ISSN 0081-5438, Proceedings of the Steklov Institute of Mathematics, 2012, Vol. 278, pp. 218–232. © Pleiades Publishing, Ltd., 2012.

# Closed Euler Elasticae

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Received February 2011

**Abstract**—Euler's classical problem on stationary configurations of an elastic rod in a plane is studied as an optimal control problem on the group of motions of a plane. We show complete integrability of the Hamiltonian system of the Pontryagin maximum principle. We prove that a closed elastica is either a circle or a figure-of-eight elastica, wrapped around itself several times. Finally, we study local and global optimality of closed elasticae: the figure-of-eight elastica is optimal only locally, while the circle is optimal globally.

**DOI:** 10.1134/S0081543812060211

## 1. INTRODUCTION

In 1744 Leonhard Euler considered the following problem on stationary configurations of an elastic rod. Given a rod in the plane with fixed endpoints and tangents at the endpoints, one should determine possible profiles of the rod under the given boundary conditions. Euler obtained ODEs for stationary configurations of the elastic rod and described their possible qualitative types. These configurations are called *Euler elasticae*. Since then, the problem on elasticae has become one of the famous problems in elasticity, calculus of variations, elliptic functions, optimal control, image inpainting, etc. (see, e.g., [8, 10, 12, 18, 19]).

We study the elastic problem as an optimal control problem on the group of motions of a plane. For the Hamiltonian system of the Pontryagin maximum principle, we present three independent commuting integrals, which means complete integrability of this system.

The main goal of this paper is to study closed elasticae and their optimality properties. We prove that a closed elastica is either a circle or a figure-of-eight elastica (see Fig. 1), wrapped around itself several times.

Elasticae are critical points of the elastic energy functional $J = \frac{1}{2}\int \kappa^2(s)\,ds$, and we wish to distinguish between local and global minima of the elastic energy. We show that the figure-of-eight elastica wrapped once is a local minimizer of $J$, while the circle wrapped once is a global minimizer of $J$.

We used the system "Mathematica" [21] to carry out complicated calculations and to produce illustrations in this paper.

Euler's elastic problem is stated as the following optimal control problem [8, 15]:

$$\dot{x} = \cos\theta, \tag{1}$$

$$\dot{y} = \sin\theta, \tag{2}$$

$$\dot{\theta} = u, \tag{3}$$

$$q = (x, y, \theta) \in M = \mathbb{R}^2_{x,y} \times S^1_\theta, \qquad u \in \mathbb{R}, \tag{4}$$

$$q(0) = q_0 = (x_0, y_0, \theta_0), \qquad q(t_1) = q_1 = (x_1, y_1, \theta_1), \qquad t_1 \text{ fixed}, \tag{5}$$

$$J = \frac{1}{2}\int_0^{t_1} u^2(t)\,dt \to \min. \tag{6}$$

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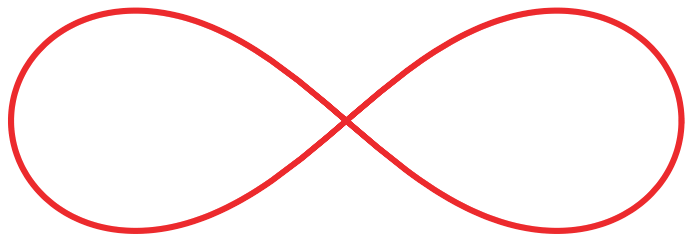

**Fig. 1.** Figure-of-eight elastica.

Admissible controls are $u(t) \in L_2[0,t_1]$, and admissible trajectories are absolutely continuous curves $q(t) \in \mathrm{AC}([0,t_1];M)$.

This is a left-invariant problem on the group of motions of a plane

$$\mathrm{SE}(2) = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; (x,y) \in \mathbb{R}^2,\ \theta \in S^1 \right\}.$$

In [15] the following results were obtained.

**Theorem 1.1** [15, Theorem 4.1]**.** *Let $q_0 = (x_0, y_0, \theta_0) \in M = \mathbb{R}^2 \times S^1$ and $t_1 > 0$. Then the time $t_1$ attainable set of system* (1)–(4) *is*

$$\mathcal{A}_{q_0}(t_1) = \{(x,y,\theta) \in M \mid (x-x_0)^2 + (y-y_0)^2 < t_1^2 \text{ or } (x,y,\theta) = (x_0 + t_1\cos\theta_0, y_0 + t_1 \sin\theta_0, \theta_0)\}.$$

**Theorem 1.2** [15, Theorems 5.3, 6.3]**.** *Let $q_1 \in \mathcal{A}_{q_0}(t_1)$.*

1. *Then there exists an optimal control for Euler's problem* (1)–(6).
2. *All optimal controls and solutions to Euler's problem satisfy the Pontryagin maximum principle and are real analytic.*

In order to write the Pontryagin maximum principle (PMP) for the elastic problem in an invariant form, we recall the basic notions of the Hamiltonian formalism [1, 8]. Let $M$ be a smooth $n$-dimensional manifold; then its cotangent bundle $T^*M$ is a smooth $2n$-dimensional manifold. The canonical projection $\pi\colon T^*M \to M$ maps a covector $\lambda \in T_q^*M$ to the base point $q \in M$. The tautological 1-form $s \in \Lambda^1(T^*M)$ on the cotangent bundle is defined as follows. Let $\lambda \in T^*M$ and $v \in T_\lambda(T^*M)$; then $\langle s_\lambda, v\rangle = \langle \lambda, \pi_* v\rangle$ (in coordinates $s = p\,dq$). The canonical symplectic structure on the cotangent bundle $\sigma \in \Lambda^2(T^*M)$ is defined as $\sigma = ds$ (in coordinates $\sigma = dp \wedge dq$). To any Hamiltonian $h \in C^\infty(T^*M)$, there corresponds a Hamiltonian vector field on the cotangent bundle $\vec{h} \in \mathrm{Vec}(T^*M)$ by the rule $\sigma_\lambda(\cdot, \vec{h}) = d_\lambda h$.

Now let $M = \mathbb{R}^2_{x,y} \times S^1_\theta$ be the state space of Euler's problem. The left-invariant vector fields

$$X_1 = \cos\theta \cdot \partial_x + \sin\theta \cdot \partial_y, \qquad X_2 = \partial_\theta, \qquad X_3 = \sin\theta \cdot \partial_x - \cos\theta \cdot \partial_y \tag{7}$$

form a basis in the tangent spaces to $M$. The Lie brackets of these vector fields are as follows:

$$[X_1, X_2] = X_3, \qquad [X_2, X_3] = X_1, \qquad [X_1, X_3] = 0. \tag{8}$$

Introduce the Hamiltonians that are linear on the fibers of $T^*M$ and correspond to these basis vector fields:

$$h_i(\lambda) = \langle \lambda, X_i\rangle, \qquad \lambda \in T^*M, \quad i = 1,2,3,$$

and the family of Hamiltonian functions

$$h_u^\nu(\lambda) = \langle \lambda, X_1 + uX_2\rangle + \frac{\nu}{2}u^2 = h_1(\lambda) + uh_2(\lambda) + \frac{\nu}{2}u^2, \qquad \lambda \in T^*M, \quad u \in \mathbb{R}, \quad \nu \in \mathbb{R},$$

the control-dependent Hamiltonian of the PMP for Euler's problem (1)–(6).

