

**CUT LOCUS AND OPTIMAL SYNTHESIS
IN THE SUB-RIEMANNIAN PROBLEM
ON THE GROUP OF MOTIONS OF A PLANE ***

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Abstract. The left-invariant sub-Riemannian problem on the group of motions (rototranslations) of a plane $SE(2)$ is considered. In the previous works [7,10], extremal trajectories were defined, their local and global optimality were studied.

In this paper the global structure of the exponential mapping is described. On this basis an explicit characterization of the cut locus and Maxwell set is obtained. The optimal synthesis is constructed.

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1. INTRODUCTION

This work completes the study of the left-invariant sub-Riemannian problem on the group of motions of a plane $SE(2) = \mathbb{R}^2 \times SO(2)$ started in [7, 10]. In visual geometric terms, this problem can be stated as follows: given two unit vectors $v_0 = (\cos \theta_0, \sin \theta_0)$, $v_1 = (\cos \theta_1, \sin \theta_1)$ attached respectively at two given points (x_0, y_0) , (x_1, y_1) in the plane, one should find an optimal motion in the plane that transfers the vector v_0 to the vector v_1 , see Fig. 1. The vector can move forward or backward and rotate simultaneously. The required motion should be optimal in the sense of minimal length in the space (x, y, θ) , where θ is the slope of the moving vector.

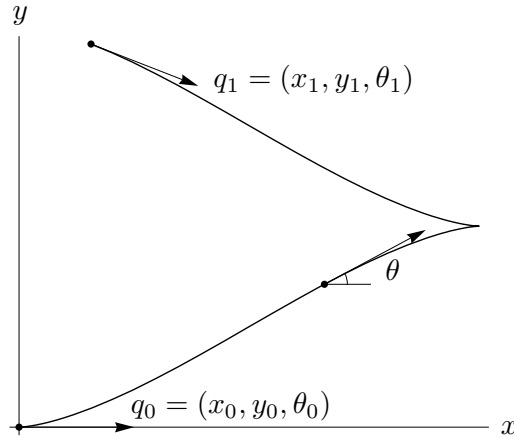


FIGURE 1. Problem statement

The corresponding optimal control problem reads as follows:

$$\dot{x} = u_1 \cos \theta, \quad \dot{y} = u_1 \sin \theta, \quad \dot{\theta} = u_2, \quad (1.1)$$

$$q = (x, y, \theta) \in M = \mathbb{R}_{x,y}^2 \times S_\theta^1, \quad u = (u_1, u_2) \in \mathbb{R}^2, \quad (1.2)$$

$$q(0) = q_0 = (0, 0, 0), \quad q(t_1) = q_1 = (x_1, y_1, \theta_1), \quad (1.3)$$

$$l = \int_0^{t_1} \sqrt{u_1^2 + u_2^2} dt \rightarrow \min, \quad (1.4)$$

or, equivalently,

$$J = \frac{1}{2} \int_0^{t_1} (u_1^2 + u_2^2) dt \rightarrow \min. \quad (1.5)$$

This problem has important relations to vision [5, 8, 9], robotics [6], and diffusion equation on SE(2) [3].

Notice that before this work a global description of the cut locus and optimal synthesis was known for left-invariant sub-Riemannian problems on the following Lie groups only: the Heisenberg group (A.M. Vershik, V.Y. Gershkovich [11]), and SO(3), SU(2) $\cong S^3$, SL(2) (U. Boscain, F. Rossi [4]).

First we recall the main results of the previous works [7, 10]. In paper [7] the normal Hamiltonian system of Pontryagin Maximum Principle was written in a triangular form in appropriate coordinates on cotangent bundle T^*M , so that its vertical subsystem takes the form of mathematical pendulum:

$$\dot{\gamma} = c, \quad \dot{c} = -\sin \gamma, \quad (\gamma, c) \in C \cong (2S_\gamma^1) \times \mathbb{R}_c, \quad (1.6)$$

$$\dot{x} = \sin \frac{\gamma}{2} \cos \theta, \quad \dot{y} = \sin \frac{\gamma}{2} \sin \theta, \quad \dot{\theta} = -\cos \frac{\gamma}{2}. \quad (1.7)$$

The phase cylinder of pendulum (1.6) decomposes into invariant subsets according to values of the energy $E = c^2/2 - \cos \gamma$:

$$C = \bigcup_{i=1}^5 C_i, \quad (1.8)$$

$$C_1 = \{\lambda \in C \mid E \in (-1, 1)\}, \quad (1.8)$$

$$C_2 = \{\lambda \in C \mid E \in (1, +\infty)\}, \quad (1.9)$$

$$C_3 = \{\lambda \in C \mid E = 1, c \neq 0\}, \quad (1.10)$$

$$C_4 = \{\lambda \in C \mid E = -1\} = \{(\gamma, c) \in C \mid \gamma = 2\pi n, c = 0\}, \quad (1.11)$$

$$C_5 = \{\lambda \in C \mid E = 1, c = 0\} = \{(\gamma, c) \in C \mid \gamma = \pi + 2\pi n, c = 0\}. \quad (1.12)$$

In the subsets C_1, C_2, C_3 elliptic coordinates (φ, k) that rectify the flow of the pendulum were introduced: φ is the phase, and k a reparametrized energy of pendulum (1.6):

$$k = \sqrt{(E+1)/2} \text{ in } C_1 \cup C_3, \quad k = \sqrt{2/(E+1)} \text{ in } C_2.$$

The Hamiltonian system (1.6), (1.7) was integrated in Jacobi's functions [12]. The equation of pendulum (1.6) has a discrete group of symmetries $G = \{\text{Id}, \varepsilon^1, \dots, \varepsilon^7\} = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ generated by reflections in the axes of coordinates γ, c , and translations $(\gamma, c) \mapsto (\gamma + 2\pi, c)$. Reflections ε^i are symmetries of the exponential mapping

$$\text{Exp} : N = C \times \mathbb{R}_+ \rightarrow M, \quad \text{Exp}(\lambda, t) = q_t.$$

The main result of work [7] is an upper bound on cut time

$$t_{\text{cut}} = \sup\{t_1 > 0 \mid q_s \text{ is optimal for } s \in [0, t_1]\}$$

along extremal trajectories q_s . It is based on the fact that a sub-Riemannian geodesic cannot be optimal after a Maxwell point, i.e., a point where two distinct geodesics of equal sub-Riemannian length meet one another. A natural idea is to look for Maxwell points corresponding to discrete symmetries of the exponential mapping. For each extremal trajectory $q_s = \text{Exp}(\lambda, s)$, we described Maxwell times $t_{\varepsilon^i}^n(\lambda)$, $i = 1, \dots, 7$, $n = 1, 2, \dots$, corresponding to discrete symmetries ε^i . The following upper bound was proved in work [7]:

$$t_{\text{cut}}(\lambda) \leq \mathbf{t}(\lambda), \quad \lambda \in C, \quad (1.13)$$

where $\mathbf{t}(\lambda) = \min(t_{\varepsilon_i}^1(\lambda))$ is the first Maxwell time corresponding to the group of symmetries G . We recall the explicit definition of the function $\mathbf{t}(\lambda)$ below in Eqs. (2.1)–(2.5).

In work [10], the local optimality of sub-Riemannian geodesics was completely characterized. Extremal trajectories corresponding to oscillating pendulum (i.e., to $\lambda \in C_1$) do not have conjugate points, thus they are locally optimal forever. In the case of rotating pendulum ($\lambda \in C_2$) the first conjugate time is bounded from below and from above by the first Maxwell times $t_{\varepsilon_2}^1$ and $t_{\varepsilon_5}^1$ respectively. For critical values of energy of the pendulum, there are no conjugate points. As a consequence, the following bound was proved in Th. 2.5 [10]:

$$\mathbf{t}(\lambda) \leq t_1^{\text{conj}}(\lambda), \quad \lambda \in C. \quad (1.14)$$

Also, in work [10] the global optimality of geodesics was studied. We constructed open dense domains in preimage and image of exponential mapping and proved that the exponential mapping transform these strata diffeomorphically. As a consequence, we showed that inequality (1.13) is in fact an equality.

In this work we obtain our further results for problem (1.1)–(1.5). We consider in detail the action of the exponential mapping at the boundary of the 3-dimensional diffeomorphic domains. This boundary is decomposed into smooth strata of dimension 2, 1, 0 so that restriction of exponential mapping to these strata is a diffeomorphism (Section 2). These results provide a detailed description of the global structure of the exponential mapping (Th. 3.1). The optimal synthesis is constructed in Th. 3.2.

In Theorems 3.4, 3.5 we characterize the global structure of the Maxwell set (the set of points q_1 connected by more than one optimal trajectory with the initial point q_0), and the cut locus (the set of points where extremal trajectories lose optimality). For each point of the Maxwell set there are exactly two optimal trajectories. The cut locus has 3 connected components $\text{Cut}_{\text{loc}}^+$, $\text{Cut}_{\text{loc}}^-$, and Cut_{glob} . The initial point q_0 is contained in the closure of the local components $\text{Cut}_{\text{loc}}^\pm$, and is separated from the global component Cut_{glob} . The global component admits a simple description:

$$\text{Cut}_{\text{glob}} = \{q = (x, y, \theta) \in M \mid \theta = \pi\},$$

while the local components are subsets of the Moebius strip:

$$\text{Cut}_{\text{loc}}^\pm \subset \{q = (x, y, \theta) \in M \mid x \cos(\theta/2) = y \sin(\theta/2) = 0\}$$

defined by some inequalities, see Th. 3.5. Embedding of the cut locus in the solid torus is shown at Fig. 18.

In Section 4 we present explicit optimal solutions for special boundary conditions.

2. STRUCTURE OF EXPONENTIAL MAPPING AT THE BOUNDARY OF OPEN STRATA

We recall some more definitions and notation introduced in the previous works [7, 10]. The function $\mathbf{t} : C \rightarrow (0, +\infty]$ on the phase cylinder of pendulum $(2S_\gamma^1) \times \mathbb{R}_c = C = \cup_{i=1}^5 C_i$ that evaluates the cut time along sub-Riemannian geodesics $\text{Exp}(\lambda, t)$, $t \in C$, is defined as follows:

$$\lambda \in C_1 \quad \Rightarrow \quad \mathbf{t}(\lambda) = 2K(k), \quad (2.1)$$

$$\lambda \in C_2 \quad \Rightarrow \quad \mathbf{t}(\lambda) = 2kp_1^1(k), \quad (2.2)$$

$$\lambda \in C_3 \quad \Rightarrow \quad \mathbf{t}(\lambda) = +\infty, \quad (2.3)$$

$$\lambda \in C_4 \quad \Rightarrow \quad \mathbf{t}(\lambda) = \pi, \quad (2.4)$$

$$\lambda \in C_5 \quad \Rightarrow \quad \mathbf{t}(\lambda) = +\infty, \quad (2.5)$$

where $p_1^1(k)$ is the first positive root of the equation $\operatorname{cn} p (\operatorname{E}(p) - p) - \operatorname{dn} p \operatorname{sn} p = 0$. Here and below we use Jacobi's functions cn , sn , dn , E , and the complete elliptic integral of the first kind K [12]. Further,

$$\begin{aligned}\widehat{M} &= M \setminus \{q_0\}, \\ \widehat{N} &= \{(\lambda, t) \in N \mid t \leq \mathbf{t}(\lambda)\}, \\ N_i &= C_i \times \mathbb{R}_+, \quad i = 1, \dots, 5, \\ \widetilde{N} &= \{(\lambda, t) \in \cup_{i=1}^3 N_i \mid t < \mathbf{t}(\lambda), \operatorname{sn} \tau \operatorname{cn} \tau \neq 0\}, \\ N' &= \{(\lambda, t) \in \cup_{i=1}^3 N_i \mid t = \mathbf{t}(\lambda) \text{ or } \operatorname{sn} \tau \operatorname{cn} \tau = 0\} \cup \widehat{N}_4 \cup N_5, \\ \widehat{N}_4 &= \widehat{N} \cap N_4, \\ \widetilde{M} &= \{q \in M \mid R_1(q)R_2(q) \sin \theta \neq 0\}, \\ M' &= \{q \in M \mid R_1(q)R_2(q) \sin \theta = 0\},\end{aligned}$$

where

$$R_1 = y \cos \frac{\theta}{2} - x \sin \frac{\theta}{2}, \quad R_2 = x \cos \frac{\theta}{2} + y \sin \frac{\theta}{2}.$$

Along with the coordinates (k, φ, t) , we use in the domains N_1, N_2, N_3 also the coordinates (k, p, τ) :

$$\begin{aligned}(\lambda, t) \in N_1 \cup N_3 &\Rightarrow \tau = (2\varphi + t)/2, & p = t/2, \\ (\lambda, t) \in N_2 &\Rightarrow \tau = (2\varphi + t)/(2k), & p = t/(2k).\end{aligned}$$

In work [10] we proved that $\operatorname{Exp} : \widetilde{N} \rightarrow \widetilde{M}$ is a diffeomorphism, and that $\operatorname{Exp}(N') \subset M'$. In this section we describe the action of the exponential mapping

$$\operatorname{Exp} : N' = \widehat{N} \setminus \widetilde{N} \rightarrow M' = \widehat{M} \setminus \widetilde{M}.$$

2.1. Decomposition of the set N'

Consider the following subsets of the set N' :

$$N_{\text{cut}} = \{(\lambda, t) \in N \mid t = \mathbf{t}(\lambda)\}, \quad (2.6)$$

$$N_{\text{conj}} = \{(\lambda, t) \in N_2 \mid t = \mathbf{t}(\lambda), \operatorname{sn} \tau = 0\}, \quad (2.7)$$

$$N_{\text{Max}} = N_{\text{cut}} \setminus N_{\text{conj}}, \quad N_{\text{rest}} = N' \setminus N_{\text{cut}}.$$

The meaning of the subscripts in N_{cut} , N_{conj} , N_{Max} , and N_{rest} is the following: we will show that $\operatorname{Exp}(N_{\text{cut}})$ is the cut locus, $\operatorname{Exp}(N_{\text{Max}})$ is the first Maxwell set, $\operatorname{Exp}(N_{\text{conj}})$ is the intersection of the cut locus with the conjugate locus (caustic), and $\operatorname{Exp}(N_{\text{rest}})$ has no special meaning in this problem (so it contains all the rest strata), see Th. 3.4. We have the following decompositions:

$$\widehat{N} = \widetilde{N} \sqcup N', \quad N' = N_{\text{cut}} \sqcup N_{\text{rest}}, \quad N_{\text{cut}} = N_{\text{Max}} \sqcup N_{\text{conj}}, \quad (2.8)$$

here and below we denote by \sqcup the union of disjoint sets.

