CONTROLLABILITY OF INVARIANT SYSTEMS ON LIE GROUPS AND HOMOGENEOUS SPACES

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Contents

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1. INTRODUCTION

The aim of this work is to give a comprehensive survey of results on controllability of right-invariant control systems on Lie groups and their homogeneous spaces. This subject is an area of active research in the mathematical control theory and the Lie semigroup theory during the last 25 years. The motivations for this study are diverse: applications in mechanics and geometry, connections with other important classes of nonlinear control systems (bilinear and affine), the work on generalization of S. Lie's theory from the group case to the semigroup case.

The structure of this work is reflected in detail in the contents. First we give definitions, state the problems, and present general results on right-invariant systems (Sections 2–4). In Sections 5–7 we give controllability conditions in the three best studied cases: homogeneous systems, compact Lie groups, and semidirect products. In the next sections we present controllability results according to group properties of the state space of a system: the semisimple case (Section 8), the nilpotent case (Section 9) and its generalization (Section 10), the solvable case and its generalizations (Sections 11–16). In Section 17 we list some works related to the subject of this survey.

2. Definitions and general properties of right-invariant systems

Throughout this work G will denote a real Lie group and L its Lie algebra, i.e., the set of right-invariant vector fields on G.

2.1. **The basic definitions.** A *right-invariant control system* Γ on a Lie group G is an arbitrary set of right-invariant vector fields on G, i.e., any subset

$$\Gamma \subset L. \tag{2.1}$$

A particular class of right-invariant systems important for applications is formed by systems *affine in control*

$$\Gamma = \left\{ A + \sum_{i=1}^{m} u_i B_i \mid u = (u_1, \dots , u_m) \in U \subset \mathbb{R}^m \right\}, \tag{2.2}$$

where A, B_1, … , B_m are some elements of L. If the control set U coincides with $\mathbb{R}^m$, then system (2.2) is an affine subspace of L.

Remark. Throughout this work we write a right-invariant control system as (2.1) or (2.2), i.e., as a collection of vector fields — a *polysystem*. In the *classical notation*, control affine systems (2.2) are written as

$$\dot{x} = A(x) + \sum_{i=1}^{m} u_i B_i(x), \qquad u = (u_1, \dots , u_m) \in U, \quad x \in G \tag{2.3}$$

with piecewise constant control functions $u_1(\cdot)$, … , $u_m(\cdot)$. Polysystem (2.1) may also be written in such classical notation via a choice of a parametrization of the set Γ.

A *trajectory* of a right-invariant system Γ on G is a continuous curve $x(t)$ in G defined on an interval $[a, b] \subset \mathbb{R}$ so that there exists a partition $a = t_0 < t_1 < \dots < t_m = b$ and vector fields $A_1, \dots , A_m$ in Γ such that the restriction of $x(t)$ to each open interval (t_{i-1}, t_i) is differentiable and $\dot{x}(t) = A_i(x(t))$ for $t \in (t_{i-1}, t_i)$, $i = 1, \dots , m$.

For any $T \geq 0$ and any x in G, the *time T reachable set* of a system Γ from the point x is the set $\mathbb{A}_\Gamma(x, T)$ of all points that can be reached from x in exactly T units of time:

$$\mathbb{A}_\Gamma(x, T) = \{x(T) \mid x(\cdot) \text{ a trajectory of } \Gamma,\ x(0) = x\}.$$

The *reachable set for the time not greater than* $T \geq 0$ is defined as

$$\mathbb{A}_\Gamma(x, \leq T) = \bigcup_{0 \leq t \leq T} \mathbb{A}_\Gamma(x, t).$$

The *reachable* (or *attainable*) *set* of a system Γ from a point $x \in G$ is the set $\mathbb{A}_\Gamma(x)$ of all terminal points $x(T)$, $T \geq 0$, of all trajectories of Γ starting at x:

$$\mathbb{A}_\Gamma(x) = \{x(T) \mid x(\cdot) \text{ a trajectory of } \Gamma,\ x(0) = x,\ T \geq 0\} = \bigcup_{T \geq 0} \mathbb{A}_\Gamma(x, T).$$

If there is no ambiguity, in the sequel we denote the reachable sets $\mathbb{A}_\Gamma(x, T)$ and $\mathbb{A}_\Gamma(x)$ respectively by $\mathbb{A}(x, T)$ and $\mathbb{A}(x)$.

A system Γ is called *controllable* if, given any pair of points x_0 and x_1 in G, the point x_1 can be reached from x_0 along a trajectory of Γ for a nonnegative time:

$$x_1 \in \mathbb{A}(x_0) \text{ for any } x_0,\ x_1 \in G,$$

or, in other words, if

$$\mathbb{A}(x) = G \text{ for any } x \in G.$$

Another property, obviously weaker than controllability, is also essential for description of reachable sets. As system Γ is called *accessible* at a point $x \in G$ if the reachable set $\mathbb{A}(x)$ has nonempty interior in G.

The *orbit* of a system Γ through a point $x \in G$ is denoted by $\mathcal{O}_\Gamma(x)$ and is defined analogously to the reachable set $\mathbb{A}(x)$, but the terminal time T may take both positive and negative values:

$$\mathcal{O}_\Gamma(x) = \{x(T) \mid x(\cdot) \text{ a trajectory of } \Gamma,\ x(0) = x,\ T \in \mathbb{R}\}.$$

If a system Γ is fixed, its orbit is denoted by $\mathcal{O}(x)$.