By Theorem 1.2, all optimal solutions to Euler's problem satisfy the Pontryagin maximum principle [13]. We write it in the following invariant form [1].

**Theorem 1.3** [1, Theorem 12.3]. *Let $u(t)$ and $q(t)$, $t \in [0, t_1]$, be an optimal control and the corresponding optimal trajectory in Euler's problem* (1)–(6). *Then there exists a curve $\lambda_t \in T^*M$, $\pi(\lambda_t) = q(t)$, $t \in [0, t_1]$, and a number $\nu \le 0$ for which the following conditions hold for almost all $t \in [0, t_1]$:*

$$\dot{\lambda}_t = \vec{h}^{\nu}_{u(t)}(\lambda_t) = \vec{h}_1(\lambda_t) + u(t)\vec{h}_2(\lambda_t), \tag{9}$$

$$h^{\nu}_{u(t)}(\lambda_t) = \max_{u\in\mathbb{R}} h^{\nu}_u(\lambda_t), \tag{10}$$

$$(\nu, \lambda_t) \ne 0. \tag{11}$$

Using the coordinates $(h_1, h_2, h_3, x, y, \theta)$, we can write the Hamiltonian system of the PMP (9) as follows:

$$\dot{h}_1 = -uh_3, \tag{12}$$

$$\dot{h}_2 = h_3, \tag{13}$$

$$\dot{h}_3 = uh_1, \tag{14}$$

$$\dot{x} = \cos\theta, \tag{15}$$

$$\dot{y} = \sin\theta, \tag{16}$$

$$\dot{\theta} = u. \tag{17}$$

Abnormal extremal trajectories are straight lines, and normal extremals satisfy the normal Hamiltonian system of the PMP with the Hamiltonian $H = \frac{1}{2}(h_1^2 + h_2^2)$, which reads as follows:

$$\dot{\lambda} = \vec{H}(\lambda) \quad \Leftrightarrow \quad \begin{cases} \dot{h}_1 = -h_2h_3, \\ \dot{h}_2 = h_3, \\ \dot{h}_3 = h_1h_2, \\ \dot{x} = \cos\theta, \\ \dot{y} = \sin\theta, \\ \dot{\theta} = h_2. \end{cases} \tag{18}$$

In the coordinates $(r, \beta, c)$ in cotangent spaces $T_q^*M$ given by

$$h_1 = -r\cos\beta, \qquad h_3 = -r\sin\beta, \qquad h_2 = c, \tag{19}$$

the vertical subsystem of the normal Hamiltonian system (18) (i.e., its first three equations) takes the form of the pendulum

$$\dot{\beta} = c, \qquad \dot{c} = -r\sin\beta, \qquad \dot{r} = 0; \tag{20}$$

this is Kirchhoff's kinetic analog of elasticae [10].

## 2. INTEGRABILITY

At the time of L. Euler's work [5] on elasticae (1742), the theory of elliptic functions did not exist, so he restricted himself to a qualitative study of the ODEs describing elasticae. These ODEs were integrated by many people, including L. Saalschütz (1880) [14] and D. Mumford (1994) [12]. Below we present our parameterization of elasticae obtained in [15].

In this section we prove the integrability of the normal Hamiltonian system (18). V. Jurdjevic [8] stated that any left-invariant optimal control problem on a 3-dimensional Lie group is integrable since it has three independent integrals in involution. P. Griffiths [6] showed that Euler's elastic problem is quasi-integrable in quadratures. However, we have not found an explicit presentation of integrals for the elastic problem in the literature. Below we follow V. Jurdjevic's approach and obtain such a presentation.

Denote the basis matrices of the Lie algebra

$$\mathrm{se}(2) = T_{\mathrm{Id}}\mathrm{SE}(2) = \mathrm{span}(e_1, e_2, e_3)$$

by

$$e_1 = E_{13}, \qquad e_2 = E_{21} - E_{12}, \qquad e_3 = -E_{23},$$
$$[e_1, e_2] = e_3, \qquad [e_2, e_3] = e_1, \qquad [e_1, e_3] = 0.$$

Here and below we denote by $E_{ij}$ the $3 \times 3$ matrix with 1 at the intersection of the $i$th row and $j$th column and zeros elsewhere. The vector fields $X_i(q) = qe_i$, $i = 1, 2, 3$, form a basis of the Lie algebra of left-invariant vector fields on SE(2):

$$X_1(q) = \begin{pmatrix} 0 & 0 & \cos\theta \\ 0 & 0 & \sin\theta \\ 0 & 0 & 0 \end{pmatrix}, \qquad X_2(q) = \begin{pmatrix} -\sin\theta & -\cos\theta & 0 \\ \cos\theta & -\sin\theta & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad X_3(q) = \begin{pmatrix} 0 & 0 & \sin\theta \\ 0 & 0 & -\cos\theta \\ 0 & 0 & 0 \end{pmatrix}.$$

The corresponding basis right-invariant vector fields $Y_i(q)$ such that $Y_i(\mathrm{Id}) = X_i(\mathrm{Id})$, $i = 1, 2, 3$, have the form $Y_i(q) = e_i q$; thus

$$Y_1(q) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad Y_2(q) = \begin{pmatrix} -\sin\theta & -\cos\theta & -y \\ \cos\theta & -\sin\theta & x \\ 0 & 0 & 0 \end{pmatrix}, \qquad Y_3(q) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

In order to write the right-invariant fields $Y_i$ in terms of the basis coordinate vector fields $\partial_x$, $\partial_y$, and $\partial_\theta$ on SE(2), we rewrite (7) as

$$\partial_x = \cos\theta \cdot X_1 + \sin\theta \cdot X_3, \qquad \partial_y = \sin\theta \cdot X_1 - \cos\theta \cdot X_3, \qquad \partial_\theta = X_2;$$

hence in the matrix representation

$$\partial_x = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \partial_y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \partial_\theta = \begin{pmatrix} -\sin\theta & -\cos\theta & 0 \\ \cos\theta & -\sin\theta & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus

$$Y_1 = \partial_x = \cos\theta \cdot X_1 + \sin\theta \cdot X_3,$$
$$Y_2 = -\partial_y = -\sin\theta \cdot X_1 + \cos\theta \cdot X_3,$$
$$Y_3 = \partial_\theta - y\partial_x + x\partial_y = (x\sin\theta - y\cos\theta)X_1 - (x\cos\theta + y\sin\theta)X_3 + X_2.$$