In order to study the structure of the exponential mapping at the set N' , we need a further decomposition into subsets N'_i , $i = 1, \dots, 58$, defined by Table 1.

| | | | | | | | | |
|-----------|----------|-----------|------------|------------|----------|-----------|------------|------------|
| N'_i | N'_1 | N'_2 | N'_3 | N'_4 | N'_5 | N'_6 | N'_7 | N'_8 |
| λ | C_1^0 | C_1^0 | C_1^0 | C_1^0 | C_1^1 | C_1^1 | C_1^1 | C_1^1 |
| τ | $(0, K)$ | $(K, 2K)$ | $(2K, 3K)$ | $(3K, 4K)$ | $(0, K)$ | $(K, 2K)$ | $(2K, 3K)$ | $(3K, 4K)$ |
| p | K | K | K | K | K | K | K | K |

| | | | | | | | | |
|-----------|------------|-----------|-----------|------------|--------------|-------------|-----------|-----------|
| N'_i | N'_9 | N'_{10} | N'_{11} | N'_{12} | N'_{13} | N'_{14} | N'_{15} | N'_{16} |
| λ | C_2^+ | C_2^+ | C_2^+ | C_2^+ | C_2^- | C_2^- | C_2^- | C_2^- |
| τ | $(3K, 4K)$ | $(0, K)$ | $(K, 2K)$ | $(2K, 3K)$ | $(-3K, -2K)$ | $(-2K, -K)$ | $(-K, 0)$ | $(0, K)$ |
| p | p_1^1 | p_1^1 | p_1^1 | p_1^1 | p_1^1 | p_1^1 | p_1^1 | p_1^1 |

| | | | | | | | | | | | | | | |
|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| N'_i | N'_{17} | N'_{18} | N'_{19} | N'_{20} | N'_{21} | N'_{22} | N'_{23} | N'_{24} | N'_{25} | N'_{26} | N'_{27} | N'_{28} | N'_{29} | N'_{30} |
| λ | C_1^0 | C_1^0 | C_1^0 | C_1^0 | C_1^1 | C_1^1 | C_1^1 | C_1^1 | C_2^+ | C_2^+ | C_2^+ | C_2^+ | C_2^- | C_2^- |
| τ | 0 | K | $2K$ | $3K$ | 0 | K | $2K$ | $3K$ | $3K$ | 0 | K | $2K$ | K | $-2K$ |
| p | K | K | K | K | K | K | K | K | p_1^1 | p_1^1 | p_1^1 | p_1^1 | p_1^1 | p_1^1 |

| | | | | | | | | | | |
|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| N'_i | N'_{31} | N'_{32} | N'_{35} | N'_{36} | N'_{37} | N'_{38} | N'_{39} | N'_{40} | N'_{41} | N'_{42} |
| λ | C_2^- | C_2^- | C_1^0 | C_1^0 | C_1^0 | C_1^0 | C_1^1 | C_1^1 | C_1^1 | C_1^1 |
| τ | $-K$ | 0 | 0 | K | $2K$ | $3K$ | 0 | K | $2K$ | $3K$ |
| p | p_1^1 | p_1^1 | $(0, K)$ | $(0, K)$ | $(0, K)$ | $(0, K)$ | $(0, K)$ | $(0, K)$ | $(0, K)$ | $(0, K)$ |

| | | | | | | | | |
|-----------|----------------|----------------|----------------|----------------|--------------|--------------|--------------|--------------|
| N'_i | N'_{47} | N'_{48} | N'_{49} | N'_{50} | N'_{51} | N'_{52} | N'_{53} | N'_{54} |
| λ | C_3^{0+} | C_3^{0-} | C_3^{1+} | C_3^{1-} | C_2^+ | C_2^+ | C_2^+ | C_2^+ |
| τ | 0 | 0 | 0 | 0 | $3K$ | 0 | K | $2K$ |
| p | $(0, +\infty)$ | $(0, +\infty)$ | $(0, +\infty)$ | $(0, +\infty)$ | $(0, p_1^1)$ | $(0, p_1^1)$ | $(0, p_1^1)$ | $(0, p_1^1)$ |

| | | | | |
|-----------|--------------|--------------|--------------|--------------|
| N'_i | N'_{55} | N'_{56} | N'_{57} | N'_{58} |
| λ | C_2^- | C_2^- | C_2^- | C_2^- |
| τ | K | $-2K$ | $-K$ | 0 |
| p | $(0, p_1^1)$ | $(0, p_1^1)$ | $(0, p_1^1)$ | $(0, p_1^1)$ |

| | | | | | | |
|-----------|-----------|-----------|------------|------------|----------------|----------------|
| N'_i | N'_{33} | N'_{34} | N'_{43} | N'_{44} | N'_{45} | N'_{46} |
| λ | C_4^0 | C_4^1 | C_4^0 | C_4^1 | C_5^0 | C_5^1 |
| t | π | π | $(0, \pi)$ | $(0, \pi)$ | $(0, +\infty)$ | $(0, +\infty)$ |

TABLE 1. Definition of sets N'_i

Images of the projections

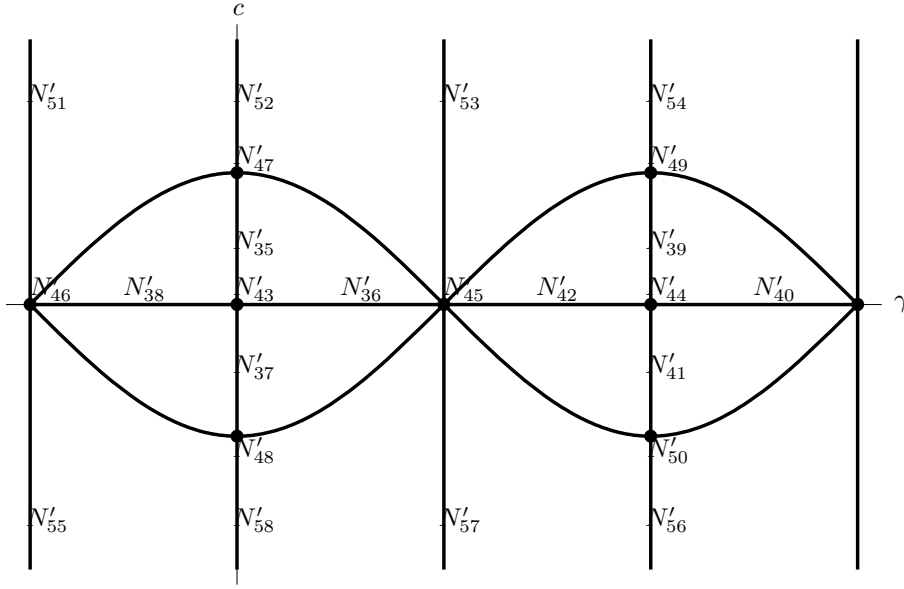
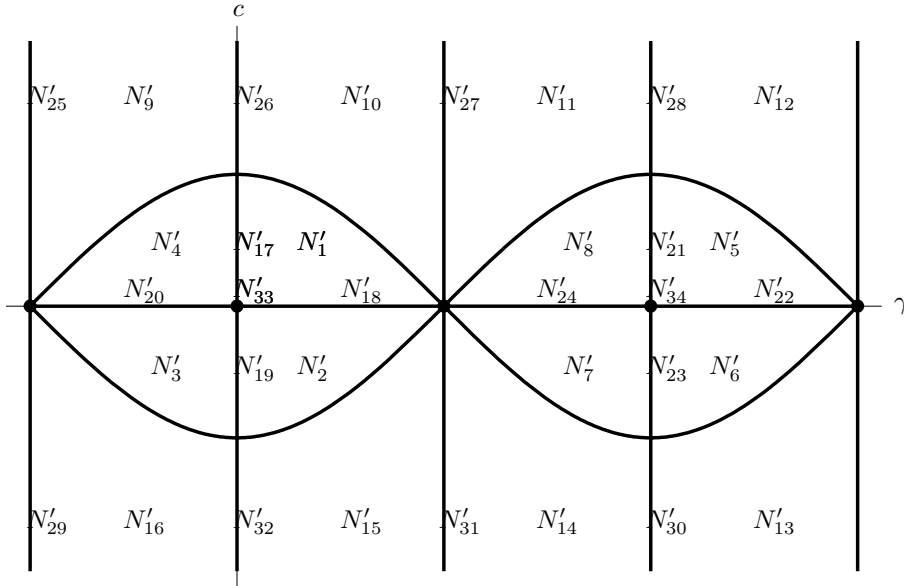
$$N'_i \cap \{t < \mathbf{t}(\lambda), \text{sn } \tau \text{ cn } \tau = 0\} \rightarrow \{p = 0\}, \quad (k, \tau, p) \mapsto (k, \tau, 0),$$

$$N'_i \cap \{t = \mathbf{t}(\lambda)\} \rightarrow \{p = 0\}, \quad (k, \tau, p) \mapsto (k, \tau, 0),$$

are shown respectively at Figs. 2, 3.

Table 1 provides a definition of the sets N'_i ; e.g., the second column of this table means that

$$N'_1 = \{(\lambda, t) \in N \mid \lambda \in C_1^0, \tau \in (0, K), p = K, k \in (0, 1)\}.$$

FIGURE 2. $N'_i \cap \{t < \mathbf{t}(\lambda), \operatorname{sn} \tau \operatorname{cn} \tau = 0\}$ FIGURE 3. $N'_i \cap \{t = \mathbf{t}(\lambda)\}$

Here we use the following decomposition of the sets C_i into connected components:

$$\begin{aligned}
 C_1 &= \cup_{i=0}^1 C_1^i, & C_1^i &= \{(\gamma, c) \in C_1 \mid \operatorname{sgn}(\cos(\gamma/2)) = (-1)^i\}, & i &= 0, 1, \\
 C_2 &= C_2^+ \cup C_2^-, & C_2^\pm &= \{(\gamma, c) \in C_2 \mid \operatorname{sgn} c = \pm 1\}, \\
 C_3 &= \cup_{i=0}^1 (C_3^{i+} \cup C_3^{i-}), \\
 & & C_3^{i\pm} &= \{(\gamma, c) \in C_3 \mid \operatorname{sgn}(\cos(\gamma/2)) = (-1)^i, \operatorname{sgn} c = \pm 1\}, & i &= 0, 1, \\
 C_4 &= \cup_{i=0}^1 C_4^i, & C_4^i &= \{(\gamma, c) \in C \mid \gamma = 2\pi i, c = 0\}, & i &= 0, 1, \\
 C_5 &= \cup_{i=0}^1 C_5^i, & C_5^i &= \{(\gamma, c) \in C \mid \gamma = \pi + 2\pi i, c = 0\}, & i &= 0, 1.
 \end{aligned}$$

Introduce the following index sets for numeration of the subsets N'_i :

$$I = \{1, \dots, 58\}, \quad C = \{1, \dots, 34\}, \quad J = \{26, 28, 30, 32\}, \quad (2.9)$$

$$R = \{35, \dots, 58\}, \quad X = C \setminus J. \quad (2.10)$$

Notice that $I = C \sqcup R$, $J \subset C$.

Lemma 2.1. (1) We have $N'_i \cap N'_j = \emptyset$ for any distinct $i, j \in I$.

(2) There are the following decompositions of subsets of the set N' :

$$N_{\text{cut}} = \cup_{i \in C} N'_i, \quad N_{\text{conj}} = \cup_{i \in J} N'_i, \quad N_{\text{rest}} = \cup_{i \in R} N'_i,$$

thus

$$N_{\text{Max}} = \cup_{i \in X} N'_i, \quad (2.11)$$

$$N' = \sqcup_{i \in I} N'_i. \quad (2.12)$$

Proof. Both statements (1), (2) follow directly from Table 1, definitions of the sets N' , N_{Max} , N_{cut} , N_{conj} , N_{rest} , and decompositions (2.8). \square

2.2. Exponential mapping of the sets N'_{35} , N'_{47} , N'_{26} , N'_{52}

2.2.1. Exponential mapping of the set N'_{35}

In order to describe the image $\text{Exp}(N'_{35})$, we will need the following function:

$$R_1^2(\theta) = 2(\text{artanh}(\sin(\theta/2)) - \sin(\theta/2)), \quad \theta \in [0, \pi]. \quad (2.13)$$

It is obvious that $R_1^2 \in C^\infty[0, \pi)$, $R_1^2(0) = 0$, $R_1^2(\theta) > 0$ for $\theta \in (0, \pi)$, $\lim_{\theta \rightarrow \pi-0} R_1^2(\theta) = +\infty$, and

$$\frac{dR_1^2}{d\theta}(0) = 0. \quad (2.14)$$

A plot of the function $R_1^2(\theta)$ is given at Fig. 4.

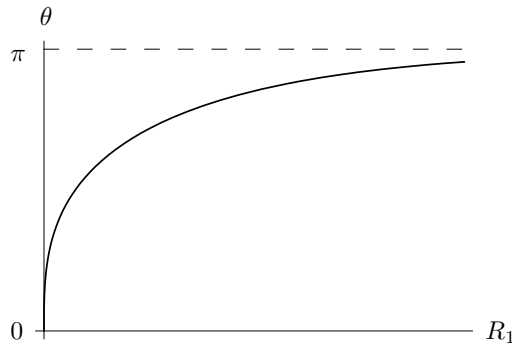


FIGURE 4. Plot of $R_1 = R_1^2(\theta)$

Define the following subset of the set M' , see Fig. 6 and Fig. 14:

$$M'_{35} = \{q \in M \mid \theta \in (\pi, 2\pi), R_1 \in (0, R_1^2(2\pi - \theta)), R_2 = 0\}.$$

Lemma 2.2. *The mapping $\text{Exp} : N'_{35} \rightarrow M'_{35}$ is a diffeomorphism of 2-dimensional manifolds.*

To be more precise, we state that $\text{Exp}(N'_{35}) = M'_{35}$ and $\text{Exp}|_{N'_{35}}$ is a diffeomorphism of the manifold N'_{35} onto the manifold M'_{35} . Below we will write such statements briefly as in Lemma 2.2.