Remark. The inversion

$$i : G \to G, \qquad i(x) = x^{-1}$$

induces an isomorphism between the Lie algebra of right-invariant vector fields on a Lie group G and the Lie algebra of left-invariant vector fields on G. Thus all problems for left-invariant control systems, including controllability, are reduced to the study of right-invariant systems.

For any subset $\Gamma \subset L$ we denote by $\operatorname{Lie}(\Gamma)$ the Lie algebra generated by Γ, i.e., the smallest subalgebra of L containing Γ.

Given any subset l of a vector space V, we denote by $\operatorname{span}(l)$ the vector subspace of V generated by l and by $\operatorname{co}(l)$ the convex hull of l.

We denote topological closure and interior of a set M by $\operatorname{cl} M$ and $\operatorname{int} M$ respectively.

The identity operator or matrix will be denoted by Id, and E_{ij} will denote the matrix with the identity ij-th entry and all other zero entries. We denote by A^{T} the transpose of a matrix A.

2.2. **Elementary properties of orbits and reachable sets.** Let $\exp : L \to G$ be the exponential mapping from the Lie algebra L into the Lie group G. Any right-invariant field $A \in L$ is complete. The trajectory of A through the group identity $e \in G$ is

$$\exp(tA), \qquad t \in \mathbb{R},$$

and

$$\exp(tA)x, \qquad t \in \mathbb{R}$$

is the trajectory of A through a point $x \in G$.

Some properties of orbits of systems of right-invariant vector fields well-known from the Lie group theory are collected in the following proposition.

Lemma 2.1. *Let $\Gamma \subset L$ be a right-invariant system and x an arbitrary point of G.*

(i) $\mathcal{O}(x) = \{\exp(t_k A_k) \cdots \exp(t_1 A_1) x \mid A_i \in \Gamma,\ t_i \in \mathbb{R},\ k \in \mathbb{N}\}$.
(ii) $\mathcal{O}(x) = \mathcal{O}(e)x$.
(iii) $\mathcal{O}(e)$ *is the connected Lie subgroup of G with Lie algebra* $\operatorname{Lie}(\Gamma)$.
(iv) $\mathcal{O}(x)$ *is the maximal integral manifold of the involutive right-invariant distribution* $\operatorname{Lie}(\Gamma)$ *on G through the point x.*

The following basic properties of attainable sets easily follow from the right-invariant property of Γ and the definition of $\mathbb{A}(x)$.

Lemma 2.2. *Let $\Gamma \subset L$ be a right-invariant system and x an arbitrary point of G.*

(i) $\mathbb{A}(x) = \{\exp(t_k A_k) \cdots \exp(t_1 A_1) x \mid A_i \in \Gamma,\ t_i \geq 0,\ k \in \mathbb{N}\}$.
(ii) $\mathbb{A}(x) = \mathbb{A}(e)x$.
(iii) $\mathbb{A}(e)$ *is a subsemigroup of G.*
(iv) $\mathbb{A}(x)$ *is an arcwise connected subset of G.*

Since all essential properties of attainable sets (including controllability, see e.g. Theorems 2.6, 2.7) are expressed in terms of the attainable set from the identity $\mathbb{A}(e)$, in the sequel we restrict ourselves by this set and denote it by $\mathbb{A}$. Analogously, we denote the orbit $\mathcal{O}(e)$ simply by $\mathcal{O}$.

2.3. **Matrix systems.** The important class of right-invariant systems that motivated the whole theory of such systems are *matrix* control systems.

Denote by $\mathrm{M}(n;\mathbb{R})$ the set of all $n \times n$ real matrices.

The *general linear group* $\mathrm{GL}(n;\mathbb{R})$ is formed by nonsingular real $n \times n$ matrices,

$$\mathrm{GL}(n;\mathbb{R}) = \{X \in \mathrm{M}(n;\mathbb{R}) \mid \det X \neq 0\}.$$

The group product in $\mathrm{GL}(n;\mathbb{R})$ is the usual matrix product, and the real analytic structure on $\mathrm{GL}(n;\mathbb{R})$ is induced by identifying $\mathrm{M}(n;\mathbb{R})$ with $\mathbb{R}^{n^2}$.

The Lie algebra of $\mathrm{GL}(n;\mathbb{R})$ is the space of all real $n \times n$ matrices

$$\mathfrak{gl}(n;\mathbb{R}) = \mathrm{M}(n;\mathbb{R})$$

with the matrix commutator

$$[A,B] = AB - BA, \qquad A, B \in \mathfrak{gl}(n;\mathbb{R}),$$

as a Lie product.

Let G be a *linear group*, i.e., a closed subgroup of $\mathrm{GL}(n;\mathbb{R})$, and let $L \subset \mathfrak{gl}(n;\mathbb{R})$ be the Lie algebra of G.

For any matrix $A \in L$, the corresponding right-invariant vector field on G is defined by the matrix product

$$A(x) = Ax, \qquad x \in G \tag{2.4}$$

(we identify a right-invariant vector field with its value at the group identity).

The exponential mapping from L to G is the matrix exponent

$$A \mapsto \exp(A) = \mathrm{Id} + A + \frac{1}{2!}A^2 + \cdots + \frac{1}{n!}A^n + \cdots, \qquad A \in L.$$

The trajectory of $A \in L$ through a point $x \in G$ is given by the matrix exponent and product

$$\exp(tA)x, \qquad t \in \mathbb{R}. \tag{2.5}$$

The right translation by an element $g \in G$

$$x \mapsto xg, \qquad x \in G$$

maps a trajectory (2.5) into a trajectory, this explains the title right-invariant for vector fields of the form (2.4).