Along with the left-invariant Hamiltonians $h_i(\lambda) = \langle \lambda, X_i \rangle$, we also consider the right-invariant Hamiltonians $g_i(\lambda) = \langle \lambda, Y_i \rangle$. In the canonical coordinates $(x, y, \theta, \psi_x, \psi_y, \psi_\theta)$ on the cotangent bundle $T^*M$ (see [1]) we have

$$h_1 = \cos\theta \cdot \psi_x + \sin\theta \cdot \psi_y, \qquad h_3 = \sin\theta \cdot \psi_x - \cos\theta \cdot \psi_y, \qquad h_2 = \psi_\theta$$

and

$$g_1 = \psi_x, \qquad g_3 = -\psi_y, \qquad g_2 = -y\psi_x + x\psi_y + \psi_\theta.$$

The left-invariant Hamiltonian $H = \frac{1}{2}(h_1^2 + h_2^2)$ Poisson-commutes with the right-invariant Hamiltonians; thus the Hamiltonian system (18) has the algebra of integrals

$$I = \mathrm{span}_{\mathbb{R}}(H, g_1, g_2, g_3).$$

The only nonzero Poisson brackets between the basis elements of this algebra are

$$\{g_1, g_2\} = -g_3, \qquad \{g_2, g_3\} = -g_2.$$

So the derived series of the Lie algebra $I$ has the form

$$I \supset I^{(1)} \supset I^{(2)} = \{0\}, \qquad I^{(1)} = \{I, I\} = \mathrm{span}_{\mathbb{R}}(g_2, g_3).$$

Thus the Lie algebra $I$ is metabelian but not abelian. The subalgebra

$$\widetilde{I} = \mathrm{span}_{\mathbb{R}}(H, g_1, g_3)$$

is abelian, i.e., the integrals $H$, $g_1$, and $g_3$ are in involution. It is easy to see that these integrals are independent on $T^*M/S$, where $S = \{\lambda \in T^*M \mid h_2 = h_3 = 0\}$. So the Hamiltonian vector field $\vec{H}$ is completely integrable [3] on $T^*M/S$. Finally, it is easy to see that the restriction of the Hamiltonian system (18) to the invariant manifold $S$ reads as follows:

$$\dot{h}_1 = \dot{h}_2 = \dot{h}_3 = 0, \qquad \dot{x} = \cos\theta, \qquad \dot{y} = \sin\theta, \qquad \dot{\theta} = 0,$$

which is obviously integrable in quadratures. To sum up, we have proved the following statement.

**Theorem 2.1.** *The normal Hamiltonian system* (18) *has a* 4*-dimensional metabelian algebra of integrals $I$ and a* 3*-dimensional abelian algebra of integrals $\widetilde{I}$. This system is completely integrable on $T^*M/S$ and is integrable in quadratures on $T^*M$.*

## 3. DESCRIPTION OF CLOSED ELASTICAE

As early as in [5], L. Euler showed that there are essentially two closed elasticae: a circle and a figure-of-eight elastica (see Fig. 1). This fact is widely known and used, although we have failed to find a proof in the literature. In this section we provide such a proof.

We wish to describe all closed elasticae, i.e., solutions to problem (1)–(6) with the periodic boundary condition $q_0 = q_1$. Since the problem is left invariant, we can assume that

$$q_0 = \mathrm{Id}, \qquad \text{i.e.,} \qquad (x_0, y_0, \theta_0) = (0, 0, 0). \tag{21}$$

It is well known [1] that periodic extremal trajectories are projections of periodic extremals:

$$\lambda_0 = \lambda_{t_1}.$$

In the coordinates (19), we get

$$\beta_0 = \beta_{t_1} \pmod{2\pi}, \qquad c_0 = c_{t_1}, \tag{22}$$

i.e., $t_1$ is a period of the pendulum (20). In order to go further, we recall the parameterization of elasticae by Jacobi's functions cn, sn, dn, and E obtained in [15].

The total energy of the pendulum (20) is

$$E = \frac{c^2}{2} - r\cos\beta \in [-r, +\infty); \tag{23}$$

this is just the Hamiltonian $H$. Consider the exponential mapping for the problem:

$$\mathrm{Exp}_{t_1}\colon\ N = T^*_{q_0}M \to M, \qquad \mathrm{Exp}_{t_1}(\lambda_0) = \pi \circ e^{t_1\vec{H}}(\lambda_0) = q(t_1).$$

The following decomposition of the preimage $N$ of the exponential mapping into invariant subsets of the field $\vec{H}$ will be very important below:

$$T^*_{q_0}M = N = \bigcup_{i=1}^{7} N_i,$$

$$\begin{aligned}
&N_1 = \{\lambda \in N \mid r \neq 0,\ E \in (-r, r)\},\\
&N_2 = \{\lambda \in N \mid r \neq 0,\ E \in (r, +\infty)\} = N_2^+ \cup N_2^-,\\
&N_3 = \{\lambda \in N \mid r \neq 0,\ E = r,\ \beta \neq \pi\} = N_3^+ \cup N_3^-,\\
&N_4 = \{\lambda \in N \mid r \neq 0,\ E = -r\},\\
&N_5 = \{\lambda \in N \mid r \neq 0,\ E = r,\ \beta = \pi\},\\
&N_6 = \{\lambda \in N \mid r = 0,\ c \neq 0\} = N_6^+ \cup N_6^-,\\
&N_7 = \{\lambda \in N \mid r = c = 0\},\\
&N_i^\pm = N_i \cap \{\lambda \in N \mid \operatorname{sgn} c = \pm 1\}, \qquad i = 2, 3, 6.
\end{aligned}$$