Proof. Formulas (5.2)–(5.6) [7] imply that in the domain N'_{35} we have the following:

$$\sin(\theta/2) = \text{sn } p, \quad \cos(\theta/2) = -\text{cn } p, \quad (2.15)$$

$$R_1 = 2(p - E(p))/k, \quad R_2 = 0. \quad (2.16)$$

By Th. 2.5 [10] (see (1.14)), the restriction $\text{Exp}|_{N'_{35}}$ is nondegenerate. Thus the set $\text{Exp}(N'_{35})$ is an open connected domain in the 2-dimensional manifold

$$S = \{q \in M \mid \theta \in (\pi, 2\pi), R_1 > 0, R_2 = 0\}.$$

On the other hand, the set N'_{35} is an open connected simply connected domain in the 2-dimensional manifold

$$T = \{\nu = (\lambda, t) \in N_1 \mid \tau = 0, p \in (0, K), k \in (0, 1)\}.$$

In the topology of T , we have

$$\begin{aligned} \partial N'_{35} &= \cup_{i=1}^4 n_i, \\ n_1 &= \{\nu \in N_1 \mid \tau = 0, p = 0, k \in [0, 1]\}, \\ n_2 &= \{\nu \in N_1 \mid \tau = 0, p \in [0, \pi/2], k = 0\}, \\ n_3 &= \{\nu \in N_1 \mid \tau = 0, p = K(k), k \in [0, 1]\}, \\ n_4 &= \{\nu \in N_1 \mid \tau = 0, p \in [0, +\infty), k = 1\}, \end{aligned}$$

see Fig. 5.

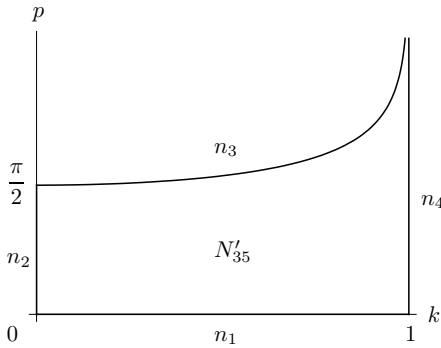


FIGURE
5. Domain N'_{35}

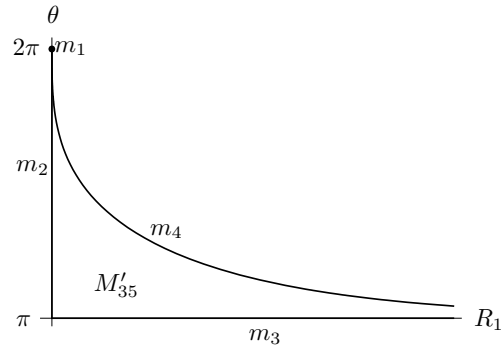


FIGURE
6. Domain M'_{35}

It follows from formulas (2.15), (2.16) that

$$\begin{aligned} \text{Exp}(n_1) &= m_1 = \{q \in M \mid \theta = 2\pi, R_1 = 0, R_2 = 0\}, \\ \text{Exp}(n_2) &= m_2 = \{q \in M \mid \theta \in [\pi, 2\pi], R_1 = 0, R_2 = 0\}, \\ \text{Exp}(n_3) &= m_3 = \{q \in M \mid \theta = \pi, R_1 > 0, R_2 = 0\}, \\ \text{Exp}(n_4) &= m_4 = \{q \in M \mid \theta \in [\pi, 2\pi], R_1 = R_1^2(2\pi - \theta), R_2 = 0\}, \end{aligned}$$

moreover, $\partial M'_{35} = \cup_{i=1}^4 m_i$, see Fig. 6.

Now we show that $\text{Exp}(N'_{35}) \subset M'_{35}$ and $\text{Exp} : N'_{35} \rightarrow M'_{35}$ is a diffeomorphism.

(a) We show that $\text{Exp}(N'_{35}) \cap M'_{35} \neq \emptyset$. Formulas (2.15), (2.16) give the following asymptotics as $k \rightarrow 0$:

$$\theta = 2\pi - 2p + o(1), \quad R_1 = k(p/2 - (\sin 2p)/4) + o(k).$$

There exists (p, k) close to $(\pi/2, 0)$ such that the corresponding point (θ, R_1) is arbitrarily close to $(0, 0)$, with $\theta > 0$, $R_1 > 0$. Thus there exists $\nu \in N'_{35}$ such that $\text{Exp}(\nu) \in M'_{35}$.

(b) We show that $\text{Exp}(N'_{35}) \neq S$. Formulas (2.15), (2.16) yield the following chain:

$$\theta \rightarrow 2\pi - 0 \quad \Rightarrow \quad \text{sn } p \rightarrow 0 \quad \Rightarrow \quad p \rightarrow 0 \quad \Rightarrow \quad R_1 \rightarrow 0.$$

Thus there exists $q \in S \setminus \text{Exp}(N'_{35})$.

(c) We prove that $\text{Exp}(N'_{35}) \subset M'_{35}$. By contradiction, suppose that there exists a point $q_1 \in \text{Exp}(N'_{35}) \setminus M'_{35}$. Since the mapping $\text{Exp}|_{N'_{35}}$ is nondegenerate, we can choose this point such that $q_1 \in \text{Exp}(N'_{35}) \setminus \text{cl}(M'_{35})$.

Choose any point $q_2 \in S \setminus \text{cl}(M'_{35})$. Connect the points q_1, q_2 by a continuous curve in S , and find at this curve a point $q_3 \in S \setminus \text{Exp}(N'_{35})$, $q_3 \notin \text{cl}(M'_{35})$ such that there exists a converging sequence $q^n \rightarrow q_3$, $q^n = \text{Exp}(\nu^n) \in \text{Exp}(N'_{35})$. Further, there exist a subsequence $\nu^{n_i} \in N'_{35}$ converging to a finite or infinite limit. If $\nu^{n_i} \rightarrow \bar{\nu} \in N'_{35}$, then $q_3 = \text{Exp}(\bar{\nu}) \in \text{int } \text{Exp}(N'_{35})$ by nondegeneracy of $\text{Exp}|_{N'_{35}}$, a contradiction. If $\nu^{n_i} \rightarrow \bar{\nu} \in \partial N'_{35}$, then

$$q_3 = \text{Exp}(\bar{\nu}) \in \text{Exp}(\partial N'_{35}) = \partial M'_{35} \subset \text{cl}(M'_{35}),$$

a contradiction. Finally, if $\nu^{n_i} \rightarrow \infty$, then at this sequence $k^{n_i} \rightarrow 1 - 0$, $p^{n_i} \rightarrow \infty$, thus $R_1(q^{n_i}) \rightarrow \infty$, a contradiction.

Consequently, $\text{Exp}(N'_{35}) \subset M'_{35}$.

(d) The mapping $\text{Exp} : N'_{35} \rightarrow M'_{35}$ is a diffeomorphism since $\text{Exp}|_{N'_{35}}$ is nondegenerate and proper, and N'_{35}, M'_{35} are connected and simply connected. \square

2.2.2. Exponential mapping of the set N'_{47}

Define the following subset of M' , see Fig. 14:

$$M'_{47} = \{q \in M \mid \theta \in (\pi, 2\pi), R_1 = R_1^2(2\pi - \theta), R_2 = 0\}.$$

Lemma 2.3. *The mapping $\text{Exp} : N'_{47} \rightarrow M'_{47}$ is a diffeomorphism of 1-dimensional manifolds.*

Proof. We pass to the limit $k \rightarrow 1 - 0$ in formulas (2.15), (2.16) and obtain for $\nu \in N'_{47}$:

$$\sin(\theta/2) = \tanh p, \quad \cos(\theta/2) = -1/\cosh p, \quad R_1 = 2(p - \tanh p), \quad R_2 = 0.$$

This coordinate representation shows that $\text{Exp} : N'_{47} \rightarrow M'_{47}$ is a diffeomorphism. \square

2.2.3. Exponential mapping of the set N'_{26}

Before the study of $\text{Exp}|_{N'_{52}}$, postponed till the next subsection, we need to consider the set N'_{26} contained in the boundary of N'_{52} . In order to parametrize regularly the image $\text{Exp}(N'_{26})$, we introduce the necessary functions.

Recall that the function $p = p_1^1(k)$, $k \in [0, 1)$, is the first positive root of the function $f_1(p) = \text{cn } p (\text{E}(p) - p) - \text{dn } p \text{ sn } p$, see Eq. (5.12) and Cor. 5.1 [7]. Define the function

$$v_1^1(k) = \text{am}(p_1^1(k), k), \quad k \in [0, 1). \quad (2.17)$$

Lemma 2.4. (1) The number $v = v_1^1(k)$ is the first positive root of the function

$$h_1(v, k) = E(v, k) - F(v, k) - \sqrt{1 - k^2 \sin^2 v} \tan v, \quad k \in [0, 1].$$

- (2) $v_1^1 \in C^\infty[0, 1)$.
- (3) $v_1^1(k) \in (\pi/2, \pi)$ for $k \in (0, 1)$; moreover, $v_1^1(0) = \pi$.
- (4) The function $v_1^1(k)$ is strictly decreasing at the segment $k \in [0, 1)$.
- (5) $\lim_{v \rightarrow 1-0} v_1^1(k) = \pi/2$, thus setting $v_1^1(1) = \pi/2$, we obtain $v_1^1 \in C[0, 1]$.
- (6) $v_1^1(k) = \pi - (\pi/2)k^2 + o(k^2)$, $k \rightarrow +0$.

Proof. (1) follows from (2.17) since $p = p_1^1$ is the first positive root of the function $f_1(p)$.

(2) follows since $p_1^1 \in C^\infty[0, 1)$ by Lemma 5.3 [7].

(3) follows since $p_1^1 \in (K, 2K)$ and $p_1^1(0) = \pi$, see Cor. 5.1 [7].

(4) We have for $v \in (\pi/2, \pi]$:

$$\begin{aligned} \frac{\partial h_1}{\partial v} &= -\sqrt{1 - k^2 \sin^2 v} / \cos^2 v < 0, \\ \frac{\partial h_1}{\partial k} &= -\frac{k}{1 - k^2} (E(v, k) - \sqrt{1 - k^2 \sin^2 v} \tan v) < 0. \end{aligned}$$

Thus $\frac{dv_1^1}{dk} = -\frac{\partial h_1 / \partial k}{\partial h_1 / \partial v} < 0$ for $k \in [0, 1)$.

(5) Monotonicity and boundedness of $v_1^1(k)$ imply that there exists a limit $\lim_{k \rightarrow 1-0} v_1^1(k) = \bar{v} \in [\pi/2, \pi)$. If $\bar{v} \in (\pi/2, \pi)$, then as $k \rightarrow 1 - 0$

$$h_1(v_1^1(k), k) \rightarrow \int_0^{\bar{v}} (|\cos t| - 1/|\cos t|) dt - \sqrt{1 - \sin^2 \bar{v}} \tan \bar{v} = \infty,$$

which contradicts the identity $h_1(v_1^1(k), k) \equiv 0$, $k \in [0, 1)$. Thus $\bar{v} = \pi/2$.

(6) As $(k, v) \rightarrow (0, \pi)$, we have $h_1(v, k) = v - \pi + (\pi/2)k^2 + o(k^2 + (v - \pi)^2)$, thus $v_1^1(k) = \pi - (\pi/2)k^2 + o(k^2)$, $k \rightarrow +0$. \square

A plot of the function $v_1^1(k)$ is given at Fig. 7.

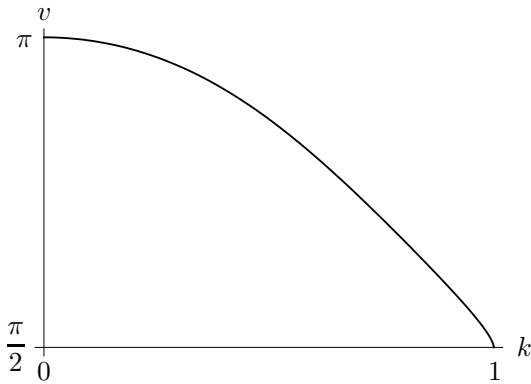


FIGURE 7. Plot of $v = v_1^1(k)$

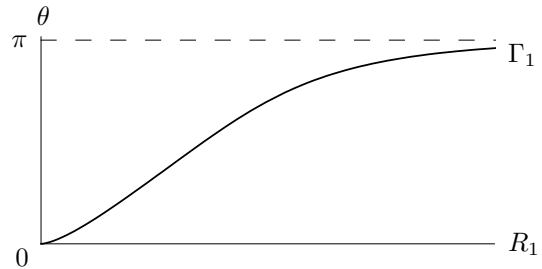


FIGURE 8. The curve Γ_1

Define the curve $\Gamma_1 \subset S = \{q \in M \mid \theta \in (0, \pi), R_1 > 0, R_2 = 0\}$ given parametrically as follows:

$$\theta = 2 \arcsin(k \sin v_1^1(k)), \quad (2.18)$$

$$R_1 = 2(F(v_1^1(k), k) - E(v_1^1(k), k)), \quad k \in [0, 1], \quad (2.19)$$

see Fig. 8.

- Lemma 2.5.** (1) *The function $k \sin v_1^1(k)$ is strictly increasing as $k \in [0, 1]$, thus the function $\theta = \theta(k)$, $k \in [0, 1]$, determined by (2.18) has an inverse function $k = k_1^1(\theta)$, $\theta \in [0, \pi]$.*
(2) $k_1^1 \in C[0, \pi] \cap C^\infty[0, \pi]$.
(3) *The function $k_1^1(\theta)$ is strictly increasing as $\theta \in [0, \pi]$.*
(4) *The curve Γ_1 is a graph of the function*

$$\begin{aligned} R_1 &= R_1^1(\theta), \quad \theta \in [0, \pi], \\ R_1^1(\theta) &= 2(F(v_1^1(k), k) - E(v_1^1(k), k)), \quad k = k_1^1(\theta). \end{aligned} \quad (2.20)$$

- (5) $R_1^1 \in C[0, \pi] \cap C^\infty(0, \pi)$.
(6) $R_1^1(\theta) = \sqrt[3]{\pi}/2 \theta^{2/3} + o(\theta^{2/3})$, $\theta \rightarrow +0$.