A right-invariant control system on a linear group G is an arbitrary set of matrices $\Gamma \subset L$.

An affine in control right-invariant system on G has form (2.2) for some matrices $A, B_1, \dots, B_m \in L$. In the classical notation, such system is written as a matrix control system

$$\dot{x} = Ax + \sum_{i=1}^{m} u_i B_i x, \qquad u = (u_1, \dots, u_m) \in U \subset \mathbb{R}^m, \quad x \in G. \tag{2.6}$$

Now we list several examples of linear groups G and their Lie algebras L. In each of these cases, G can be viewed as a state space of a right-invariant system $\Gamma \subset L$; in the affine in control case, see (2.2) or (2.6), matrices A, $B_1, \dots, B_m$ may be chosen arbitrarily in L.

Example 2.1. The general linear group $\mathrm{GL}(n;\mathbb{R})$ has Lie algebra $\mathfrak{gl}(n;\mathbb{R})$. Its dimension is equal to n^2. Notice that $\mathrm{GL}(n;\mathbb{R})$ is not connected: it has two connected components.

Example 2.2. The connected component of the identity in $\mathrm{GL}(n;\mathbb{R})$ is the group of all real $n \times n$ matrices with positive determinant

$$\mathrm{GL}_+(n;\mathbb{R}) = \{X \in \mathrm{M}(n;\mathbb{R}) \mid \det X > 0\}.$$

The Lie algebra of the group $\mathrm{GL}_+(n;\mathbb{R})$ is $\mathfrak{gl}(n;\mathbb{R})$.

Example 2.3. The *special linear group* is the group of all real $n \times n$ unimodular matrices

$$\mathrm{SL}(n;\mathbb{R}) = \{X \in \mathrm{M}(n;\mathbb{R}) \mid \det X = 1\}.$$

It is a connected (n^2-1)-dimensional Lie group, and its Lie algebra $\mathfrak{sl}(n;\mathbb{R})$ consists of all $n \times n$ traceless matrices:

$$\mathfrak{sl}(n;\mathbb{R}) = \{A \in \mathrm{M}(n;\mathbb{R}) \mid \operatorname{tr} A = 0\}.$$

Example 2.4. The *special orthogonal group* is formed by all real $n \times n$ orthogonal unimodular matrices:

$$\mathrm{SO}(n;\mathbb{R}) = \{X \in \mathrm{M}(n;\mathbb{R}) \mid X^{\mathrm{T}} = X^{-1}, \ \det X = 1\}.$$

It is a connected Lie group of dimension $n(n-1)/2$, and its Lie algebra $\mathfrak{so}(n;\mathbb{R})$ consists of all real $n \times n$ skew-symmetric matrices:

$$\mathfrak{so}(n;\mathbb{R}) = \{A \in \mathrm{M}(n;\mathbb{R}) \mid A^{\mathrm{T}} = -A\}.$$

2.4. **Normal accessibility.** If a point y in G is reachable (or accessible) from a point x in G, then there exist elements $A_1, \dots, A_m$ in Γ and $t = (t_1, \dots, t_m) \in \mathbb{R}^m$ with positive coordinates such that

$$y = \exp(t_m A_m) \cdots \exp(t_1 A_1) x.$$

The following stronger notion turns out important in the study of topological properties of reachable sets and controllability.

Definition 2.1. A point y in G is called *normally accessible* from a point x in G by Γ if there exist elements $A_1, \dots, A_m$ in Γ and $\hat{t} \in \mathbb{R}^m$ with positive coordinates $\hat{t}_1, \dots, \hat{t}_m$ such that the mapping $F(t_1, \dots, t_m) = \exp(t_m A_m) \cdots \exp(t_1 A_1) x$ as a mapping from $\mathbb{R}^m$ into G satisfies the following:

(i) $F(\hat{t}) = y$.
(ii) The rank of the differential dF at $\hat{t}$ is equal to the dimension of G.

The point y is said to be normally accessible from x by $A_1, \dots, A_m$.

Theorem 2.1. *If* $\mathrm{Lie}(\Gamma) = L$*, then in any neighborhood* O *of the identity* $e \in G$ *there are points normally accessible from* e *by* Γ*. Consequently, the set* $\operatorname{int} \mathbb{A} \cap O$ *is nonempty.*

Proof. Denote $n = \dim L = \dim \mathrm{Lie}(\Gamma)$. If $n = 0$, everything is clear. Assume $n > 0$ and fix a neighborhood O of the identity e.

There exists a nonzero element $A_1 \in \Gamma$. The curve

$$M_1 = \{\exp(t_1 A_1) \mid 0 < t_1 < \varepsilon_1\}$$

is a smooth one-dimensional manifold contained in the neighborhood O for sufficiently small positive ε_1. If $n = 1$, then any point in M_1 is normally accessible from e by A_1 since the mapping $F_1(t_1) = \exp(t_1 A_1)$ has rank 1 in the interval $I_1 = (0, \varepsilon_1)$.