In the domain $N_1 \cup N_2 \cup N_3$ we use elliptic coordinates of the form $(\varphi, k, r)$. On the set $N_1 \cup N_2^+ \cup N_3^+$, the coordinate $\varphi$ is equal to the time of motion of the generalized pendulum (20) from a point $(\beta = 0,\ c = c_0 > 0,\ r)$ to a point $(\beta, c, r)$, while on the set $N_2^- \cup N_3^-$ the time of motion is calculated from a point $(\beta = 0,\ c = c_0 < 0,\ r)$. The elliptic coordinates $(\varphi, k, r)$ are defined as follows (below $K(k)$ is the complete elliptic integral of the first kind):

$$\lambda = (\beta, c, r) \in N_1 \quad \Rightarrow \quad \begin{cases} \sin \dfrac{\beta}{2} = k \operatorname{sn}(\sqrt{r}\varphi, k),\\[2mm] \dfrac{c}{2} = k\sqrt{r} \operatorname{cn}(\sqrt{r}\varphi, k),\\[2mm] \cos \dfrac{\beta}{2} = \operatorname{dn}(\sqrt{r}\varphi, k), \end{cases}$$

$$k = \sqrt{\frac{E + r}{2r}} \in (0, 1), \qquad \sqrt{r}\varphi \pmod{4K(k)} \in [0, 4K(k)];$$

$$\lambda = (\beta, c, r) \in N_2^\pm \quad \Rightarrow \quad \begin{cases} \sin \dfrac{\beta}{2} = \pm \operatorname{sn}\left(\dfrac{\sqrt{r}\varphi}{k}, k\right),\\[2mm] \dfrac{c}{2} = \pm \dfrac{\sqrt{r}}{k} \operatorname{dn}\left(\dfrac{\sqrt{r}\varphi}{k}, k\right),\\[2mm] \cos \dfrac{\beta}{2} = \operatorname{cn}\left(\dfrac{\sqrt{r}\varphi}{k}, k\right), \end{cases}$$

$$k = \sqrt{\frac{2r}{E + r}} \in (0, 1), \qquad \sqrt{r}\varphi \pmod{2K(k)k} \in [0, 2K(k)k], \qquad \pm = \operatorname{sgn} c;$$

$$\lambda = (\beta, c, r) \in N_3^\pm \quad \Rightarrow \quad \begin{cases} \sin \dfrac{\beta}{2} = \pm \tanh(\sqrt{r}\varphi),\\[2mm] \dfrac{c}{2} = \pm \dfrac{\sqrt{r}}{\cosh(\sqrt{r}\varphi)},\\[2mm] \cos \dfrac{\beta}{2} = \dfrac{1}{\cosh(\sqrt{r}\varphi)}, \end{cases} \qquad k = 1, \qquad \varphi \in \mathbb{R}, \qquad \pm = \operatorname{sgn} c.$$

In the domain $N_2$ it will also be convenient to use the coordinates $(k, \psi, r)$, where

$$\psi = \frac{\varphi}{k}, \qquad \sqrt{r}\psi \pmod{2K(k)} \in [0, 2K(k)].$$

In the elliptic coordinates $(\varphi, k, r)$ in the domain $N_1 \cup N_2 \cup N_3$, the vertical subsystem (20) of the normal Hamiltonian system $\dot{\lambda} = \vec{H}(\lambda)$ rectifies:

$$\dot{\varphi} = 1, \qquad \dot{k} = 0, \qquad \dot{r} = 0;$$

thus it has solutions

$$\varphi_t = \varphi + t, \qquad k = \text{const}, \qquad r = \text{const}.$$

Then expressions for the vertical coordinates $(\beta, c, r)$ are immediately given by the above formulas for the elliptic coordinates. For $\lambda \in \bigcup_{i=4}^{7} N_i$, the vertical subsystem degenerates and is easily integrated. So we obtain the following description of the solution $(\beta_t, c_t, r)$ to the vertical subsystem (20) with the initial condition $(\beta_t, c_t, r)|_{t=0} = (\beta, c, r)$:

$$\lambda \in N_1 \quad \Rightarrow \quad \begin{cases} \sin \dfrac{\beta_t}{2} = k \operatorname{sn}(\sqrt{r}\varphi_t), \\ \cos \dfrac{\beta_t}{2} = \operatorname{dn}(\sqrt{r}\varphi_t), \\ \dfrac{c_t}{2} = k\sqrt{r} \operatorname{cn}(\sqrt{r}\varphi_t); \end{cases}$$

$$\lambda \in N_2^{\pm} \quad \Rightarrow \quad \begin{cases} \sin \dfrac{\beta_t}{2} = \pm \operatorname{sn}\left(\dfrac{\sqrt{r}\varphi_t}{k}\right), \\ \cos \dfrac{\beta_t}{2} = \operatorname{cn}\left(\dfrac{\sqrt{r}\varphi_t}{k}\right), \\ \dfrac{c_t}{2} = \pm \dfrac{\sqrt{r}}{k} \operatorname{dn}\left(\dfrac{\sqrt{r}\varphi_t}{k}\right); \end{cases}$$

$$\lambda \in N_3^{\pm} \quad \Rightarrow \quad \begin{cases} \sin \dfrac{\beta_t}{2} = \pm \tanh(\sqrt{r}\varphi_t), \\ \cos \dfrac{\beta_t}{2} = \dfrac{1}{\cosh(\sqrt{r}\varphi_t)}, \\ \dfrac{c_t}{2} = \pm \dfrac{\sqrt{r}}{\cosh(\sqrt{r}\varphi_t)}; \end{cases}$$

$$\lambda \in N_4 \quad \Rightarrow \quad \beta_t \equiv 0, \quad c_t \equiv 0;$$

$$\lambda \in N_5 \quad \Rightarrow \quad \beta_t \equiv \pi, \quad c_t \equiv 0;$$

$$\lambda \in N_6 \quad \Rightarrow \quad \beta_t = ct + \beta, \quad c_t \equiv c;$$

$$\lambda \in N_7 \quad \Rightarrow \quad c_t \equiv 0, \quad r \equiv 0.$$

It is easy to see from the above parameterization of trajectories of the pendulum (20) that the periodic boundary conditions (22) occur in the following cases:

(1) if $\lambda \in N_1$, then $t_1 = Tn$, $T = \frac{4K}{\sqrt{r}}$;

(2) if $\lambda \in N_2$, then $t_1 = Tn$, $T = \frac{2Kk}{\sqrt{r}}$;

(3) if $\lambda \in N_6$, then $t_1 = Tn$, $T = \frac{2\pi}{|c|}$,

where $n \in \mathbb{N}$ and $T$ is the period of oscillations of the pendulum.

Recall the parameterization of elasticae obtained in [15].