Proof. (1) As $k \in [0, 1]$, we have:

$$\begin{aligned} v_1^1(k) &\downarrow, \quad v_1^1(k) \in [\pi/2, \pi], \\ \sin v_1^1(k) &\uparrow, \quad k \sin v_1^1(k) \uparrow, \quad 2 \arcsin(kv_1^1(k)) \uparrow. \end{aligned}$$

- (2) follows from items (2), (5) of Lemma 2.4.
(3) follows from item (1) of this lemma.
(4) follows from (2.18), (2.19).
(5) follows from item (2) of this lemma.
(6) As $k \rightarrow +0$, we have

$$v_1^1(k) = \pi - (\pi/2)k^2 + o(k^2), \quad \sin v_1^1(k) = (\pi/2)k^2 + o(k^2),$$

and for the functions (2.18), (2.19)

$$\theta = \pi k^3 + o(k^3), \quad R_1 = (\pi/2)k^2 + o(k^2).$$

Thus as $\theta \rightarrow +0$, we have

$$k_1^1(\theta) = \sqrt[3]{\theta/\pi} + o(\sqrt[3]{\theta}), \quad R_1^1(\theta) = \sqrt[3]{\pi}/2 \theta^{2/3} + o(\theta^{2/3}).$$

□

Define the following subset of M' , see Fig. 14:

$$M'_{26} = \{q \in M \mid \theta \in (\pi, 2\pi), R_1 = R_1^1(2\pi - \theta), R_2 = 0\}.$$

Lemma 2.6. *The mapping $\text{Exp} : N'_{26} \rightarrow M'_{26}$ is a diffeomorphism of 1-dimensional manifolds.*

Proof. For $\nu \in N'_{26}$ we obtain from formulas (5.7)–(5.12) [7]:

$$\begin{aligned}\sin(\theta/2) &= k \operatorname{sn} p_1^1(k) = k \sin v_1^1(k), \\ \cos(\theta/2) &= -\operatorname{dn} p_1^1(k) = -\sqrt{1 - k^2 \sin^2 v_1^1(k)}, \\ R_1 &= 2(p_1^1(k) - E(p_1^1(k))) = 2(F(v_1^1(k), k) - E(v_1^1(k), k)), \\ R_2 &= 0.\end{aligned}$$

Thus $\operatorname{Exp}(N'_{26}) = M'_{26}$. Moreover, the mapping $\operatorname{Exp} : N'_{26} \rightarrow M'_{26}$ decomposes into the chain

$$\begin{aligned}N'_{26} &\xrightarrow{(*)} \Gamma_1 \xrightarrow{(**)} M'_{26}, \\ (*) &: k \mapsto (\theta = 2 \arcsin(k \sin v_1^1(k)), R_1 = 2(F(v_1^1(k), k) - E(v_1^1(k), k)), R_2 = 0), \\ (**) &: (\theta, R_1, R_2) \mapsto (2\pi - \theta, R_1, R_2).\end{aligned}$$

The mapping $(*)$ is a diffeomorphism by Lemma 2.5. Thus $\operatorname{Exp} : N'_{26} \rightarrow M'_{26}$ is a diffeomorphism. \square

2.2.4. Exponential mapping of the set N'_{52}

Lemma 2.7. (1) *The functions $R_1^1(\theta)$, $R_1^2(\theta)$ defined in (2.20), (2.13) satisfy the inequality*

$$R_1^2(\theta) < R_1^1(\theta), \quad \theta \in (0, \pi).$$

(2) *The mapping $\operatorname{Exp} : N'_{52} \rightarrow M'_{52}$ is a diffeomorphism of 2-dimensional manifolds, where*

$$M'_{52} = \{q \in M \mid \theta \in (\pi, 2\pi), R_1 \in (R_1^2(2\pi - \theta), R_1^1(2\pi - \theta)), R_2 = 0\},$$

see Fig. 14.

Proof. We have

$$N'_{52} = \{\nu \in N_2^+ \mid \tau = 0, v \in (0, v_1^1(k)), k \in (0, 1)\},$$

where $v = \operatorname{am}(p, k)$. Thus

$$N'_{52} \subset T = \{\nu \in N_2^+ \mid \tau = 0, v \in [0, \pi], k \in [0, 1]\},$$

and in the 2-dimensional topology of T

$$\begin{aligned}\partial N'_{52} &= \cup_{i=1}^4 n_i, \\ n_1 &= \{\nu \in N_2^+ \mid \tau = 0, v = 0, k \in [0, 1]\}, \\ n_2 &= \{\nu \in N_2^+ \mid \tau = 0, v \in [0, \pi], k = 0\}, \\ n_3 &= \{\nu \in N_2^+ \mid \tau = 0, v = v_1^1(k), k \in [0, 1]\}, \\ n_4 &= \{\nu \in N_2^+ \mid \tau = 0, v \in [0, \pi/2], k = 1\},\end{aligned}$$

see Fig. 9.

By formulas (5.7)–(5.12) [7], the exponential mapping in the domain N'_{52} reads as follows:

$$\begin{aligned}\sin(\theta/2) &= k \sin v, & \cos(\theta/2) &= -\sqrt{1 - k^2 \sin^2 v}, \\ R_1 &= 2(F(v, k) - E(v, k)), & R_2 &= 0.\end{aligned}$$

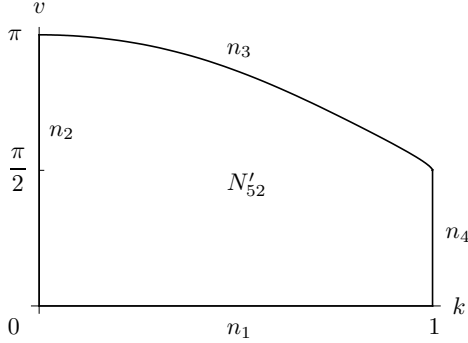


FIGURE
9. Domain N'_{52}

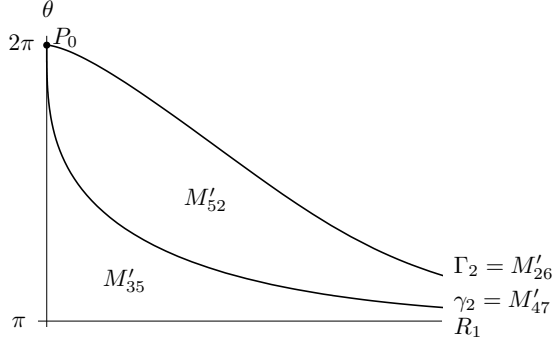


FIGURE
10. Curves γ_2
and Γ_2

Thus

$$\begin{aligned} \text{Exp}(N'_{52}) &\subset S = \{q \in M \mid \theta \in (\pi, 2\pi), R_1 > 0, R_2 = 0\}, \\ \text{Exp}(n_1) = \text{Exp}(n_2) = P_0 &= \{q \in M \mid \theta = 2\pi, R_1 = 0, R_2 = 0\}, \\ \text{Exp}(n_3) = \overline{M'_{26}} = \overline{\Gamma_2}, &\quad \Gamma_2 := M'_{26}, \\ \text{Exp}(n_4) = \overline{M'_{47}} = \overline{\gamma_2}, &\quad \gamma_2 := M'_{47}. \end{aligned}$$

By Th. 2.5 [10] (see (1.14)), the mapping $\text{Exp}|_{N'_{52}}$ is nondegenerate, thus $\text{Exp}(N'_{52})$ is an open connected domain in S , with $\partial \text{Exp}(N'_{52}) \subset \text{Exp}(\partial N'_{52}) = \overline{\Gamma_2} \cup \overline{\gamma_2}$.

The curves $\overline{\Gamma_2}$ and $\overline{\gamma_2}$ intersect one another at the point P_0 . We show that they have no other intersection points. By contradiction, assume that the curves $\overline{\Gamma_2}$ and $\overline{\gamma_2}$ have intersection points distinct from P_0 , then the domain $\text{Exp}(N'_{52})$ is bounded by finite arcs of the curves $\overline{\Gamma_2}$ and $\overline{\gamma_2}$, i.e., there exists a point $P_1 \in \gamma_2 \cap \Gamma_2$, $P_1 \neq P_0$, such that $\partial \text{Exp}(N'_{52}) = P_0 \gamma_2 P_1 \cup P_0 \Gamma_2 P_1$. Then $\overline{\text{Exp}(N'_{52})}$ does not contain the curves γ_2, Γ_2 . This is a contradiction to the diffeomorphic property of the mapping $\text{Exp} : n_3 = N'_{26} \rightarrow \Gamma_2 = M'_{26}$ and $\text{Exp} : n_4 = N'_{47} \rightarrow \gamma_2 = M'_{47}$, see Lemmas 2.6 and 2.3 respectively.

Consequently, $\overline{\gamma_2} \cap \overline{\Gamma_2} = P_0$, and the domain $\text{Exp}(N'_{52})$ is bounded by the curves $\overline{\gamma_2}, \overline{\Gamma_2}$.

The equalities $\frac{dR_1^1}{d\theta}(0) = +\infty$, $\frac{dR_1^2}{d\theta}(0) = 0$ (see Lemma 2.5 and Eq. (2.14)) imply that $R_1^2(\theta) < R_1^1(\theta)$ for sufficiently small $\theta > 0$. Further, the representations

$$\begin{aligned} \Gamma_2 = M'_{26} &= \{q \in M \mid \theta \in (\pi, 2\pi), R_1 = R_1^1(2\pi - \theta), R_2 = 0\}, \\ \gamma_2 = M'_{47} &= \{q \in M \mid \theta \in (\pi, 2\pi), R_1 = R_1^2(2\pi - \theta), R_2 = 0\} \end{aligned}$$

imply the required inequality

$$R_1^2(\theta) < R_1^1(\theta), \quad \theta \in (0, \pi),$$

and the equality $\text{Exp}(N'_{52}) = M'_{52}$.

Since the mapping $\text{Exp} : N'_{52} \rightarrow M'_{52}$ is nondegenerate and proper, and the domains N'_{52}, M'_{52} are open, connected, and simply connected, it follows that this mapping is a diffeomorphism. \square

The mutual disposition of the curves $\gamma_2 = M'_{47}$ and $\Gamma_2 = M'_{26}$ is shown at Fig. 10 and Fig. 14.

| | | | | | | |
|----------|----------------|----------------|----------------|----------------|-----------------------------------|-----------------------------|
| M'_i | $M'_1 = M'_6$ | $M'_2 = M'_5$ | $M'_3 = M'_8$ | $M'_4 = M'_7$ | $M'_9 = M'_{10}$ | $M'_{11} = M'_{12}$ |
| θ | π | π | π | π | $(\pi, 2\pi)$ | $(0, \pi)$ |
| R_1 | $(0, +\infty)$ | $(-\infty, 0)$ | $(-\infty, 0)$ | $(0, +\infty)$ | $(R_1^1(2\pi - \theta), +\infty)$ | $(-\infty, -R_1^1(\theta))$ |
| R_2 | $(-\infty, 0)$ | $(-\infty, 0)$ | $(0, +\infty)$ | $(0, +\infty)$ | 0 | 0 |

| | | | | | |
|----------|----------------------------|------------------------------------|---------------------|---------------------|---------------------|
| M'_i | $M'_{13} = M'_{14}$ | $M'_{15} = M'_{16}$ | $M'_{17} = M'_{23}$ | $M'_{18} = M'_{22}$ | $M'_{19} = M'_{21}$ |
| θ | $(0, \pi)$ | $(\pi, 2\pi)$ | π | π | π |
| R_1 | $(R_1^1(\theta), +\infty)$ | $(-\infty, -R_1^1(2\pi - \theta))$ | $(0, +\infty)$ | 0 | $(-\infty, 0)$ |
| R_2 | 0 | 0 | 0 | $(-\infty, 0)$ | 0 |

| | | | | | | | |
|----------|---------------------|---------------------|------------------------|------------------|---------------------|-----------------|-------------------------|
| M'_i | $M'_{20} = M'_{24}$ | $M'_{25} = M'_{27}$ | M'_{26} | M'_{28} | $M'_{29} = M'_{31}$ | M'_{30} | M'_{32} |
| θ | π | 0 | $(\pi, 2\pi)$ | $(0, \pi)$ | 0 | $(0, \pi)$ | $(\pi, 2\pi)$ |
| R_1 | 0 | $(-\infty, 0)$ | $R_1^1(2\pi - \theta)$ | $-R_1^1(\theta)$ | $(0, +\infty)$ | $R_1^1(\theta)$ | $-R_1^1(2\pi - \theta)$ |
| R_2 | $(0, +\infty)$ | 0 | 0 | 0 | 0 | 0 | 0 |

| | | | | | | |
|----------|---------------------|-----------------------------|----------------|------------------------------|----------------|-----------------------|
| M'_i | $M'_{33} = M'_{34}$ | M'_{35} | M'_{36} | M'_{37} | M'_{38} | M'_{39} |
| θ | π | $(\pi, 2\pi)$ | $(\pi, 2\pi)$ | $(\pi, 2\pi)$ | $(\pi, 2\pi)$ | $(0, \pi)$ |
| R_1 | 0 | $(0, R_1^1(2\pi - \theta))$ | 0 | $(-R_1^1(2\pi - \theta), 0)$ | 0 | $(-R_1^2(\theta), 0)$ |
| R_2 | 0 | 0 | $(-\infty, 0)$ | 0 | $(0, +\infty)$ | 0 |

| | | | | | | | | |
|----------|----------------|----------------------|----------------|---------------|------------|----------------|----------------|------------------------|
| M'_i | M'_{40} | M'_{41} | M'_{42} | M'_{43} | M'_{44} | M'_{45} | M'_{46} | M'_{47} |
| θ | $(0, \pi)$ | $(0, \pi)$ | $(0, \pi)$ | $(\pi, 2\pi)$ | $(0, \pi)$ | 0 | 0 | $(\pi, 2\pi)$ |
| R_1 | 0 | $(0, R_1^2(\theta))$ | 0 | 0 | 0 | 0 | 0 | $R_1^2(2\pi - \theta)$ |
| R_2 | $(-\infty, 0)$ | 0 | $(0, +\infty)$ | 0 | 0 | $(0, +\infty)$ | $(-\infty, 0)$ | 0 |

| | | | | | | |
|----------|-------------------------|------------------|-----------------|----------------|--|----------------|
| M'_i | M'_{48} | M'_{49} | M'_{50} | M'_{51} | M'_{52} | M'_{53} |
| θ | $(\pi, 2\pi)$ | $(0, \pi)$ | $(0, \pi)$ | 0 | $(\pi, 2\pi)$ | 0 |
| R_1 | $-R_1^2(2\pi - \theta)$ | $-R_1^1(\theta)$ | $R_1^2(\theta)$ | $(-\infty, 0)$ | $(R_1^2(2\pi - \theta), R_1^1(2\pi - \theta))$ | $(-\infty, 0)$ |
| R_2 | 0 | 0 | 0 | $(-\infty, 0)$ | 0 | $(0, +\infty)$ |

| | | | | | |
|----------|------------------------------------|----------------|----------------------------------|----------------|--|
| M'_i | M'_{54} | M'_{55} | M'_{56} | M'_{57} | M'_{58} |
| θ | $(0, \pi)$ | 0 | $(0, \pi)$ | 0 | $(\pi, 2\pi)$ |
| R_1 | $(-R_1^1(\theta), -R_1^2(\theta))$ | $(0, +\infty)$ | $(R_1^2(\theta), R_1^1(\theta))$ | $(0, +\infty)$ | $(-R_1^1(2\pi - \theta), -R_1^2(2\pi - \theta))$ |
| R_2 | 0 | $(-\infty, 0)$ | 0 | $(0, +\infty)$ | 0 |