If $n > 1$, there exists an element $A_2 \in \Gamma$ such that the right-invariant field A_2 is not tangent to M_1 at any point of M_1: if this were the case for any $A_2 \in \Gamma$, then $\dim \mathrm{Lie}(\Gamma) = 1$, a contradiction. That is why the set

$$M_2 = \{\exp(t_2 A_2) \exp(t_1 A_1) \mid 0 < t_i < \varepsilon_i, \ i = 1, 2\}$$

is a smooth two-dimensional manifold, which belongs to O for sufficiently small positive $\varepsilon_1, \varepsilon_2$. Moreover, the mapping $F_2(t_1, t_2) = \exp(t_2 A_2) \exp(t_1 A_1)$ has rank 2 in the domain $I_2 = (0, \varepsilon_1) \times (0, \varepsilon_2)$. If $n = 2$, the theorem is proved since then any point of M_2 is normally accessible from e by A_1, A_2.

If $n > 2$, we proceed by induction. For any dimension $k < n$ and some elements $A_1, \dots, A_k \in \Gamma$ we construct a k-dimensional smooth manifold

$$M_k = \{\exp(t_k A_k) \cdots \exp(t_1 A_1) \mid 0 < t_i < \varepsilon_i, \ i = 1, \dots, k\}$$

contained in the neighborhood O for sufficiently small positive $\varepsilon_1, \dots, \varepsilon_k$, so that the mapping $F_k(t_1, \dots, t_k) = \exp(t_k A_k) \cdots \exp(t_1 A_1)$ has rank k in the domain $I_k = (0, \varepsilon_1) \times \cdots \times (0, \varepsilon_k)$. Then any point in M_n is normally accessible from e by $A_1, \dots, A_n$.

The image of the box I_n by the mapping F_n is an open set contained in $\mathbb{A}$ and O, thus $\operatorname{int} \mathbb{A} \cap O \supset F_n(I_n)$. ☐

If the Lie algebra generated by Γ does not coincide with the whole Lie algebra L, then Γ may be considered as a right-invariant system on the orbit $\mathcal{O}$. By item

(iv) of Lemma 2.1, $\mathrm{Lie}(\Gamma)$ coincides with the Lie algebra of the Lie group $\mathcal{O}$, thus the previous theorem implies the following relationship between the attainable set $\mathbb{A}$ and the orbit $\mathcal{O}$.

Lemma 2.3. (i) *The attainable set* $\mathbb{A}$ *is contained in the orbit* $\mathcal{O}$*.*
(ii) *For any neighborhood* O *of the identity* e *in the topology of the orbit* $\mathcal{O}$*, the intersection* $\mathrm{int}_{\mathcal{O}}\, \mathbb{A} \cap O$ *is nonempty.*
(iii) *Moreover,* $\mathrm{cl}\, \mathrm{int}_{\mathcal{O}}\, \mathbb{A} \supset \mathbb{A}$*.*

(We denote by $\mathrm{int}_{\mathcal{O}}$ the interior of a subset of the orbit $\mathcal{O}$ in the topology of $\mathcal{O}$.)

Proof. Item (i) is straightforward. Item (ii) follows from Theorem 2.1: since $\mathrm{Lie}(\Gamma)$ is the Lie algebra of $\mathcal{O}$, one should replace in this theorem G by $\mathcal{O}$. To prove inclusion (iii), take any point x in $\mathbb{A}$ and choose any neighborhood U of x in $\mathcal{O}$. We have to show that the intersection $\mathrm{int}_{\mathcal{O}}\, \mathbb{A} \cap U$ is nonempty. There exists a neighborhood O of e in $\mathcal{O}$ such that $Ox \subset U$. By item (ii), there is a point y in $\mathrm{int}_{\mathcal{O}}\, \mathbb{A} \cap O$. Then $yx \in \mathrm{int}_{\mathcal{O}}\, \mathbb{A} \cap U$. □

2.5. **Basic controllability conditions.**

Theorem 2.2. *A necessary condition for a right-invariant system* Γ *on* G *to be controllable is that the Lie group* G *be connected.*

Proof. The reachable set $\mathbb{A}$ is arcwise connected, see Lemma 2.2. □

Remark. In view of the previous theorem, in the sequel *all Lie groups are assumed connected*, unless otherwise explicitly stated.

The fundamental necessary controllability condition given in the following proposition is usually referred to as *the rank condition* or *the Lie algebra rank condition* (*LARC*).

Theorem 2.3. *A necessary condition for a right-invariant system* Γ *on* G *to be controllable is that* Γ *generates* L *as a Lie algebra:* $\mathrm{Lie}(\Gamma) = L$*. If* $\mathrm{Lie}(\Gamma) = L$*, then the attainable set* $\mathbb{A}$ *has a nonempty interior.*

Proof. If $\mathbb{A} = G$, then the more so $\mathcal{O} = G$. By Lemma 2.1, $\mathrm{Lie}(\Gamma) = L$.
If $\mathrm{Lie}(\Gamma) = L$, then Theorem 2.1 gives $\mathrm{int}\, \mathbb{A} \neq \emptyset$. □

In general, the rank condition is not sufficient for controllability, but it is equivalent to accessibility.