If $\lambda \in N_1$, then

$$\sin\frac{\theta_t}{2} = k\,\mathrm{dn}(\sqrt{r}\varphi)\,\mathrm{sn}(\sqrt{r}\varphi_t) - k\,\mathrm{sn}(\sqrt{r}\varphi)\,\mathrm{dn}(\sqrt{r}\varphi_t),$$

$$\cos\frac{\theta_t}{2} = \mathrm{dn}(\sqrt{r}\varphi)\,\mathrm{dn}(\sqrt{r}\varphi_t) + k^2\,\mathrm{sn}(\sqrt{r}\varphi)\,\mathrm{sn}(\sqrt{r}\varphi_t),$$

$$\begin{aligned} x_t &= \frac{2}{\sqrt{r}}\,\mathrm{dn}^2(\sqrt{r}\varphi)\bigl(\mathrm{E}(\sqrt{r}\varphi_t) - \mathrm{E}(\sqrt{r}\varphi)\bigr) + \frac{4k^2}{\sqrt{r}}\,\mathrm{dn}(\sqrt{r}\varphi)\,\mathrm{sn}(\sqrt{r}\varphi)\bigl(\mathrm{cn}(\sqrt{r}\varphi) - \mathrm{cn}(\sqrt{r}\varphi_t)\bigr) \\ &\quad + \frac{2k^2}{\sqrt{r}}\,\mathrm{sn}^2(\sqrt{r}\varphi)\bigl(\sqrt{r}t + \mathrm{E}(\sqrt{r}\varphi) - \mathrm{E}(\sqrt{r}\varphi_t)\bigr) - t, \end{aligned}$$

$$\begin{aligned} y_t &= \frac{2k}{\sqrt{r}}\bigl(2\,\mathrm{dn}^2(\sqrt{r}\varphi) - 1\bigr)\bigl(\mathrm{cn}(\sqrt{r}\varphi) - \mathrm{cn}(\sqrt{r}\varphi_t)\bigr) \\ &\quad - \frac{2k}{\sqrt{r}}\,\mathrm{sn}(\sqrt{r}\varphi)\,\mathrm{dn}(\sqrt{r}\varphi)\bigl(2\bigl(\mathrm{E}(\sqrt{r}\varphi_t) - \mathrm{E}(\sqrt{r}\varphi)\bigr) - \sqrt{r}t\bigr). \end{aligned}$$

If $\lambda \in N_2^{\pm}$, then $\psi_t = \frac{\varphi_t}{k}$ and

$$\sin\frac{\theta_t}{2} = \pm\bigl(\mathrm{cn}(\sqrt{r}\psi)\,\mathrm{sn}(\sqrt{r}\psi_t) - \mathrm{sn}(\sqrt{r}\psi)\,\mathrm{cn}(\sqrt{r}\psi_t)\bigr),$$

$$\cos\frac{\theta_t}{2} = \mathrm{cn}(\sqrt{r}\psi)\,\mathrm{cn}(\sqrt{r}\psi_t) + \mathrm{sn}(\sqrt{r}\psi)\,\mathrm{sn}(\sqrt{r}\psi_t),$$

$$\begin{aligned} x_t &= \frac{1}{\sqrt{r}}\bigl(1 - 2\,\mathrm{sn}^2(\sqrt{r}\psi)\bigr)\left(\frac{2}{k}\bigl(\mathrm{E}(\sqrt{r}\psi_t) - \mathrm{E}(\sqrt{r}\psi)\bigr) - \frac{2-k^2}{k^2}\sqrt{r}t\right) \\ &\quad + \frac{4}{k\sqrt{r}}\,\mathrm{cn}(\sqrt{r}\psi)\,\mathrm{sn}(\sqrt{r}\psi)\bigl(\mathrm{dn}(\sqrt{r}\psi) - \mathrm{dn}(\sqrt{r}\psi_t)\bigr), \end{aligned}$$

$$\begin{aligned} y_t &= \pm\biggl(\frac{2}{k\sqrt{r}}\bigl(2\,\mathrm{cn}^2(\sqrt{r}\psi) - 1\bigr)\bigl(\mathrm{dn}(\sqrt{r}\psi) - \mathrm{dn}(\sqrt{r}\psi_t)\bigr) \\ &\quad - \frac{2}{\sqrt{r}}\,\mathrm{sn}(\sqrt{r}\psi)\,\mathrm{cn}(\sqrt{r}\psi)\left(\frac{2}{k}\bigl(\mathrm{E}(\sqrt{r}\psi_t) - \mathrm{E}(\sqrt{r}\psi)\bigr) - \frac{2-k^2}{k^2}\sqrt{r}t\right)\biggr). \end{aligned}$$

If $\lambda \in N_6$, then

$$\theta_t = ct, \qquad x_t = \frac{\sin ct}{c}, \qquad y_t = \frac{1-\cos ct}{c}.$$

By virtue of the Hamiltonian system (18),

$$(\beta - \theta)^{\cdot} = c - c = 0;$$

thus the first of the periodic conditions (22) in $T^*_{q_0}M$ yields $\theta_0 = \theta_{t_1}$, so the periodic conditions (22), (21) in $T^*M$ are equivalent to

$$x_{t_1} = y_{t_1} = 0.$$

In the coordinates

$$P = x\sin\frac{\theta}{2} - y\cos\frac{\theta}{2}, \qquad Q = x\cos\frac{\theta}{2} + y\sin\frac{\theta}{2},$$

we get

$$P_{t_1} = Q_{t_1} = 0. \tag{24}$$

1. Let $\lambda \in N_1$. Then

$$P_t = \frac{4k \operatorname{dn}\tau \operatorname{sn}\tau\big(\operatorname{dn} p \operatorname{sn} p - \operatorname{cn} p\,(p - 2\operatorname{E}(p))\big)}{\sqrt{r}(1 - k^2 \operatorname{sn}^2 p \operatorname{sn}^2 \tau)},$$

$$Q_t = -2\frac{-2\operatorname{E}(p) + p + k^2\big(4\operatorname{E}(p) + (p - 2\operatorname{E}(p))\operatorname{sn}^2 p - 2(p + \operatorname{cn} p \operatorname{sn} p \operatorname{dn} p)\big)\operatorname{sn}^2\tau}{\sqrt{r}(1 - k^2 \operatorname{sn}^2 p \operatorname{sn}^2 \tau)},$$

$$p = \frac{\sqrt{r}t}{2}, \qquad \tau = \sqrt{r}\left(\varphi + \frac{t}{2}\right).$$

If $t_1 = Tn$, $T = \frac{4K}{\sqrt{r}}$, then

$$p_1 = 2Kn, \qquad \operatorname{sn} p_1 = 0, \qquad \operatorname{cn} p_1 = \pm 1,$$

$$P_{t_1} = \pm\frac{4k}{\sqrt{r}} \operatorname{dn}\tau \operatorname{sn}\tau(2\operatorname{E}(p_1) - p_1),$$

$$Q_{t_1} = -\frac{2}{\sqrt{r}}(2k^2 p \operatorname{sn}\tau - 1)(2\operatorname{E}(p_1) - p_1),$$

$$2\operatorname{E}(p_1) - p_1 = 2n(2E(k) - K(k)).$$

Here and below $E(k)$ is the complete elliptic integral of the second kind. Consequently, equations (24) are equivalent to the equation

$$2E(k) - K(k) = 0, \qquad k \in (0,1). \tag{25}$$

**Proposition 3.1** [15, Proposition 11.5]**.** *The function $2E(k) - K(k)$ decreases from $\frac{\pi}{2}$ to $-\infty$ on the interval $[0,1)$. Equation (25) has a unique root $k_0 \in \left(\frac{1}{\sqrt{2}}, 1\right)$. Moreover,*

$$k \in [0, k_0) \quad \Rightarrow \quad 2E - K > 0,$$

$$k \in (k_0, 1) \quad \Rightarrow \quad 2E - K < 0.$$

To sum up, in the case $\lambda \in N_1$ equations (24) with $t_1 = Tn = \frac{4K}{\sqrt{r}}n$ are equivalent to the equality $k = k_0$. The corresponding periodic elastica is the figure of eight (Fig. 1) wrapped $n$ times.