TABLE 2. Definition of sets M'_i

2.3. Decomposition of the set M'

Now we have the functions $R_1^2(\theta) < R_1^1(\theta)$ required for definition of the following decomposition:

$$M' = \cup_{i=1}^{58} M'_i, \quad (2.21)$$

where the subsets M'_i are defined by Table 2. Notice that some of the sets M'_i coincide between themselves, unlike the sets N'_i , see (2.12). A precise definition of coinciding M'_i is given below in Th. 3.1, item (1).

The structure of decomposition (2.21) in the surfaces $\{\theta = 0\}$, $\{\theta = \pi\}$, $\{R_1 = 0\}$, $\{R_2 = 0\}$ is shown respectively at Figs. 11, 12, 13, 14.

2.4. Exponential mapping of the sets N'_{36} , N'_{53} , N'_{18} , N'_{33} , N'_{34} , N'_{17} , N'_{27} , N'_1 , N'_{10}

2.4.1. Exponential mapping of the set N'_{36}

Lemma 2.8. *The mapping $\text{Exp} : N'_{36} \rightarrow M'_{36}$ is a diffeomorphism of 2-dimensional manifolds.*

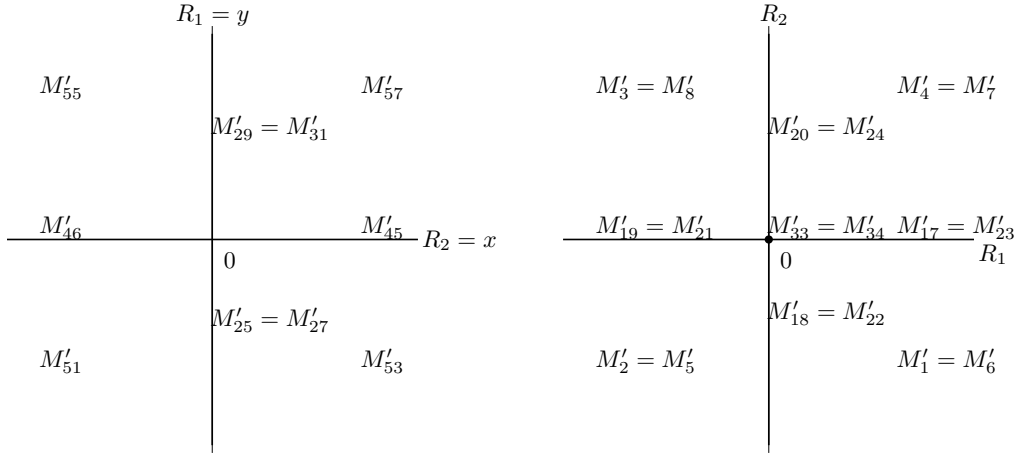


FIGURE 11. Decomposition of surface $\{\theta = 0\}$

FIGURE 12. Decomposition of surface $\{\theta = \pi\}$

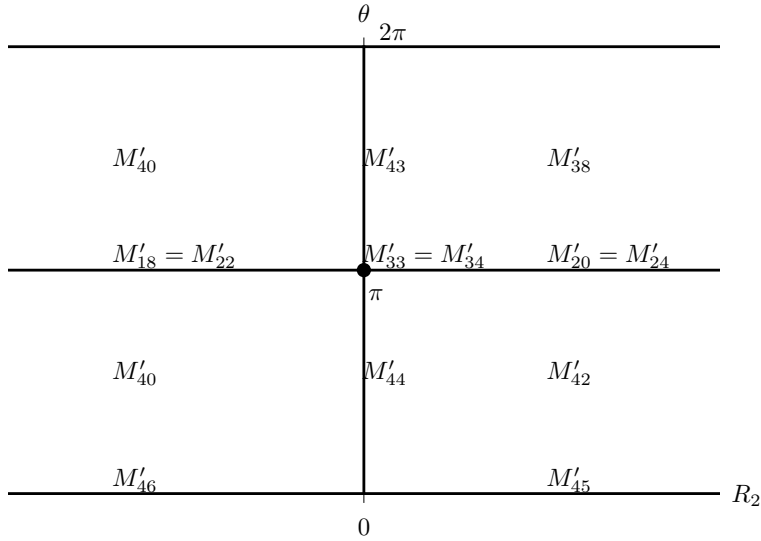
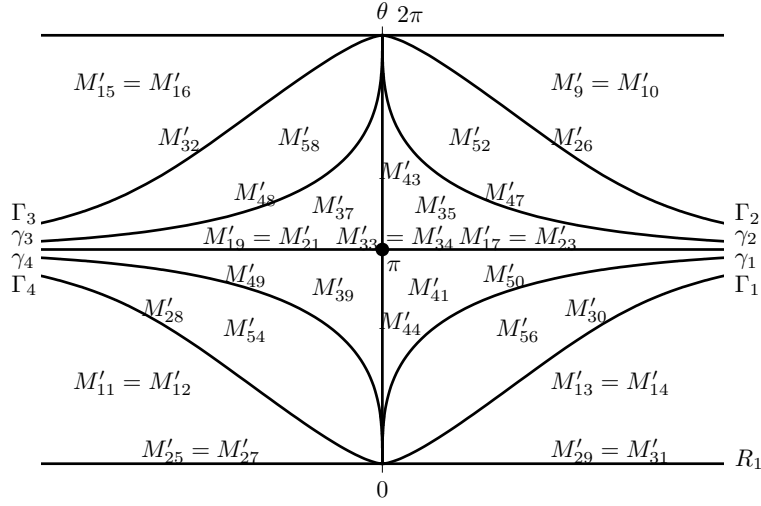


FIGURE 13. Decomposition of surface $\{R_1 = 0\}$

Proof. By formulas (5.2)–(5.6) [7], exponential mapping in the domain N'_{36} reads as follows:

$$\begin{aligned} \sin(\theta/2) &= \sqrt{1 - k^2} \operatorname{sn} p / \operatorname{dn} p, & \cos(\theta/2) &= -\operatorname{cn} p / \operatorname{dn} p, \\ R_1 &= 0, & R_2 &= -2f_2(p, k) / (k \operatorname{dn} p), \end{aligned}$$

where $f_2(p, k) = k^2 \operatorname{cn} p \operatorname{sn} p + \operatorname{dn} p (p - E(p)) > 0$ by Lemma 5.2 [7], thus $\operatorname{Exp}(N'_{36}) \subset M'_{36}$.

FIGURE 14. Decomposition of surface $\{R_2 = 0\}$

In the topology of the manifold $\{R_1 = 0\}$, we have:

$$\begin{aligned} \partial N'_{36} &= \cup_{i=1}^4 n_i, \\ n_1 &= \{\nu \in N_1^0 \mid \tau = K, p = 0, k \in [0, 1]\}, \\ n_2 &= \{\nu \in N_1^0 \mid \tau = K, p \in [0, \pi/2], k = 0\}, \\ n_3 &= \{\nu \in N_1^0 \mid \tau = K, p = K, k \in [0, 1]\}, \\ n_4 &= \{\nu \in N_1^0 \mid \tau = K, p \in [0, +\infty), k = 1\}, \end{aligned}$$

see Fig. 15.

Further, we have $\text{Exp}(n_i) = m_i$, $i = 1, \dots, 4$, where

$$\begin{aligned} m_1 &= \{q \in M \mid \theta = 2\pi, R_1 = 0, R_2 = 0\}, \\ m_2 &= \{q \in M \mid \theta \in [\pi, 2\pi], R_1 = 0, R_2 = 0\}, \\ m_3 &= \{q \in M \mid \theta = \pi, R_1 = 0, R_2 \in (-\infty, 0]\}, \\ m_4 &= \{q \in M \mid \theta = 2\pi, R_1 = 0, R_2 \in (-\infty, 0]\}, \end{aligned}$$

see Fig. 16.

The mapping $\text{Exp} : N'_{36} \rightarrow M'_{36}$ is nondegenerate and proper, the domains N'_{36} , M'_{36} are open (in the 2-dimensional topology), connected and simply connected, thus it is a diffeomorphism. \square

2.4.2. Exponential mapping of the set N'_{53}

Lemma 2.9. *The mapping $\text{Exp} : N'_{53} \rightarrow M'_{53}$ is a diffeomorphism of 2-dimensional manifolds.*

Proof. The argument follows similarly to the proof of Lemma 2.8 via the following coordinate representation of the exponential mapping in the domain N'_{53} :

$$\theta = 0, \quad R_1 = -2\sqrt{1 - k^2}(p - E(p))/\text{dn } p < 0, \quad R_2 = -2kf_1(p, k)/\text{dn } p,$$

where $f_1(p, k) = \text{cn } p(E(p) - p) - \text{dn } p \text{sn } p < 0$ for $p \in (0, p_1^1)$, see Cor. 5.1 [7]. \square

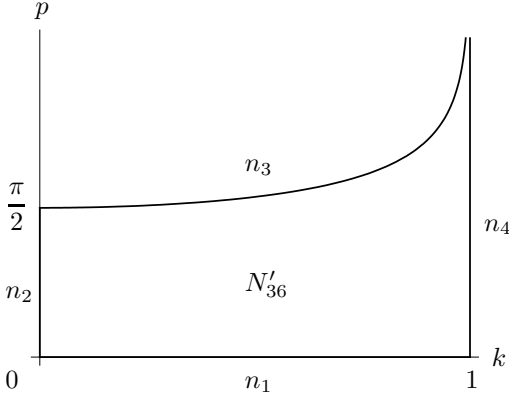


FIGURE
15. Domain N'_{36}

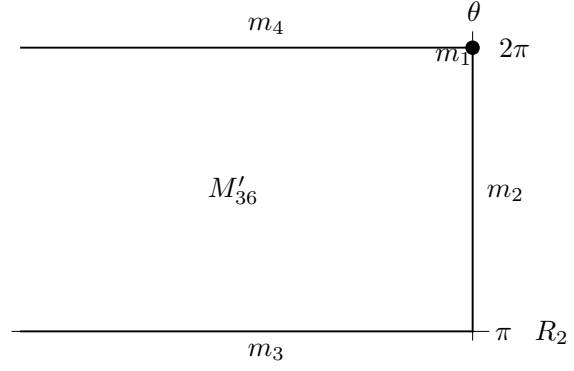


FIGURE
16. Domain M'_{36}

2.4.3. Exponential mapping of the set N'_{18}

Lemma 2.10. *The mapping $\text{Exp} : N'_{18} \rightarrow M'_{18}$ is a diffeomorphism of 1-dimensional manifolds.*

Proof. By formulas (5.2)–(5.6) [7], we have in the set N'_{18} :

$$\theta = \pi, \quad R_1 = 0, \quad R_2 = -(2/k)(K(k) - E(k)),$$

and the diffeomorphic property of $\text{Exp}|_{N'_{18}}$ follows as usual from its nondegeneracy and properness, and topological properties of the sets N'_{18} , M'_{18} . \square

2.4.4. Exponential mapping of the sets N'_{33} , N'_{34}

Lemma 2.11. *The mappings $\text{Exp} : N'_{33} \rightarrow M'_{33}$ and $\text{Exp} : N'_{34} \rightarrow M'_{34}$ are diffeomorphisms (bijections) of 0-dimensional manifolds.*

Proof. Obvious. \square

2.4.5. Exponential mapping of the set N'_{17}

Lemma 2.12. *The mapping $\text{Exp} : N'_{17} \rightarrow M'_{17}$ is a diffeomorphism of 1-dimensional manifolds.*

Proof. The statement follows as in the proof of Lemma 2.10 via the following coordinate representation of the exponential mapping in the domain N'_{17} :

$$\theta = \pi, \quad R_1 = (2/k)(K(k) - E(k)), \quad R_2 = 0.$$

\square

2.4.6. Exponential mapping of the set N'_{27}

Lemma 2.13. *The mapping $\text{Exp} : N'_{27} \rightarrow M'_{27}$ is a diffeomorphism of 2-dimensional manifolds.*

Proof. Formulas (5.7)–(5.12) [7] yield:

$$\theta = \pi, \quad R_1 = -2\sqrt{1-k^2} (p - E(p)) / \text{dn } p|_{p=p_1^1(k)}, \quad R_2 = 0.$$

Since $\tau = K$, then Lemma 2.4 [10] implies that the mapping $\text{Exp}|_{N'_{27}}$ is nondegenerate. Then the diffeomorphic property of $\text{Exp} : N'_{27} \rightarrow M'_{27}$ follows as usual. \square

