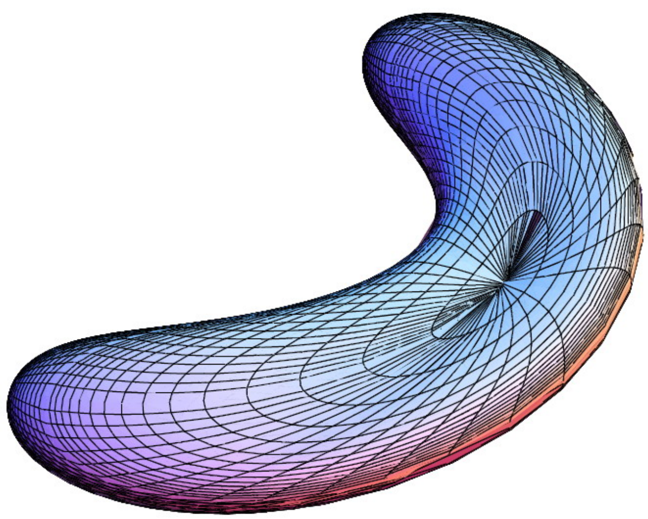

FIGURE 14. Sub-Riemannian sphere $S_{\pi/2}$, global view

invariant problems, such as the nilpotent sub-Riemannian problem with the growth vector (2,3,5), the ball-plate problem [7], and others.

Acknowledgements

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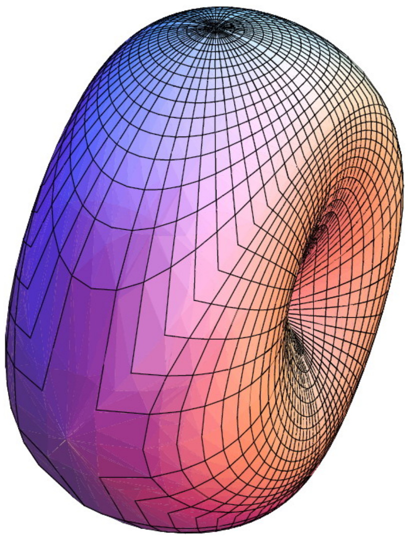

FIGURE 15. Sub-Riemannian sphere S_π

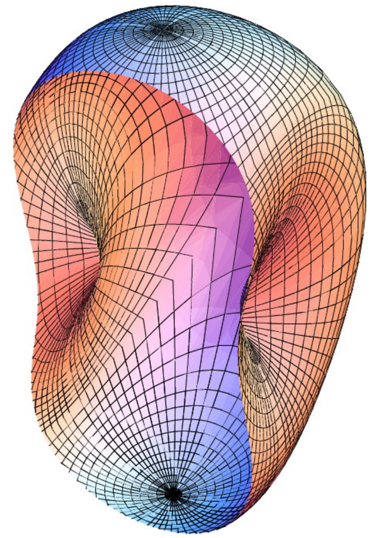

FIGURE 16. Sub-Riemannian sphere S_π with cut-out

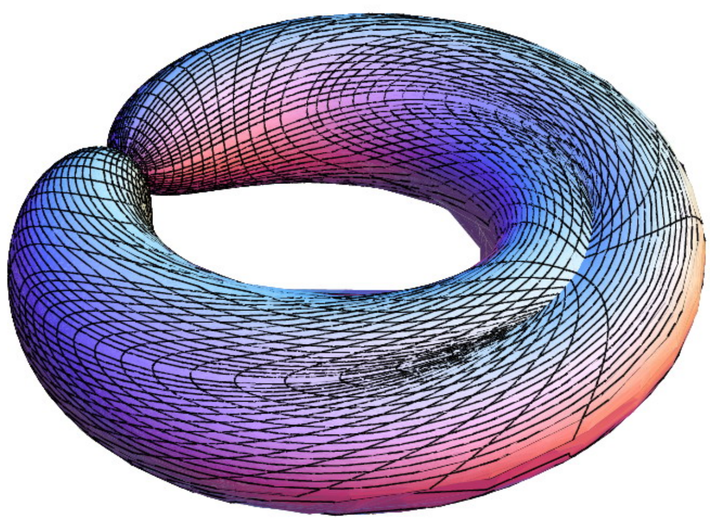

FIGURE 17. Sub-Riemannian sphere S_π, global view

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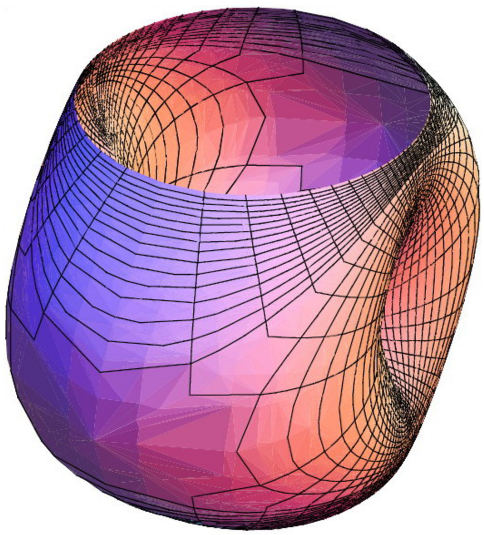