Theorem 2.4. *A right-invariant system* Γ *on* G *is accessible at the identity* (*and thus at any point in* G) *if and only if* $\mathrm{Lie}(\Gamma) = L$*.*

Proof. Necessity. If the reachable set $\mathbb{A}$ has a nonempty interior in G, then the same holds for the orbit $\mathcal{O}$. By Lemma 2.1, we obtain $\mathrm{Lie}(\Gamma) = L$.
Sufficiency. If $\mathrm{Lie}(\Gamma) = L$, then $\mathrm{int}\, \mathbb{A}$ is nonempty by Theorem 2.1. □

A system $\Gamma \subset L$ is said to *have the full rank* if the rank condition $\mathrm{Lie}(\Gamma) = L$ holds.

Theorem 2.5. *A right-invariant system* Γ *on a connected Lie group* G *is controllable if and only if*
(i) *The attainable set* $\mathbb{A}$ *is a subgroup of* G *and*
(ii) $\mathrm{Lie}(\Gamma) = L$*.*

Proof. Necessity. Item (i) is obvious, and item (ii) follows from the rank condition.

Sufficiency. If $\mathbb{A}$ is a subgroup, then for any exponential $\exp(tA)$, $A \in \Gamma$, $t \geq 0$, its inverse $\exp(-tA)$ is also in $\mathbb{A}$. Thus the attainable set A coincides with the orbit $\mathcal{O}$. But since Γ has the full rank, its orbit coincides with the whole group G (see Lemma 2.1, item (iv)). Consequently, $\mathbb{A} = G$. □

Theorem 2.6. *A right-invariant system* Γ *is controllable on a connected Lie group* G *if and only if it is controllable from the identity, i.e.,* $\mathbb{A} = G$.

Proof. Apply item (ii) of Lemma 2.2. □

For a full-rank system Γ, its attainable set $\mathbb{A}$ has a nonempty interior in G. But in general, the identity e may lie on the boundary of $\mathbb{A}$.

Theorem 2.7. *A right-invariant system* Γ *is controllable on a connected Lie group* G *if and only if the group identity* e *is contained in the interior of* $\mathbb{A}$.

Proof. Necessity is obvious, and sufficiency follows since for a connected Lie group G, an arbitrary neighborhood of the identity e generates G as a semigroup. □

The following controllability condition is fundamental since it shows that in the study of controllability of full-rank systems, one can replace the attainable set $\mathbb{A}$ by its closure $\operatorname{cl} \mathbb{A}$.

Theorem 2.8. *If the reachable set* $\mathbb{A}$ *is dense in a connected Lie group* G *and* $\operatorname{Lie}(\Gamma) = L$, *then* Γ *is controllable on* G.

Proof. Consider the time-reversed system

$$-\Gamma = \{-A \mid A \in \Gamma\},$$

its trajectories are trajectories of Γ passed in the reverse time. The attainable set of $-\Gamma$ is

$$\mathbb{A}_{-\Gamma} = \{\exp(-t_k A_k) \cdots \exp(-t_1 A_1) \mid A_i \in \Gamma,\ t_i \geq 0,\ k \in \mathbb{N}\} = \mathbb{A}^{-1}. \tag{2.7}$$

Since the system $-\Gamma$ has the full rank: $\operatorname{Lie}(-\Gamma) = \operatorname{Lie}(\Gamma) = L$, its attainable set has a nonempty interior and thus contains an open set O_1.

On the other hand, since Γ has the full rank, there is a point x in G that has a neighborhood $O(x)$ contained in $\mathbb{A}$.

Closure of the attainable set from x is everywhere dense: $\operatorname{cl} \mathbb{A}(x) = \operatorname{cl}(\mathbb{A} \cdot x) = G$, thus there exists a point $y \in \mathbb{A}(x) \cap O_1$. We have $y \in \mathbb{A} \cdot x$, hence $yx^{-1} \in \mathbb{A}$. Taking into account the inclusion $O(x) \subset \mathbb{A}$ and the semigroup property of $\mathbb{A}$, we obtain that the neighborhood $O(y) = yx^{-1} \cdot O(x)$ of the point y is contained in $\mathbb{A}$. But $y \in O_1 \subset \mathbb{A}^{-1}$, thus $y^{-1} \in \mathbb{A}$, and the neighborhood of the identity $O(e) = y^{-1} \cdot O(y)$ is contained in $\mathbb{A}$. By Theorem 2.7, $\mathbb{A} = G$. □

Theorem 2.9. *A right-invariant system* Γ *is controllable on a connected Lie group* G *if and only if the identity* e *is normally accessible from* e *by some elements* $A_1, \dots, A_m$ *in* Γ.

Proof. Necessity. By Theorem 2.1, there exists a point $x \in G$ normally accessible by some fields $A_1, \dots, A_m \in \Gamma$ from e. Since Γ is controllable, the time-reversed system $-\Gamma$ is also controllable, thus $e = \exp(t_k A_k) \cdots \exp(t_{m+1} A_{m+1}) x$ for some $A_{m+1}, \dots, A_k \in \Gamma$ and some $t_{m+1}, \dots, t_k > 0$. Then e is normally accessible from e by the fields $A_1, \dots, A_k$.