2.4.7. *Exponential mapping of the set N'_1*

Lemma 2.14. *The mapping $\text{Exp} : N'_1 \rightarrow M'_1$ is a diffeomorphism of 2-dimensional manifolds.*

Proof. Formulas (5.2)–(5.6) [7] yield:

$$\begin{aligned}\theta &= \pi, & R_1 &= 2(K(k) - E(k)) \text{cn } \tau / (k \text{dn } \tau) > 0, \\ R_2 &= -2\sqrt{1 - k^2}(K(k) - E(k)) \text{sn } \tau / (k \text{dn } \tau) < 0,\end{aligned}$$

and the statement follows as usual since the mapping $\text{Exp}|_{N'_1}$ is nondegenerate and proper. \square

2.4.8. *Exponential mapping of the set N'_{10}*

Lemma 2.15. *The mapping $\text{Exp} : N'_{10} \rightarrow M'_{10}$ is a diffeomorphism of 2-dimensional manifolds.*

Proof. By formulas (5.7)–(5.12) [7] we get:

$$\begin{aligned}\sin(\theta/2) &= k \text{sn } p_1^1 \text{cn } \tau / \sqrt{\Delta} > 0, & \cos(\theta/2) &= -\text{dn } p_1^1 / \sqrt{\Delta} < 0, \\ R_1 &= 2(p - E(p)) \text{dn } \tau / \sqrt{\Delta} \Big|_{p=p_1^1} > 0, & R_2 &= 0,\end{aligned}$$

where $\Delta = 1 - k^2 \text{sn}^2 p \text{sn}^2 \tau$, and the statement follows by standard argument since $\text{Exp}|_{N'_{10}}$ is nondegenerate and proper. \square

2.5. Action of the group of reflections in the preimage and image of the exponential mapping

In order to extend the results of the preceding subsections to all 58 pairs (N'_i, M'_i) , $i \in I$, we describe the action of the group of reflections $G = \{\text{Id}, \varepsilon^1, \dots, \varepsilon^7\}$ on these sets.

| | | | | | |
|--------------------|-----------|-----------|-----------|-----------|-----------|
| D | N'_{35} | N'_{47} | N'_{52} | N'_{17} | N'_{26} |
| $\varepsilon^1(D)$ | N'_{37} | N'_{48} | N'_{58} | N'_{19} | N'_{32} |
| $\varepsilon^4(D)$ | N'_{39} | N'_{49} | N'_{54} | N'_{21} | N'_{28} |
| $\varepsilon^5(D)$ | N'_{41} | N'_{50} | N'_{56} | N'_{23} | N'_{30} |

TABLE 3. Action of $\varepsilon^1, \varepsilon^4, \varepsilon^5$ on $N'_{35}, N'_{47}, N'_{52}, N'_{17}, N'_{26}$

| | | |
|--------------------|-----------|-----------|
| D | N'_{36} | N'_{18} |
| $\varepsilon^2(D)$ | N'_{38} | N'_{20} |
| $\varepsilon^4(D)$ | N'_{40} | N'_{22} |
| $\varepsilon^6(D)$ | N'_{42} | N'_{24} |

TABLE 4. Action of $\varepsilon^2, \varepsilon^4, \varepsilon^6$ on N'_{36}, N'_{18}

| | | |
|--------------------|-----------|-----------|
| D | N'_{53} | N'_{27} |
| $\varepsilon^1(D)$ | N'_{57} | N'_{31} |
| $\varepsilon^2(D)$ | N'_{51} | N'_{25} |
| $\varepsilon^3(D)$ | N'_{55} | N'_{29} |

TABLE 5. Action of $\varepsilon^1, \varepsilon^2, \varepsilon^3$ on N'_{53}, N'_{27}

Theorem 2.1. *Tables 3, 4, 5, 9 and 6, 7, 8, 10 define diffeomorphisms between the corresponding manifolds N'_i and M'_i .*

Proof. Follows from definitions of the manifolds N'_i and M'_i (Subsections 2.1 and 2.3) and Propositions 4.3, 4.4 [7] describing action of the reflections $\varepsilon^i \in G$ in the image and preimage of the exponential mapping. Moreover, in the coordinates (θ, R_1, R_2) action of the reflections is described by Table 11. \square

| | | | | | |
|--------------------|-----------|-----------|-----------|-----------|-----------|
| D | M'_{35} | M'_{47} | M'_{52} | M'_{17} | M'_{26} |
| $\varepsilon^1(D)$ | M'_{37} | M'_{48} | M'_{58} | M'_{19} | M'_{32} |
| $\varepsilon^4(D)$ | M'_{39} | M'_{49} | M'_{54} | M'_{21} | M'_{28} |
| $\varepsilon^5(D)$ | M'_{41} | M'_{50} | M'_{56} | M'_{23} | M'_{30} |

TABLE 6. Action of $\varepsilon^1, \varepsilon^4, \varepsilon^5$ on $M'_{35}, M'_{47}, M'_{52}, M'_{17}, M'_{26}$

| | | |
|--------------------|-----------|-----------|
| D | M'_{36} | M'_{18} |
| $\varepsilon^2(D)$ | M'_{38} | M'_{20} |
| $\varepsilon^4(D)$ | M'_{40} | M'_{22} |
| $\varepsilon^6(D)$ | M'_{42} | M'_{24} |

TABLE 7. Action of $\varepsilon^2, \varepsilon^4, \varepsilon^6$ on M'_{36}, M'_{18}

| | | |
|--------------------|-----------|-----------|
| D | M'_{53} | M'_{27} |
| $\varepsilon^1(D)$ | M'_{57} | M'_{31} |
| $\varepsilon^2(D)$ | M'_{51} | M'_{25} |
| $\varepsilon^3(D)$ | M'_{55} | M'_{29} |

TABLE 8. Action of $\varepsilon^1, \varepsilon^2, \varepsilon^3$ on M'_{53}, M'_{27}

| | | |
|--------------------|--------|-----------|
| D | N'_1 | N'_{10} |
| $\varepsilon^1(D)$ | N'_2 | N'_{15} |
| $\varepsilon^2(D)$ | N'_4 | N'_9 |
| $\varepsilon^3(D)$ | N'_3 | N'_{16} |
| $\varepsilon^4(D)$ | N'_5 | N'_{12} |
| $\varepsilon^5(D)$ | N'_6 | N'_{13} |
| $\varepsilon^6(D)$ | N'_8 | N'_{11} |
| $\varepsilon^7(D)$ | N'_7 | N'_{14} |

TABLE 9. Action of $\varepsilon^1, \dots, \varepsilon^7$ on N'_1, N'_{10}

| | | |
|--------------------|--------|-----------|
| D | M'_1 | M'_{10} |
| $\varepsilon^1(D)$ | M'_2 | M'_{15} |
| $\varepsilon^2(D)$ | M'_4 | M'_9 |
| $\varepsilon^3(D)$ | M'_3 | M'_{16} |
| $\varepsilon^4(D)$ | M'_5 | M'_{12} |
| $\varepsilon^5(D)$ | M'_6 | M'_{13} |
| $\varepsilon^6(D)$ | M'_8 | M'_{11} |
| $\varepsilon^7(D)$ | M'_7 | M'_{14} |

TABLE 10. Action of $\varepsilon^1, \dots, \varepsilon^7$ on M'_1, M'_{10}

| | | | | | | | |
|----------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| | ε^1 | ε^2 | ε^3 | ε^4 | ε^5 | ε^6 | ε^7 |
| R_1 | $-R_1$ | R_1 | $-R_1$ | $-R_1$ | R_1 | $-R_1$ | R_1 |
| R_2 | R_2 | $-R_2$ | $-R_2$ | R_2 | R_2 | $-R_2$ | $-R_2$ |
| θ | θ | θ | θ | $2\pi - \theta$ | $2\pi - \theta$ | $2\pi - \theta$ | $2\pi - \theta$ |

TABLE 11. Action of $\varepsilon^1, \dots, \varepsilon^7$ on $M = \{(R_1, R_2, \theta)\}$

2.6. The final result for exponential mapping of the sets N'_i

Theorem 2.2. For any $i \in I$, the mapping $\text{Exp} : N'_i \rightarrow M'_i$ is a diffeomorphism of manifolds of appropriate dimension 2, 1, or 0.

Proof. For $i \in \{35, 47, 26, 52, 36, 53, 18, 17, 27, 1, 10\}$ the statement follows from Lemmas 2.2, 2.3, 2.6, 2.7, 2.8, 2.9, 2.10, 2.12, 2.13, 2.14, 2.15 respectively.

For $i \in \{33, 34\}$ the statement was proved in Lemma 2.11.

For all the rest i the statement follows from the above lemmas and Th. 2.1 since the reflections $\varepsilon^i \in G$ are symmetries of the exponential mapping, see Props. 4.5 [7]. \square

2.7. Reflections ε^k as permutations

In addition to the index sets I, C, J, R, X introduced in (2.9), (2.10), we will need also the set

$$T = \{(i, j) \in I \times I \mid i < j, M'_i = M'_j\}.$$

From the definition of the sets M'_i in Subsec. 2.3 we obtain the explicit representation:

$$T = \{(1, 6), (2, 5), (3, 8), (4, 7), (9, 10), (11, 12), (13, 14), (15, 16), \\ (17, 23), (18, 22), (19, 21), (20, 24), (25, 27), (29, 31), (33, 34)\}.$$

Notice that $X = \{i \in I \mid \exists j \in I : (i, j) \in T \text{ or } (j, i) \in T\}$.

Now we show that reflections $\varepsilon^k \in G$ permute elements in any pair $(i, j) \in T$.

We will need multiplication Table 12 in the group G , which follows from definitions of the reflections ε^k (Sec. 4 [7]). The lower diagonal entries of the table are not filled since G is Abelian.

| | | | | | | | |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| | ε^1 | ε^2 | ε^3 | ε^4 | ε^5 | ε^6 | ε^7 |
| ε^1 | Id | ε^3 | ε^2 | ε^5 | ε^4 | ε^7 | ε^6 |
| ε^2 | | Id | ε^1 | ε^6 | ε^7 | ε^4 | ε^5 |
| ε^3 | | | Id | ε^7 | ε^6 | ε^5 | ε^4 |
| ε^4 | | | | Id | ε^1 | ε^2 | ε^3 |
| ε^5 | | | | | Id | ε^3 | ε^2 |
| ε^6 | | | | | | Id | ε^1 |
| ε^7 | | | | | | | Id |

TABLE 12. Multiplication table in the group G

Lemma 2.16. *For any $(i, j) \in T$ there exists a reflection $\varepsilon^k \in G$ such that the following diagram is commutative:*

$$\begin{array}{ccc} N'_i & \xrightarrow{\text{Exp}} & M'_i \\ \varepsilon^k \downarrow & & \downarrow \text{Id} \\ N'_j & \xrightarrow{\text{Exp}} & M'_j \end{array} \qquad \begin{array}{ccc} (\lambda, t) & \xrightarrow{\text{Exp}} & q_t \\ \varepsilon^k \downarrow & & \downarrow \text{Id} \\ (\lambda^k, t) & \xrightarrow{\text{Exp}} & q_t \end{array}$$

Proof. From definitions of the sets N'_i (Subsec. 2.1) and the reflections ε^k (Sec. 4 [7]), Tables 3, 4, 5, 9, 12 and Propos. 4.5 [7], we obtain the following indices k of required symmetries ε^k for pairs $(i, j) \in T$:

$$\begin{aligned} (i, j) \in \{(1, 6), (2, 5), (3, 8), (4, 7), (17, 23), (19, 21)\} &\Rightarrow k = 5, \\ (i, j) \in \{(9, 10), (15, 16), (11, 12), (13, 14), (25, 27), (29, 31)\} &\Rightarrow k = 2, \\ (i, j) \in \{(33, 34), (18, 22), (20, 24)\} &\Rightarrow k = 4. \end{aligned}$$

□

3. SOLUTION TO OPTIMAL CONTROL PROBLEM

In this section we present the final results of this study of the sub-Riemannian problem on SE(2).

3.1. Global structure of the exponential mapping

We say that a mapping $F : X \rightarrow Y$ is double if any point $y \in Y$ has exactly two preimages:

$$\forall y \in Y \quad F^{-1}(y) = \{x_1, x_2\}, \quad x_1 \neq x_2.$$

Theorem 3.1. (1) *There is the following decomposition of preimage of the exponential mapping $\text{Exp} : \widehat{N} \rightarrow \widehat{M}$:*

$$\begin{aligned} \widehat{N} &= \widetilde{N} \sqcup N', \\ \widetilde{N} &= \sqcup_{i=1}^8 D_i, \\ N' &= N_{\text{Max}} \sqcup N_{\text{conj}} \sqcup N_{\text{rest}}, \\ N_{\text{Max}} &= \sqcup_{i \in X} N'_i, \\ N_{\text{conj}} &= \sqcup_{i \in J} N'_i, \\ N_{\text{rest}} &= \sqcup_{i \in R} N'_i, \end{aligned}$$

and in the image of the exponential mapping:

$$\begin{aligned} \widehat{M} &= \widetilde{M} \sqcup M', \\ \widetilde{M} &= \sqcup_{i=1}^8 M_i, \\ M' &= M_{\text{Max}} \sqcup M_{\text{conj}} \sqcup M_{\text{rest}}, \\ M_{\text{Max}} &= \cup_{i \in X} M'_i, \\ M'_i \cap M'_j &\neq \emptyset, \quad i < j \quad \Rightarrow \quad (i, j) \in T, \quad \{i, j\} \subset X, \\ (i, j) \in T &\quad \Rightarrow \quad M'_i = M'_j, \\ M_{\text{conj}} &= \sqcup_{i \in J} M'_i, \\ M_{\text{rest}} &= \sqcup_{i \in R} M'_i. \end{aligned}$$

(2) *In terms of these decompositions the exponential mapping $\text{Exp} : \widehat{N} \rightarrow \widehat{M}$ has the following structure:*

$$\text{Exp} : D_i \rightarrow M_i \text{ is a diffeomorphism } \forall i = 1, \dots, 8, \quad (3.1)$$

$$\text{Exp} : N'_i \rightarrow M'_i \text{ is a diffeomorphism } \forall i \in I. \quad (3.2)$$

Thus

$$\text{Exp} : \widetilde{N} \rightarrow \widetilde{M} \text{ is a bijection,} \quad (3.3)$$

$$\text{Exp} : N_{\text{Max}} \rightarrow M_{\text{Max}} \text{ is a double mapping,} \quad (3.4)$$

$$\text{Exp} : N_{\text{conj}} \rightarrow M_{\text{conj}} \text{ is a bijection,} \quad (3.5)$$

$$\text{Exp} : N_{\text{rest}} \rightarrow M_{\text{rest}} \text{ is a bijection.} \quad (3.6)$$

(3) *Any point $q \in \widetilde{M}$ ($q \in M_{\text{conj}}$, $q \in M_{\text{rest}}$) has a unique preimage $\nu = \text{Exp}^{-1}(q)$ for the mapping $\text{Exp}|_{\widetilde{N}} : \widehat{N} \rightarrow \widehat{M}$. Moreover, $\nu \in \widetilde{N}$ (resp., $\nu \in N_{\text{conj}}$, $\nu \in N_{\text{rest}}$).*

(4) *Any point $q \in M_{\text{Max}}$ has exactly two preimages $\{\nu', \nu''\} = \text{Exp}^{-1}(q)$ for the mapping $\text{Exp}|_{\widehat{N}} : \widehat{N} \rightarrow \widehat{M}$. Moreover, $\nu', \nu'' \in N_{\text{Max}}$ and $\nu'' = \varepsilon^k(\nu')$ for some $\varepsilon^k \in G$.*

The domains D_i and M_i were defined in Tables 1 and 2 [10].