2. Let $\lambda \in N_2$. Then

$$P_t = -\frac{4 \operatorname{cn}\tau \operatorname{sn}\tau \cdot f(p,k)}{k\sqrt{r}(1 - k^2 \operatorname{sn}^2 p \operatorname{sn}^2\tau)},$$

$$f(p,k) = k^2 \operatorname{cn} p \operatorname{sn} p - \operatorname{dn} p\left(2\operatorname{E}(p) + (k^2 - 2)p\right), \qquad p = \frac{\sqrt{r}t}{2k}, \qquad \tau = \frac{\sqrt{r}}{2}\left(2\psi + \frac{t}{k}\right).$$

**Proposition 3.2** [15, Proposition 11.9]**.** *The function $f(p)$ has no roots $p \neq 0$.*

By virtue of Proposition 3.2, the equality $P_t = 0$ is equivalent to $\operatorname{cn}\tau \operatorname{sn}\tau = 0$. If $\operatorname{sn}\tau = 0$ and $p_1 = 2Kn$, then

$$Q_{t_1} = \frac{2}{k\sqrt{r}}g(k), \qquad g(k) = 2E(k) + (k^2 - 2)K(k).$$

If $\operatorname{cn}\tau = 0$ and $p_1 = 2Kn$, then

$$Q_{t_1} = -\frac{2}{k\sqrt{r}}g(k).$$

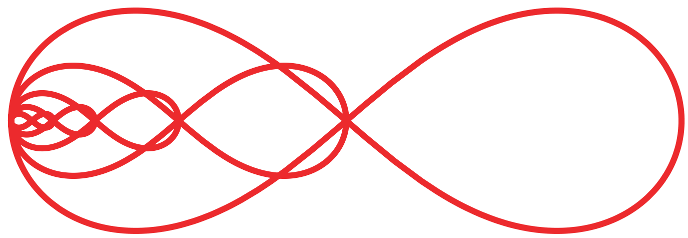

**Fig. 2.** Contractible closed elasticae.

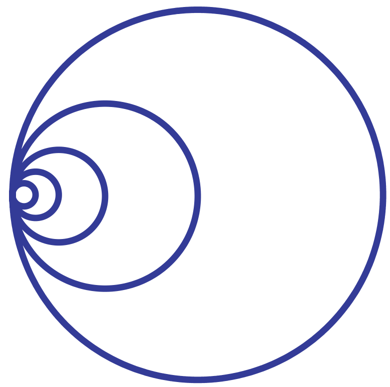

**Fig. 3.** Noncontractible closed elasticae.

**Lemma 3.1.** *$g(k) < 0$ for all $k \in (0, 1)$.*

**Proof.** Consider the auxiliary function $h(k) = \frac{g(k)}{K(k)}$. Since

$$h'(k) = -\frac{2[E(k) - (1-k^2)K(k)]^2}{k(1-k^2)K^2(k)} < 0$$

and $h(0) = 0$, it follows that $h(k) < 0$ and $g(k) < 0$ for all $k \in (0, 1)$. $\square$

To sum up, in the case $\lambda \in N_2$ equalities (24) with $t_1 = Tn = \frac{2Kk}{\sqrt{r}} n$ are impossible.

3. Finally, let $\lambda \in N_6$ and $t_1 = Tn = \frac{2\pi}{|c|} n$. Then it is obvious that $x_{t_1} = y_{t_2} = 0$. The circle is oriented positively (counterclockwise) if $c > 0$ and negatively (clockwise) if $c > 0$. The corresponding closed elastica is the circle wrapped $n$ times. We have proved the following statement.

**Theorem 3.1.** *Any closed elastica belongs to one of the following classes:*

(1) *figure of eight wrapped $n$ times* ($n \in \mathbb{N}$);

(2) *circle wrapped $n$ times* ($n \in \mathbb{N}$), *oriented positively or negatively.*

Different types of closed elasticae are shown in Figs. 2 and 3.

## 4. OPTIMALITY OF CLOSED ELASTICAE

**4.1. Optimality on $\widetilde{\mathrm{SE}}(2)$.** In this section we study local and global optimality of closed elasticae. In order to separate closed elasticae into homotopy classes, we lift the elastic problem (1)–(6) from $\mathrm{SE}(2) = \mathbb{R}^2_{x,y} \times S^1_\theta$ to its simply connected covering $\widetilde{\mathrm{SE}}(2) = \mathbb{R}^2_{x,y} \times \mathbb{R}_\theta$. The lift reads as the initial problem (1)–(6) with condition (4) replaced by

$$q = (x, y, \theta) \in \widetilde{M} = \mathbb{R}^2_{x,y} \times \mathbb{R}_\theta, \qquad u \in \mathbb{R}. \tag{26}$$

We refer to problem (1)–(3), (26), (5), (6) as the elastic problem on $\widetilde{\mathrm{SE}}(2)$, or just as the lifted problem. The trajectories $(x_t, y_t, \theta_t)$ of the lifted problem are trajectories of the initial problem, but the angle $\theta_t$ varies in $\mathbb{R}$, not in $S^1 = \mathbb{R}/2\pi\mathbb{Z}$. If a point $(x, y, \theta) \in \widetilde{M}$ is attainable for the lifted system from a point $q_0 = (x_0, y_0, \theta_0) \in \widetilde{M}$ for a time $t_1 > 0$ and $(x - x_0)^2 + (y - y_0)^2 < t_1^2$, then any point $(x, y, \theta + 2\pi n) \in M$, $n \in \mathbb{Z}$, is also time $t_1$ attainable from $q_0$ since one can add any number of small circles to the corresponding trajectory of the initial system. Thus the time $t_1$ attainable set of the lifted system is

$$\widetilde{\mathcal{A}}_{q_0}(t_1) = \big\{(x, y, \theta) \in \widetilde{M} \mid (x - x_0)^2 + (y - y_0)^2 < t_1 \ \text{ or } \ (x, y, \theta) = (x_0 + t_1 \cos\theta_0, y_0 + t_1 \sin\theta_0, \theta_0)\big\}$$

by virtue of Theorem 1.1.