Sufficiency follows from Theorem 2.7 since a normally accessible point is in the interior of the attainable set. □

The preceding result easily implies that controllability of right-invariant systems is preserved under small perturbations. More precisely, let $\rho(\cdot,\cdot)$ be a distance in the Lie algebra L and $d(\cdot,\cdot)$ the corresponding Hausdorff distance between subsets of L:

$$d(\Gamma_1,\Gamma_2) = \max\left\{ \sup_{A_1\in\Gamma_1} \inf_{A_2\in\Gamma_2} \rho(A_1,A_2),\ \sup_{A_2\in\Gamma_2} \inf_{A_1\in\Gamma_1} \rho(A_1,A_2) \right\}.$$

Theorem 2.10. *If a right-invariant system $\Gamma \subset L$ is controllable, then there exists $\varepsilon > 0$ such that any system $\Gamma' \subset L$ is controllable provided that $d(\Gamma,\Gamma') < \varepsilon$.*

Proof. If Γ is controllable, then the identity e is normally accessible from e by some $A_1,\dots,A_m \in \Gamma$. For a sufficiently small $\varepsilon > 0$, any system Γ' with $d(\Gamma,\Gamma') < \varepsilon$ contains elements $A'_1,\dots,A'_m$ such that $\rho(A_i,A'_i) < \varepsilon$, $i = 1,\dots,m$. Then e is normally accessible from e by $A'_1,\dots,A'_m$. □

2.6. **Notes.** Control systems with a Lie group as a state space are studied in the mathematical control theory since the early 1970-ies.

Brockett [32] considered applied problems leading to control systems on matrix groups and their homogeneous spaces; e.g., a model of DC to DC conversion and the rigid body control raise control problems on the group of rotations of the three-space $\mathrm{SO}(3;\mathbb{R})$ and on $\mathrm{SO}(3;\mathbb{R}) \times \mathbb{R}^3$ respectively. The natural framework for such problems are matrix control systems of the form

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{m} u_i(t)B_i x(t), \quad u_i(t) \in \mathbb{R}, \tag{2.8}$$

where $x(t)$ and A, $B_1,\dots,B_m$ are $n \times n$ matrices.

The systematic mathematical study of control systems on Lie groups was initiated by Jurdjevic and Sussmann [76]. They noticed that the passage from the matrix system (2.8) to the more general right-invariant system

$$\dot{x}(t) = A(x(t)) + \sum_{i=1}^{m} u_i(t)B_i(x(t)), \quad x(t) \in G, \quad u(t) \in \mathbb{R},$$

where A, $B_1,\dots,B_m$ are right-invariant vector fields on a Lie group G, "in no essential way affects the nature of the problem". The basic properties of attainable set and orbits of right-invariant systems were established in work [76].

The notion of normal accessibility (for arbitrary nonlinear systems) is due to Sussmann [133].

3. Control systems subordinated to a group action

3.1. Transitive actions, homogeneous spaces, and controllability.

Definition 3.1. A Lie group G is said to act on an analytic manifold M if there exists an analytic mapping $\theta : G \times M \to M$ that satisfies

(1) $\theta(g_2 g_1, x) = \theta(g_2, \theta(g_1, x))$ for any g_1, g_2 in G and any x in M, and
(2) $\theta(e, x) = x$ for any x in M.

For each $g \in G$, consider the analytic diffeomorphism $\theta_g : M \to M$ given by $\theta_g(x) = \theta(g, x)$ (the inverse to θ_g is given by $\theta_{g^{-1}}$). The mapping $g \mapsto \theta_g$ is called an *action* of G on M. Any action is a homomorphism from the group G into the group of analytic diffeomorphisms of M. For any element $A \in L$, $\theta_{\exp tA}$ is a one-parameter group of diffeomorphisms of M with the generator $\theta_*(A)$ — an analytic vector field on M:

$$\theta_*(A)(x) = \left.\frac{d}{dt}\right|_{t=0} \theta_{\exp tA}(x), \quad x \in M, \ A \in L.$$

Such vector fields $\theta_*(A)$, $A \in L$ are called *subordinated to the action θ of G*. They form the finite-dimensional Lie algebra

$$\theta_*(L) = \{\theta_*(A) \mid A \in L\}$$

of complete vector fields on M.

Definition 3.2. A system of vector fields $\mathcal{F}$ on M is called *subordinated to the action θ* if $\mathcal{F}$ is contained in $\theta_*(L)$. If $\mathcal{F} = \theta_*(\Gamma)$ for some right-invariant system $\Gamma \subset L$, then $\mathcal{F}$ is called *induced* by Γ.

A Lie group G is said to act *transitively* on M if for any $x \in M$ the orbit $\{\theta_g(x) \mid g \in G\}$ coincides with the whole M. A manifold that admits a transitive action of a Lie group is called a *homogeneous space* of this Lie group. Homogeneous spaces are exactly manifolds that can be represented as quotients of Lie groups. If θ is a transitive action of G on M, consider the isotropy group H at a given point $x \in M$:

$$H = \{g \in G \mid \theta_g(x) = x\}.$$

H is a closed subgroup of G, and the manifold M is diffeomorphic to the left coset space G/H with the diffeomorphism $G/H \to M$ given by $gH \mapsto \theta_g(x)$.

Given a right-invariant system Γ on a Lie group G that acts on a manifold M, one can construct the system on M induced by Γ. The following proposition is a controllability result related to this construction.

Theorem 3.1. *Let θ be an action of a connected Lie group on a manifold M, $\Gamma \subset L$ a right-invariant system on G, and $\mathcal{F} = \theta_*(\Gamma)$ the induced system on M.*

(i) *For any point x in M, the reachable set of $\mathcal{F}$ from x is*

$$\mathbb{A}_{\mathcal{F}}(x) = \theta_{\mathbb{A}_\Gamma}(x) = \{\theta_g(x) \mid g \in \mathbb{A}_\Gamma\}.$$

(ii) *Assume that the action θ is transitive. If Γ is controllable on G, then $\mathcal{F}$ is controllable on M.*

(iii) *$\mathcal{F}$ is controllable on M if and only if the semigroup $\mathbb{A}_\Gamma$ acts transitively on M.*

Proof. (i) For any trajectory $g(t)$ of Γ and any point x in M, the curve $\theta_{g(t)}(x)$ is a trajectory of $\mathcal{F}$; moreover, any trajectory of $\mathcal{F}$ is obtained in such a way.