Proof. Equalities in item (1) follow immediately from definitions of the corresponding decompositions.

(2) Property (3.1) was proved in Th. 3.1 [10], and property (3.3) is its corollary, with account of item (1). Property (3.2) was proved in Th. 3.2, and properties (3.4)–(3.6) are its corollaries, with account of item (1).

(3) The statement follows from (3.3), (3.4), (3.5), (3.6), with account of item (1).

(4) The statement follows from (3.4), (3.5), (3.6), and Lemma 2.16. \square

3.2. Optimal synthesis

Theorem 3.2. *Let $q \in \widehat{M} = M \setminus \{q_0\}$.*

- (1) *Let $q \in \widetilde{M} \sqcup M_{\text{conj}} \sqcup M_{\text{rest}} = \widehat{M} \setminus M_{\text{Max}}$. Denote $\nu = (\lambda, t) = \text{Exp}^{-1}(q) \in \widetilde{N} \sqcup N_{\text{conj}} \sqcup N_{\text{rest}} = \widehat{N} \setminus N_{\text{Max}}$. Then $q_s = \text{Exp}(\lambda, s)$, $s \in [0, t]$, is the unique optimal trajectory connecting q_0 with q . If $q \in \widetilde{M} \sqcup M_{\text{rest}}$, then $t < \mathbf{t}(\lambda)$; if $q \in M_{\text{conj}}$, then $t = \mathbf{t}(\lambda) = t_1^{\text{conj}}(\lambda)$.*
- (2) *Let $q \in M_{\text{Max}}$. Denote $\{\nu', \nu''\} = \text{Exp}^{-1}(q) \subset N_{\text{Max}}$, $\nu' = (\lambda', t) \neq \nu'' = (\lambda'', t)$. Then there exist exactly two distinct optimal trajectories connecting q_0 and q ; namely, $q'_s = \text{Exp}(\lambda', s)$ and $q''_s = \text{Exp}(\lambda'', s)$, $s \in [0, t]$. Moreover, $t = \mathbf{t}(\lambda) < t_1^{\text{conj}}(\lambda)$.*
- (3) *An optimal trajectory $q_s = \text{Exp}(\lambda, s)$ is generated by the optimal controls*

$$u_1(s) = \sin(\gamma_s/2), \quad u_2(s) = -\cos(\gamma_s/2),$$

where γ_s is the solution to the equation of pendulum $\ddot{\gamma}_s = -\sin \gamma_s$ with the initial condition $(\gamma_0, \dot{\gamma}_0) = \lambda$.

Proof. For any point $q \in \widehat{M}$ there exists an optimal trajectory $q_s = \text{Exp}(\lambda, s)$, $s \in [0, t]$, $\nu = (\lambda, t) \in N$, such that $q_t = q$ and $t \leq t_{\text{cut}}(\lambda)$. By Th. 5.4 [7], we have $t \leq \mathbf{t}(\lambda)$, thus $\nu \in \widehat{N}$.

(1) If $q \in \widetilde{M} \sqcup M_{\text{conj}} \sqcup M_{\text{rest}}$, then by Th. 3.1, there exists a unique $\nu = (\lambda, t) \in \widehat{N}$ such that $q = \text{Exp}(\nu)$, moreover, $\nu \in \widetilde{N} \sqcup N_{\text{conj}} \sqcup N_{\text{rest}}$. Consequently, $q_s = \text{Exp}(\lambda, s)$, $s \in [0, t]$, is a unique optimal trajectory connecting q_0 with q .

The inequality $t < \mathbf{t}(\lambda)$ for $\nu = (\lambda, t) \in \widetilde{N} \sqcup N_{\text{rest}}$, and the equality $t = \mathbf{t}(\lambda) = t_1^{\text{conj}}(\lambda)$ for $\nu \in N_{\text{conj}}$ follow from definitions of the sets \widetilde{N} , N_{rest} , N_{conj} .

(2) If $q \in M_{\text{Max}}$, then the statement follows similarly to item (1) from Th. 3.1 and definition of the set N_{Max} .

(3) The expressions for optimal controls were obtained in Sec. 2 [7]. \square

It follows from the definition of cut time that for any $\lambda \in C$ and $t \in (0, t_{\text{cut}}(\lambda))$, the trajectory $q(s) = \text{Exp}(\lambda, s)$ is optimal at the segment $s \in [0, t]$. For the case of finite $t_{\text{cut}}(\lambda)$, we obtain a similar statement for $t = t_{\text{cut}}(\lambda)$.

Theorem 3.3. *If $t_{\text{cut}}(\lambda) < +\infty$, then the extremal trajectory $\text{Exp}(\lambda, s)$ is optimal for $s \in [0, t_{\text{cut}}(\lambda)]$.*

Proof. Let $t_{\text{cut}}(\lambda) = \mathbf{t}(\lambda) < +\infty$, i.e., $\lambda \in C_1 \cup C_2 \cup C_4$, and let $t = t_{\text{cut}}(\lambda)$. Then $(\lambda, t) \in N_{\text{Max}}$, and the statement follows from item (2) of Th. 3.2. \square

3.3. Cut locus

Now we are able to describe globally the first Maxwell set

$$\text{Max} = \{q \in M \mid \exists t > 0, \exists \text{ optimal trajectories } q_s \neq q'_s, s \in [0, t], \text{ such that } q_t = q'_t = q\}, \quad (3.7)$$

the cut locus

$$\text{Cut} = \{\text{Exp}(\lambda, t) \mid \lambda \in C, t = t_{\text{cut}}(\lambda)\}, \quad (3.8)$$

and its intersection with caustic (the first conjugate locus)

$$\text{Conj} = \{\text{Exp}(\lambda, t) \mid \lambda \in C, t = t_1^{\text{conj}}(\lambda)\}. \quad (3.9)$$

Theorem 3.4. (1) $\text{Max} = M_{\text{Max}}$,
 (2) $\text{Cut} = M_{\text{cut}}$,
 (3) $\text{Cut} \cap \text{Conj} = M_{\text{conj}}$.

Proof. Items (1), (2) follow from Theorem 3.2 and Corollary 3.3.

(3) Let $q \in M_{\text{conj}}$, and let $q_s = \text{Exp}(\lambda, s)$, $s \in [0, \mathbf{t}(\lambda)]$, be the optimal trajectory connecting q_0 with q . Thus there are no conjugate points at the interval $(0, \mathbf{t}(\lambda))$. By item (1) of Th. 3.2, we have $t_1^{\text{conj}}(\lambda) = \mathbf{t}(\lambda)$. Thus $M_{\text{conj}} \subset \text{Conj}$, and in view of item (2) of this theorem we get $M_{\text{conj}} \subset \text{Cut} \cap \text{Conj}$. Now we prove that $\text{Cut} \cap \text{Conj} \subset M_{\text{conj}}$, i.e., $M_{\text{cut}} \cap \text{Conj} \subset M_{\text{conj}}$.

Fix any point $q \in M_{\text{cut}}$. Then $q = \text{Exp}(\lambda, t)$ for some $(\lambda, t) \in N_{\text{cut}} = N_{\text{Max}} \sqcup N_{\text{conj}}$. In order to complete the proof, we assume that $(\lambda, t) \in N_{\text{Max}}$ and show that $q \notin \text{Conj}$. Since $(\lambda, t) \in N_{\text{Max}}$, then $t = \mathbf{t}(\lambda)$. We prove that $t < t_1^{\text{conj}}(\lambda)$.

If $\lambda \in C_1 \cup C_4$, then $t_1^{\text{conj}}(\lambda) = +\infty$ by Th. 2.5 [10] (see (1.14)).

Let $\lambda \in C_2$. If $\text{sn } \tau = 0$, then $(\lambda, t) \in N_{\text{conj}}$, which is impossible since $(\lambda, t) \in N_{\text{Max}}$. And if $\text{sn } \tau \neq 0$, then $t < t_1^{\text{conj}}(\lambda)$ by Propos. 2.2 [10].

The inclusion $\text{Cut} \cap \text{Conj} \subset M_{\text{conj}}$ follows. \square

Theorem 3.5. *The cut locus has 3 connected components:*

$$\text{Cut} = \text{Cut}_{\text{glob}} \sqcup \text{Cut}_{\text{loc}}^+ \sqcup \text{Cut}_{\text{loc}}^- \quad (3.10)$$

$$\text{Cut}_{\text{glob}} = \{q \in M \mid \theta = \pi\}, \quad (3.11)$$

$$\text{Cut}_{\text{loc}}^+ = \{q \in M \mid \theta \in (-\pi, \pi), R_2 = 0, R_1 > R_1^1(|\theta|)\}, \quad (3.12)$$

$$\text{Cut}_{\text{loc}}^- = \{q \in M \mid \theta \in (-\pi, \pi), R_2 = 0, R_1 < -R_1^1(|\theta|)\}, \quad (3.13)$$

where the function R_1^1 is defined by Eq. (2.20). The initial point q_0 is contained in the closure of the components $\text{Cut}_{\text{loc}}^+$, $\text{Cut}_{\text{loc}}^-$, and is separated from the component Cut_{glob} .

Proof. By Theorems 3.4, 3.1 and Lemma 2.1, we have

$$\text{Cut} = M_{\text{cut}} = \text{Exp}(N_{\text{cut}}) = \cup_{i \in C} M'_i.$$

Denote

$$C_{\text{glob}} = \{1, \dots, 8, 17, \dots, 24, 33, 34\},$$

$$C_{\text{loc}}^+ = \{13, \dots, 16, 29, \dots, 32\},$$

$$C_{\text{loc}}^- = \{9, \dots, 12, 25, \dots, 28\}.$$

Then

$$\cup_{i \in C_{\text{glob}}} M'_i = \text{Cut}_{\text{glob}}, \quad \cup_{i \in C_{\text{loc}}^+} M'_i = \text{Cut}_{\text{loc}}^+, \quad \cup_{i \in C_{\text{loc}}^-} M'_i = \text{Cut}_{\text{loc}}^-,$$

and decomposition (3.10)–(3.13) follows.

The topological properties of Cut_{glob} , $\text{Cut}_{\text{loc}}^\pm$ follow from equalities (3.10)–(3.13). \square

The cut locus in rectifying coordinates (R_1, R_2, θ) is presented at Fig. 17, notice that here the horizontal planes $\theta = 0$ and $\theta = 2\pi$ should be identified. Global embedding of the cut locus to the solid torus (diffeomorphic image of the state space $M = \text{SE}(2)$) is shown at Fig. 18.

The curve $\Gamma = \text{Cut} \cap \text{Conj}$ has the following asymptotics near the initial point q_0 :

$$R_1 = R_1^1(\theta) = \sqrt[3]{\pi/2} \theta^{2/3} + o(\theta^{2/3}), \quad \theta \rightarrow 0, \quad R_2 = 0,$$