In the same way as in [15, Sect. 5], it follows that for any $q_1 \in \widetilde{\mathcal{A}}_{q_0}$ the lifted problem has an optimal control $u \in L_\infty[0, t_1]$, which satisfies the PMP. The extremal trajectories for the lifted problem coincide with the extremal trajectories of the initial problem, with the angle $\theta_t \in \mathbb{R}$, not in $S^1$. The periodic boundary condition for the initial system

$$(x_{t_1}, y_{t_1}) = (0, 0), \qquad \theta_{t_1} = 0 \ (\mathrm{mod}\ 2\pi) \tag{27}$$

splits into the countable family of boundary conditions for the lifted system

$$(x_{t_1}, y_{t_1}) = (0, 0), \qquad \theta_{t_1} = 2\pi n, \quad n \in \mathbb{Z}. \tag{28}$$

For any $n \in \mathbb{Z}$, $n \neq 0$, the lifted problem with the boundary condition (28) has a unique extremal trajectory, the circle wrapped $n$ times (counterclockwise for $n > 0$ and clockwise for $n < 0$). By virtue of the existence of an optimal solution, this extremal trajectory is optimal. For $n = 0$, the lifted problem with the boundary condition (28) has a countable number of extremal trajectories, the figure of eight wrapped $k$ times, $k \in \mathbb{N}$. For the figure-of-eight elastica, the first conjugate time is its period (see [16]); thus the $k$ times wrapped figure of eight with $k > 1$ is not locally optimal. So the only candidate for a global minimizer is the figure of eight wrapped once. By virtue of existence, this trajectory is globally optimal for the lifted problem. We have proved the following statement.

**Proposition 4.1.** *Consider the lifted elastic problem with a boundary condition* (28).

*If* $n \in \mathbb{Z}$, $n \neq 0$, *then the global solution is the circle wrapped* $n$ *times.*

*If* $n = 0$, *then the global solution is the figure of eight wrapped once. Consequently, the figure of eight is locally optimal on* $\widetilde{\mathrm{SE}}(2)$.

**4.2. Optimality on** SE(2)**.** Local optimality on SE(2) coincides with local optimality on $\widetilde{\mathrm{SE}}(2)$; thus the circle wrapped $n$ times, $n \in \mathbb{Z}$, $n \neq 0$, is locally optimal (which was already known to Max Born [4] due to the absence of conjugate points on non-inflectional elasticae, see also [16]). Moreover, the figure-of-eight elastica is locally optimal, although in a critical way, since its endpoint is the first conjugate point. In order to find a globally optimal trajectory on SE(2) with the periodic boundary condition (27), consider all locally optimal trajectories:

- the circle wrapped $n$ times, $n \in \mathbb{Z}$, $n \neq 0$;
- the figure of eight wrapped once.

We have to compare the elastic energy of these trajectories with given length $l = t_1$:

- $J_0$, elastic energy of the circle;
- $J_{n\cdot 0}$, elastic energy of the circle wrapped $n$ times, $n \in \mathbb{Z}$, $n \neq 0$ ($J_{1\cdot 0} = J_0$);
- $J_8$, elastic energy of the figure of eight wrapped once.

Since

$$J_{n\cdot 0} = n^2 J_0 > J_0, \qquad n \in \mathbb{Z}, \quad n \neq 0, 1,$$

the circle wrapped $n$ times, $n \neq 0, \pm 1$, is not globally optimal. It remains to compare $J_0$ with $J_8$.

Notice that J. Langer and D. Singer [9] proved Theorem 3.1 by considering the anti-gradient flow for the elastic energy functional $\int k^2(s)\,ds$ on the space of smooth closed immersed curves $\gamma \subset \mathbb{R}^2$. Moreover, they showed that in each regular homotopy class of immersed curves in $\mathbb{R}^2$ there exists a unique locally optimal closed elastica.

4.2.1. *Auxiliary lemmas.*

**Lemma 4.1.** *Let $f \in C[a,b] \cap D(a,b)$, $f(a)+f(b) \ge 0$, and $f'(b-t) < f'(a+t)$ for all $t \in \left(0, \frac{b-a}{2}\right)$. Then*

$$\int_a^b f(x)\,dx > 0.$$

**Proof.** We have $\int_a^b f(x)\,dx = \int_0^l (f(a+t)+f(b-t))\,dt$, $l = \frac{b-a}{2}$. We show that the function $\varphi(t) = f(a+t)+f(b-t)$ is positive for each $t \in (0,l)$. Since $\varphi \in C[0,l] \cap D(0,l)$ and $\varphi'(t) = f'(a+t) - f'(b-t) > 0$, the function $\varphi(t)$ increases for $t \in [0,l]$. But $\varphi(0) = f(a)+f(b) \ge 0$; thus $\varphi(t) > 0$ for $t \in (0,l)$, and the proof is complete. □

**Lemma 4.2.** *We have*

$$2E\left(\frac{\sqrt{3}}{2}\right) - K\left(\frac{\sqrt{3}}{3}\right) > 0.$$

**Proof.** From the definition of the functions $E(k)$ and $K(k)$ [20], we have

$$2E\left(\frac{\sqrt{3}}{2}\right) - K\left(\frac{\sqrt{3}}{2}\right) = \int_0^{\frac{\pi}{2}} \frac{1-\frac{3}{2}\sin^2 x}{\sqrt{1-\frac{3}{4}\sin^2 x}}\,dx. \tag{29}$$

We prove that the integral in (29) is positive by virtue of Lemma 4.1 with

$$a = 0, \qquad b = \frac{\pi}{2}, \qquad f(x) = \frac{1-\frac{3}{2}\sin^2 x}{\sqrt{1-\frac{3}{4}\sin^2 x}}.$$

Since $f(0) = 1$ and $f\left(\frac{\pi}{2}\right) = 1$, it remains to prove that

$$f'\left(\frac{\pi}{2} - t\right) < f'(t), \qquad 0 < t < \frac{\pi}{4}. \tag{30}$$

We have

$$f'(x) = \frac{9}{16}\sin 2x \frac{\sin^2 x - 2}{\left(1-\frac{3}{4}\sin^2 x\right)^{3/2}};$$

thus the required inequality (30) is equivalent to

$$\frac{\cos^2 t - 2}{\left(1-\frac{3}{4}\cos^2 t\right)^{3/2}} < \frac{\sin^2 t - 2}{\left(1-\frac{3}{4}\sin^2 t\right)^{3/2}}, \qquad 0 < t < \frac{\pi}{4}.$$