(ii) If $\mathbb{A}_\Gamma = G$, then $\mathbb{A}_{\mathcal{F}}(x) = M$ since the orbit of θ coincides with M.

(iii) Sufficiency follows as in (ii). And necessity is obtained from the description of the reachable set $\mathbb{A}_{\mathcal{F}}(x)$ in (i). □

Important applications of Theorem 3.1 are related to the linear action of linear groups $G \subset \mathrm{GL}(n; \mathbb{R})$ on the vector space $\mathbb{R}^n$. In this case the induced systems are bilinear or, more generally, affine systems.

3.2. Bilinear systems.

3.2.1. *Induced vector fields and systems.* For the linear action of the group $\mathrm{GL}(n;\mathbb{R})$ on the vector space $\mathbb{R}^n$

$$\theta_g(x) = gx, \quad g \in \mathrm{GL}(n;\mathbb{R}),\ x \in \mathbb{R}^n,$$

the induced vector fields are linear:

$$\theta_*(A)(x) = Ax, \quad A \in \mathfrak{gl}(n;\mathbb{R}),\ x \in \mathbb{R}^n.$$

Given any elements $A, B_1, \dots, B_m \in \mathfrak{gl}(n;\mathbb{R})$ and a control set $U \subset \mathbb{R}^m$, consider a right-invariant system on $\mathrm{GL}(n;\mathbb{R})$ affine in control:

$$\Gamma = \left\{ A + \sum_{i=1}^m u_i B_i \mid u = (u_1, \dots, u_m) \in U \subset \mathbb{R}^m \right\}.$$

Then the induced system is the set of linear vector fields on $\mathbb{R}^n$:

$$\mathcal{F} = \left\{ A + \sum_{i=1}^m u_i B_i \mid u = (u_1, \dots, u_m) \in U \subset \mathbb{R}^m \right\}.$$

Passing from polysystems to control systems in the classical notation, we obtain a bilinear system:

$$\dot{x} = Ax + \sum_{i=1}^m u_i B_i x, \quad u = (u_1, \dots, u_m) \in U \subset \mathbb{R}^m, \quad x \in \mathbb{R}^n.$$

3.2.2. *Bilinear systems on* $\mathbb{R}^n \setminus \{0\}$. Assume that the action of a connected linear group $G \subset \mathrm{GL}(n;\mathbb{R})$ is transitive on the punctured vector space $M = \mathbb{R}^n \setminus \{0\}$. The typical examples are the groups $\mathrm{GL}_+(n;\mathbb{R})$ and $\mathrm{SL}(n;\mathbb{R})$. Let L be the Lie algebra of G. The Lie algebras in the previous examples are respectively $\mathfrak{gl}(n;\mathbb{R})$ and $\mathfrak{sl}(n;\mathbb{R})$.

For this case Theorem 3.1 implies the following.

Corollary 3.1. *If a right-invariant system*

$$\Gamma = \left\{ A + \sum_{i=1}^m u_i B_i \mid u = (u_1, \dots, u_m) \in U \subset \mathbb{R}^m \right\} \subset L$$

is controllable on a linear group G that acts transitively on $\mathbb{R}^n \setminus \{0\}$, then the bilinear system

$$\dot{x} = Ax + \sum_{i=1}^m u_i B_i x, \quad u = (u_1, \dots, u_m) \in U \subset \mathbb{R}^m, \quad x \in \mathbb{R}^n \setminus \{0\}$$

is controllable on $\mathbb{R}^n \setminus \{0\}$.

3.2.3. *Bilinear systems on* S^{n-1}. Now consider the case of a connected linear group whose action is transitive on the unit sphere

$$S^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\},$$

e.g. the group $\mathrm{SO}(n;\mathbb{R})$ of rotations of the n-space. Let L be the Lie algebra of G. In the previous example the Lie algebra $\mathfrak{so}(n;\mathbb{R})$ is formed by $n \times n$ skew-symmetric matrices.

Then Theorem 3.1 yields the following.

Corollary 3.2. *If a right-invariant system*

$$\Gamma = \left\{ A + \sum_{i=1}^{m} u_i B_i \mid u = (u_1, \dots, u_m) \in U \subset \mathbb{R}^m \right\} \subset L$$

is controllable on a linear group G that acts transitively on S^{n-1}, then the bilinear system

$$\dot{x} = Ax + \sum_{i=1}^{m} u_i B_i x, \quad u = (u_1, \dots, u_m) \in U \subset \mathbb{R}^m, \quad x \in S^{n-1}$$

is controllable on the sphere S^{n-1}.