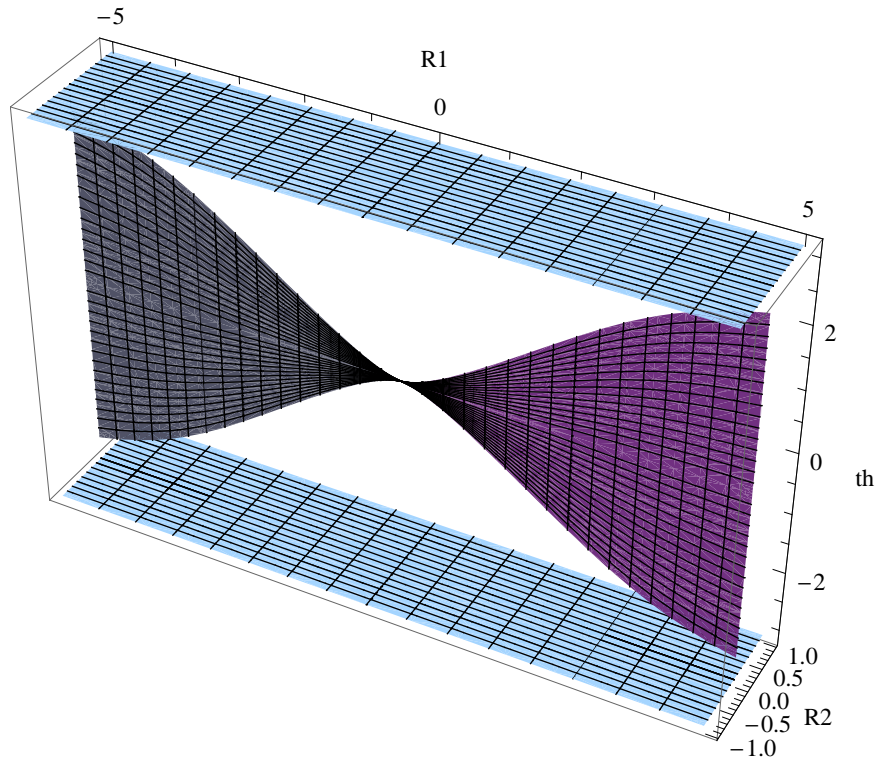
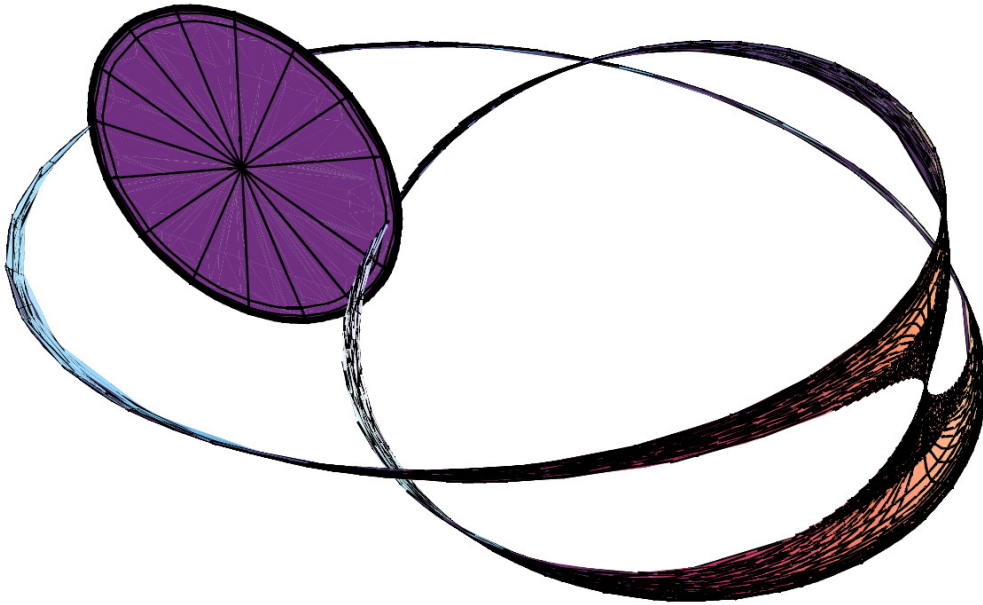
FIGURE 17. Cut locus in rectifying coordinates (R_1, R_2, θ) 

FIGURE 18. Cut locus: global view

see item (5) of Lemma 2.5. This agrees with the result on asymptotics of cut and conjugate loci for contact sub-Riemannian structures in \mathbb{R}^3 obtained by A. Agrachev [1] and by El-H. Ch. El-Alaoui, J.-P. Gauthier, I. Kupka [2].

4. EXPLICIT OPTIMAL SOLUTIONS FOR SPECIAL TERMINAL POINTS

In this section we describe optimal solutions for particular terminal points $q_1 = (x_1, y_1, \theta_1)$. Where applicable, we interpret the optimal trajectories in terms of the corresponding optimal motion of a car in the plane.

For generic terminal points, we developed a software in computer system Mathematica [13] for numerical evaluation of solutions to the problem.

4.1. $x_1 \neq 0, y_1 = 0, \theta_1 = 0$

In this case $\nu \in N_5$, and the optimal trajectory is

$$x_t = t \operatorname{sgn} x_1, \quad y_t = 0, \quad \theta_t = 0, \quad t \in [0, t_1], \quad t_1 = |x_1|,$$

the car moves uniformly forward or backward along a segment.

4.2. $x_1 = 0, y_1 = 0, |\theta_1| \in (0, \pi)$

We have $\nu \in N_4$, and the optimal solution is

$$x_t = 0, \quad y_t = 0, \quad \theta_t = t \operatorname{sgn} \theta_1, \quad t \in [0, t_1], \quad t_1 = |\theta_1|,$$

the car rotates uniformly around itself by the angle θ_1 .

4.3. $x_1 = 0, y_1 = 0, \theta_1 = \pi$

We have $\nu \in N_4$, and there are two optimal solutions:

$$x_t = 0, \quad y_t = 0, \quad \theta_t = \pm t, \quad t \in [0, t_1], \quad t_1 = \pi,$$

the car rotates uniformly around itself clockwise or counterclockwise by the angle π , see Fig. 19.

4.4. $x_1 \neq 0, y_1 = 0, \theta_1 = \pi$

There are two optimal solutions:

$$\begin{aligned} x_t &= (\operatorname{sgn} x_1)/k(t + E(k) - E(K + t, k)), & y_t &= (s/k)(\sqrt{1 - k^2} - \operatorname{dn}(K + t, k)), \\ \theta_t &= s \operatorname{sgn} x_1(\pi/2 - \operatorname{am}(K + t, k)), & s &= \pm 1, \quad t \in [0, t_1], \quad t_1 = 2K, \end{aligned}$$

and $k \in (0, 1)$ is the root of the equation

$$(2/k)(K(k) - E(k)) = |x_1|,$$

see Fig. 21.

4.5. $x_1 = 0, y_1 \neq 0, \theta_1 = \pi$

There are two optimal solutions:

$$\begin{aligned} x_t &= s(1 - \operatorname{dn}(t, k))/k, & y_t &= (\operatorname{sgn} y_1/k)(t - E(t, k)), \\ \theta_t &= s \operatorname{sgn} y_1 \operatorname{am}(t, k), & s &= \pm 1, \quad t \in [0, t_1], \quad t_1 = 2K, \end{aligned}$$

and $k \in (0, 1)$ is the root of the equation

$$(2/k)(K(k) - E(k)) = |y_1|,$$

see Fig. 22.

4.6. $x_1 = 0, y_1 \neq 0, \theta_1 = 0$

There are two optimal solutions given by formulas for (x_t, y_t, θ_t) for the case $\lambda \in C_2$ in Subsec. 3.3 [7] for the following values of parameters:

$$t \in [0, t_1], \quad t_1 = 2kp_1^1(k),$$

with the function $p_1^1(k)$ defined in Lemma 5.3 [7],

$$s_2 = -\operatorname{sgn} y_1, \quad \psi = \pm K(k) - p_1^1(k),$$

and $k \in (0, 1)$ is the root of the equation

$$2(p_1^1(k) - E(p_1^1(k), k))\sqrt{1 - k^2} / \operatorname{dn}(p_1^1(k), k) = |y_1|,$$

see Fig. 23.

4.7. $(x_1, y_1) \neq 0, \theta_1 = \pi$

Introduce the polar coordinates $x_1 = \rho_1 \cos \chi_1, y_1 = \rho_1 \sin \chi_1$. There are two optimal solutions given by formulas for (x_t, y_t, θ_t) for the case $\lambda \in C_1$ in Subsec. 3.3 [7] for the following values of parameters:

$$t \in [0, t_1], \quad t_1 = 2K(k),$$

and $k \in (0, 1)$ is the root of the equation

$$\begin{aligned} 2(p_1^1(k) - E(p_1^1(k), k))\sqrt{1 - k^2} / \operatorname{dn}(p_1^1(k), k) &= \rho_1, \\ s_1 &= \pm 1, \quad \varphi = s_1 F(\pi/2 - \chi_1, k), \end{aligned}$$

see Figs. 20, 21, 22. In the cases $y_1 = 0$ and $x_1 = 0$ we get respectively the cases considered in Subsecs. 4.4 and 4.5.

4.8. $y_1 \neq 0, \theta_1 = 0$

There is a unique optimal solution given by formulas for (x_t, y_t, θ_t) for the case $\lambda \in C_2$ in Subsec. 3.3 [7] for the following values of parameters:

$$s_2 = -\operatorname{sgn} y_1,$$

$k \in (0, 1)$ and $p \in (0, p_1^1(k))$ are solutions to the system of equations

$$\begin{aligned} s(\operatorname{sgn} y_1)2kf_1(p, k) / \operatorname{dn}(p, k) &= x_1, \quad s = \pm 1, \\ 2(p - E(p))\sqrt{1 - k^2} / \operatorname{dn}(p, k) &= |y_1|, \end{aligned}$$

and

$$t \in [0, t_1], \quad t_1 = 2kp, \quad \psi = sK(k) - p.$$

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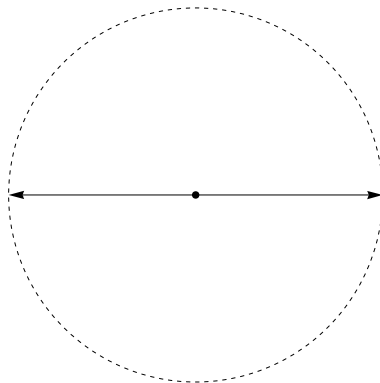


FIGURE 19. Cut point for $\lambda \in C_4$ (Optimal solutions for $x_1 = y_1 = 0, \theta_1 = \pi$)

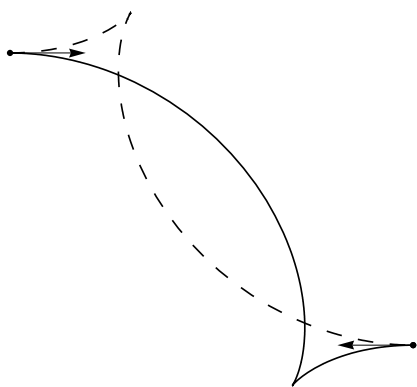


FIGURE 20. Cut point for $\lambda \in C_1$, generic case (Optimal solutions for $\theta_1 = \pi$)

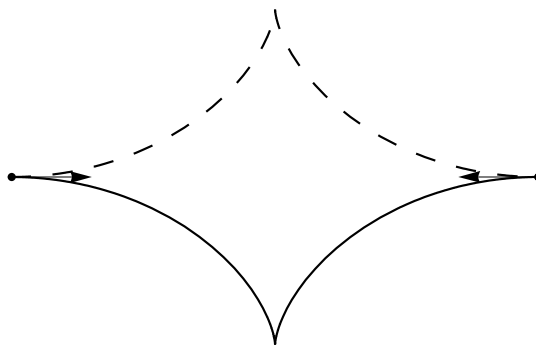


FIGURE 21. Cut point for $\lambda \in C_1$, symmetric case with cusp (Optimal solutions for $x_1 \neq 0, y_1 = 0, \theta_1 = \pi$)

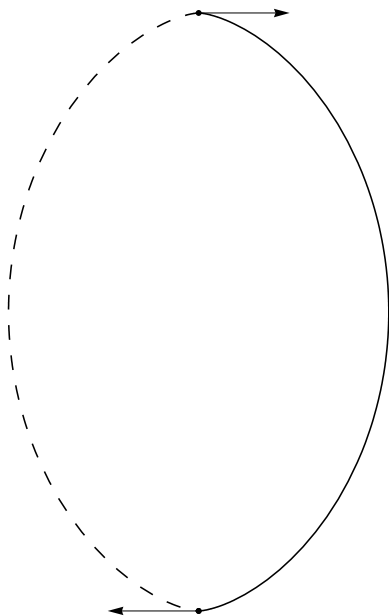


FIGURE 22. Cut point for $\lambda \in C_1$, symmetric case without cusp (Optimal solutions for $x_1 = 0$, $y_1 \neq 0$, $\theta_1 = \pi$)

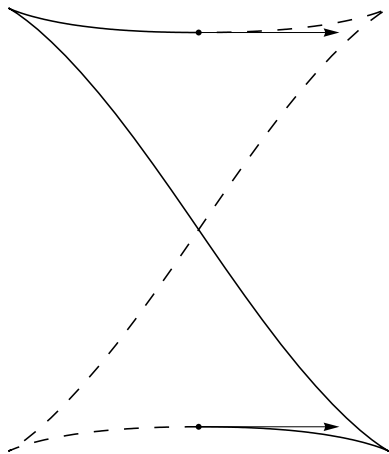


FIGURE 23. Optimal solutions for $x_1 = 0$, $y_1 \neq 0$, $\theta_1 = 0$

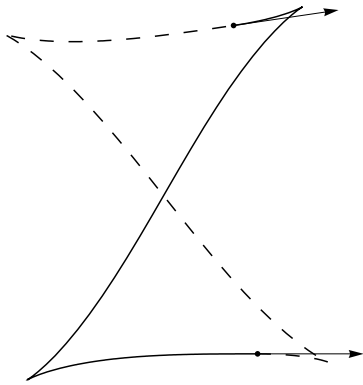


FIGURE 24. Cut point for $\lambda \in C_2$, generic case

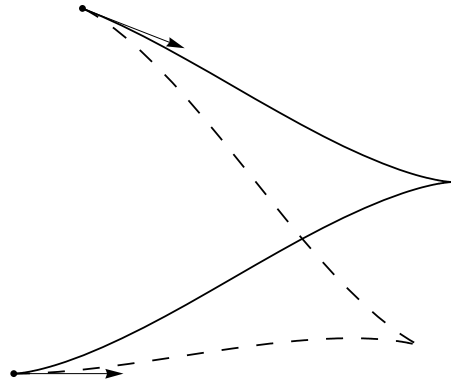


FIGURE 25. Cut point for $\lambda \in C_2$, special case with one cusp

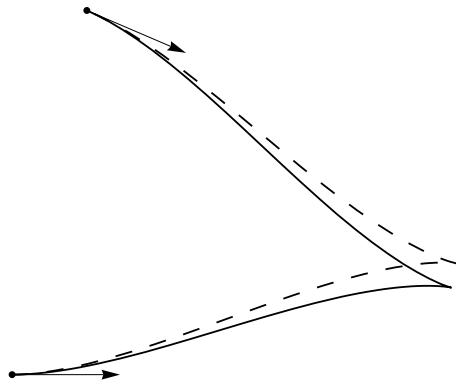


FIGURE 26. Cut point for $\lambda \in C_2$ approaching conjugate point

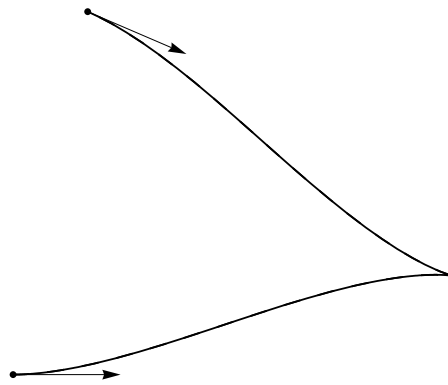


FIGURE 27. Cut point for $\lambda \in C_2$ coinciding with conjugate point

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