In terms of the variable $y = \sin^2 t$, the preceding inequality is equivalent to

$$\frac{2-y}{\left(1-\frac{3}{4}y\right)^{3/2}} < \frac{1+y}{\left(\frac{3}{4}y+\frac{1}{4}\right)^{3/2}} \qquad \Leftrightarrow \qquad (3y+1)^{3/2}(2-y) < (4-3y)^{3/2}(2+y),$$

where $0 < y < \frac{1}{2}$. But the last inequality follows from the inequalities

$$\sqrt{3y+1} < \sqrt{4-3y}, \qquad (3y+1)(2-y) < (4-3y)(1+y)$$

for $0 < y < \frac{1}{2}$, which are easily verified. This completes the proof of the lemma. □

4.2.2. *Elastic energy of the circle.* The circle of length $l = 2\pi R$ has curvature $k = \frac{1}{R} = \frac{2\pi}{l}$ and elastic energy

$$J_0 = \frac{1}{2}\int_0^l k^2\,dt = \frac{2\pi}{l}.$$

4.2.3. *Elastic energy of the figure-of-eight elastica.* The figure-of-eight elastica of length $l$ has elastic energy

$$J_8 = \frac{1}{2}\int_0^l k_t^2\,dt = \frac{1}{2}\int_0^l c_t^2\,dt = 2k^2 r\int_0^l \operatorname{cn}^2(\sqrt{r}\varphi_t)\,dt = 2\sqrt{r}\big[\mathrm{E}(\sqrt{r}(\varphi+l)) - \mathrm{E}(\sqrt{r}\varphi) - (1-k^2)\sqrt{r}l\big].$$

For one period of the figure of eight, we have

$$l = \frac{4K}{\sqrt{r}}, \qquad k = k_0, \qquad 2E(k) - K(k) = 0;$$

thus

$$\mathrm{E}(\sqrt{r}(\varphi+l)) - \mathrm{E}(\sqrt{r}\varphi) = \mathrm{E}(\sqrt{r}\varphi + 4K) - \mathrm{E}(\sqrt{r}\varphi) = \mathrm{E}(4K) = 4\,\mathrm{E}(K) = 4E(k) = 2K(k).$$

Consequently,

$$J_8 = 4(2k^2-1)\sqrt{r}K|_{k=k_0} = \frac{16(2k_0^2-1)K_0^2}{l}, \qquad K_0 = K(k_0).$$

4.2.4. *Comparison of the circle and the figure-of-eight elastica.* The ratio of elastic energies of the circle and the figure-of-eight elastica of the same length is

$$\frac{J_8}{J_0} = \frac{8(2k_0^2-1)K_0^2}{\pi^2}.$$

In order to prove that $J_8 > J_0$, we introduce the function

$$\delta(k) := \frac{8}{\pi^2}(2k^2-1)K(k), \qquad k \in (0,1).$$

**Lemma 4.3.** *We have* $\delta(k_0) > 1$.

**Proof.** For any $k \in (0,1)$ we have $K(k) > \frac{\pi}{2}$; thus $\delta(k) > 2(2k^2-1)$. By Lemma 4.4, we have $k_0 > \frac{\sqrt{3}}{2}$; thus $\delta(k_0) > 1$. □

**Lemma 4.4.** *We have* $k_0 \in \big(\frac{\sqrt{3}}{2}, 1\big)$.

**Proof.** The function $\beta(k) := 2E(k) - K(k)$ decreases for $k = [0,1)$. Since $\beta(k_0) = 0$, it remains to prove that $\beta\big(\frac{\sqrt{3}}{2}\big) > 0$, which is Lemma 4.2. □

Summing up, we obtain the following statement.

**Proposition 4.2.** *The circle has less elastic energy than the figure-of-eight elastica of the same length*:

$$J_8 > J_0.$$

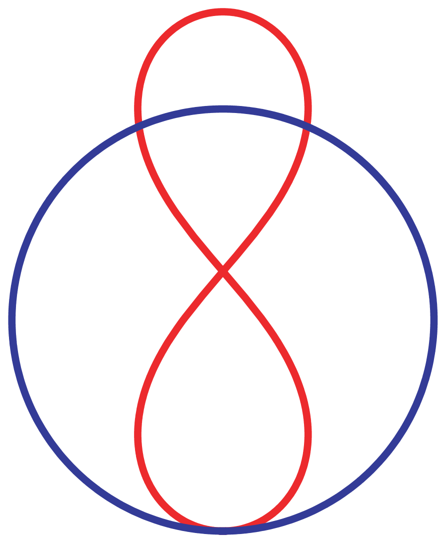

**Fig. 4.** $0 < 8$.

Numerical computation using Mathematica [21] yields the ratio

$$\frac{J_8}{J_0} \approx 2.8. \tag{31}$$

Proposition 4.2 can be reformulated as "$8 > 0$" (see Fig. 4). The ratio $\frac{8}{0}$ would be unanimously evaluated to $\infty$, which is not the correct answer in the context: equality (31) shows that "$\frac{8}{0} \approx 2.8$."

Notice that the circle wrapped $n$ times has elastic energy $J_{n \cdot 0} = \frac{2\pi^2}{l} n^2 = n^2 J_0$; thus the figure of eight has less energy than the circle wrapped two times:

$$\frac{J_8}{J_{2\cdot 0}} = \frac{1}{4}\frac{J_8}{J_0} \approx \frac{2.8}{4} = 0.7,$$

so finally

$$J_0 < J_8 < J_{2\cdot 0}.$$

## 5. CONCLUDING REMARKS

We believe that the methods developed in this work can be useful in two directions. First, the procedure of lifting the elastic problem from $\mathrm{SE}(2)$ to $\widetilde{\mathrm{SE}}(2)$ may be applied to the study of local and global optimality of extremal trajectories both for Euler's elastic problem (with general boundary conditions) and for other related optimal control problems (the plate–ball problem [7, 11, 17] and the sub-Riemannian problem on the Engel group [2]). Second, the method of studying integrability might be useful for free nilpotent sub-Riemannian problems.[1]

## ACKNOWLEDGMENTS

The author is grateful to A.A. Agrachev for fruitful discussions of the subject of this work.

Lemmas 4.4, 4.2, and 4.1 were proved by A.Yu. Popov. The author wishes to thank him for collaboration.

The work was supported by the Ministry of Education and Science of Russia within the federal program "Scientific and Scientific–Pedagogical Personnel of Innovative Russia," contract no. 12.740.11.0684.

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*This article was submitted by the author in English*