FIGURE 18. Sub-Riemannian sphere $S_{3\pi/2}$

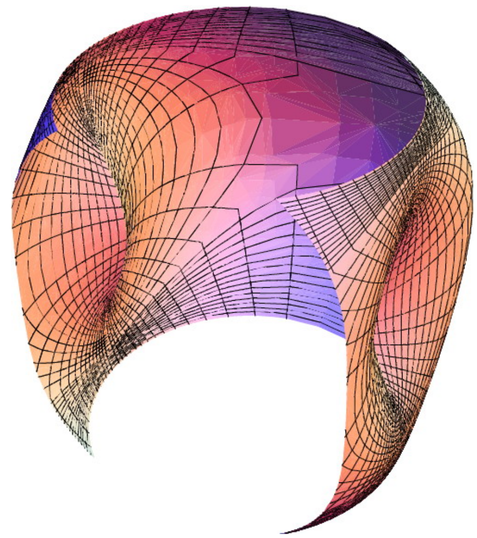

FIGURE 19. Sub-Riemannian sphere $S_{3\pi/2}$ with cut-out

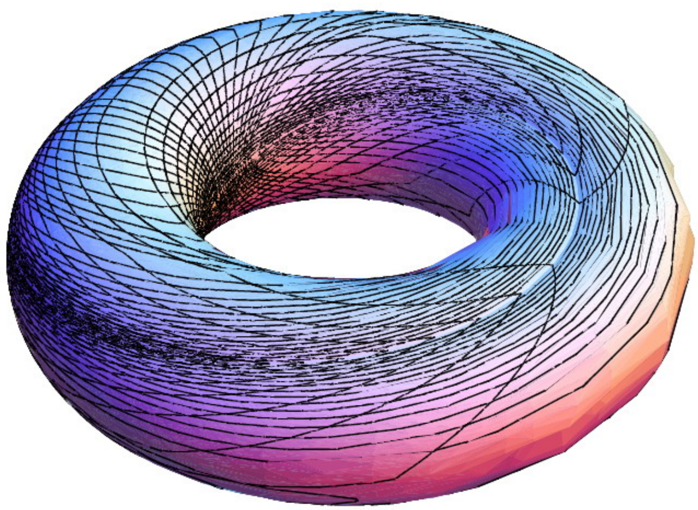

FIGURE 20. Sub-Riemannian sphere $S_{3\pi/2}$, global view

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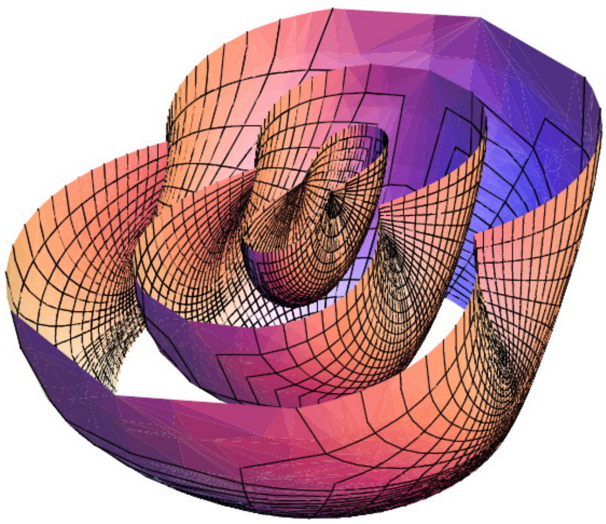

Figure 21. Sub-Riemannian hemi-spheres $S_{\pi/2}$, S_{π}, $S_{3\pi/2}$ for $\theta < 0$

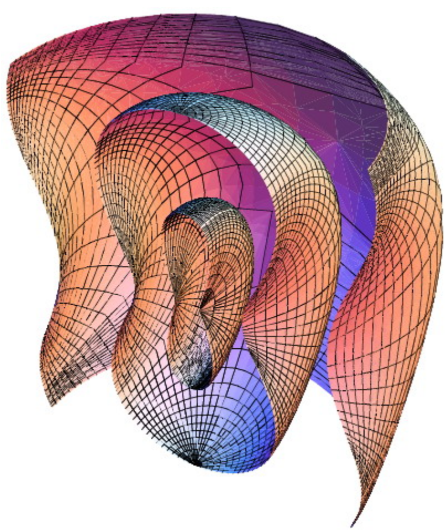

Figure 22. Sub-Riemannian hemi-spheres $S_{\pi/2}$, S_{π}, $S_{3\pi/2}$ for $R_2 > 0$

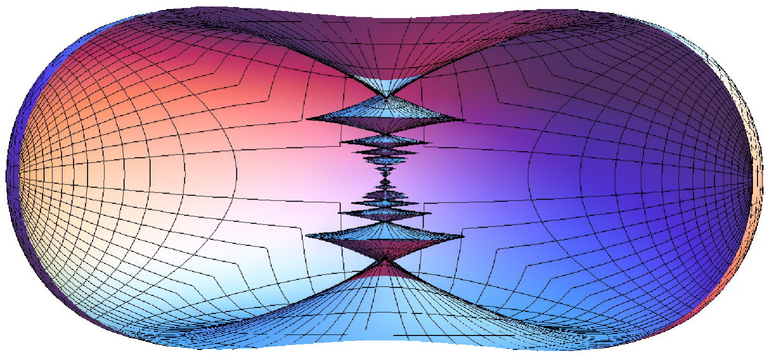

FIGURE 23. Wavefront W_π with cut-out

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