3.3. **Affine systems.**

3.3.1. *Induced vector fields and systems.* Let $\mathrm{Aff}(n;\mathbb{R})$ be the group of reversible affine transformations of the n-space. It is a semidirect product of the group of translations of the n-space with the general linear group:

$$\mathrm{Aff}(n;\mathbb{R}) = \mathbb{R}^n \ltimes \mathrm{GL}(n;\mathbb{R}).$$

This group can be represented as a subgroup of $\mathrm{GL}(n+1;\mathbb{R})$ by matrices of the form

$$\overline{X} = \begin{pmatrix} X & x \\ 0 & 1 \end{pmatrix}, \quad X \in \mathrm{GL}(n;\mathbb{R}),\ x \in \mathbb{R}^n.$$

Embedding $\mathbb{R}^n$ into $\mathbb{R}^{n+1}$ as the hyperplane

$$\mathbb{R}^n \times \{v_{n+1} = 1\} = \{(v_1, \dots, v_n, 1)^{\mathrm{T}} \in \mathbb{R}^{n+1} \mid (v_1, \dots, v_n)^{\mathrm{T}} \in \mathbb{R}^n\},$$

we obtain the affine mapping in $\mathbb{R}^n$ defined by an element $\overline{X} \in \mathrm{Aff}(n;\mathbb{R})$:

$$\begin{pmatrix} v \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} X & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v \\ 1 \end{pmatrix} = \begin{pmatrix} Xv + x \\ 1 \end{pmatrix}.$$

That is, the group $\mathrm{Aff}(n;\mathbb{R})$ acts on $\mathbb{R}^n$ as follows:

$$\theta_{\overline{X}}(v) = Xv + x, \quad \overline{X} \in \mathrm{Aff}(n;\mathbb{R}), \quad v \in \mathbb{R}^n.$$

The Lie algebra $\mathfrak{aff}(n;\mathbb{R})$ of the affine group is represented by the matrices

$$\overline{A} = \begin{pmatrix} A & a \\ 0 & 0 \end{pmatrix}, \quad A \in \mathfrak{gl}(n;\mathbb{R}),\ a \in \mathbb{R}^n.$$

The one-parameter subgroup in $\mathrm{Aff}(n;\mathbb{R})$ corresponding to $\overline{A} \in \mathfrak{aff}(n;\mathbb{R})$ is

$$\exp t \begin{pmatrix} A & a \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} e^{tA} & \frac{e^{tA} - \mathrm{Id}}{A} a \\ 0 & 1 \end{pmatrix},$$

where

$$\frac{e^{tA} - \mathrm{Id}}{A} = t \,\mathrm{Id} + \frac{t^2}{2!} A + \dots + \frac{t^n}{n!} A^{n-1} + \cdots.$$

The corresponding flow in $\mathbb{R}^n$ is

$$\theta_{\exp(t\overline{A})}(v) = e^{tA} v + \frac{e^{tA} - \mathrm{Id}}{A} a,$$

thus the induced vector field is an affine field on $\mathbb{R}^n$:

$$\theta_*(\overline{A})(v) = Av + a, \quad v \in \mathbb{R}^n.$$

Now let G be a connected linear subgroup of $\mathrm{Aff}(n;\mathbb{R})$ that acts transitively on $\mathbb{R}^n$, e.g. the group of reversible affine transformations of the n-space that preserve orientation

$$\mathrm{Aff}_+(n;\mathbb{R}) = \mathbb{R}^n \ltimes \mathrm{GL}_+(n;\mathbb{R}) = \left\{ \begin{pmatrix} X & x \\ 0 & 1 \end{pmatrix} \mid X \in \mathrm{GL}_+(n;\mathbb{R}),\ x \in \mathbb{R}^n \right\},$$

or the group of Euclidean motions of the n-space

$$\mathrm{E}(n;\mathbb{R}) = \mathbb{R}^n \ltimes \mathrm{SO}(n;\mathbb{R}) = \left\{ \begin{pmatrix} X & x \\ 0 & 1 \end{pmatrix} \mid X \in \mathrm{SO}(n;\mathbb{R}),\ x \in \mathbb{R}^n \right\}.$$

Let L be the Lie algebra of G; in the previous cases respectively we have the Lie algebras

$$\mathfrak{aff}(n;\mathbb{R}) = \mathbb{R}^n \leftthreetimes \mathfrak{gl}(n;\mathbb{R}) = \left\{ \begin{pmatrix} A & a \\ 0 & 0 \end{pmatrix} \mid A \in \mathfrak{gl}(n;\mathbb{R}),\ a \in \mathbb{R}^n \right\},$$

and

$$\mathfrak{e}(n;\mathbb{R}) = \mathbb{R}^n \leftthreetimes \mathfrak{so}(n;\mathbb{R}) = \left\{ \begin{pmatrix} A & a \\ 0 & 0 \end{pmatrix} \mid A \in \mathfrak{so}(n;\mathbb{R}),\ a \in \mathbb{R}^n \right\}.$$

A right-invariant system on the Lie group G affine in control

$$\Gamma = \left\{ \overline{A} + \sum_{i=1}^{m} u_i \overline{B}_i \mid u = (u_1, \dots, u_m) \in U \subset \mathbb{R}^m \right\} \subset L \tag{3.1}$$

with

$$\overline{A} = \begin{pmatrix} A & a \\ 0 & 0 \end{pmatrix}, \quad \overline{B}_i = \begin{pmatrix} B_i & b_i \\ 0 & 0 \end{pmatrix}, \ i = 1, \dots, m,$$

induces the following *affine* control system:

$$\dot{x} = Ax + a + \sum_{i=1}^{m} u_i (B_i x + b_i), \quad u = (u_1, \dots, u_m) \in U \subset \mathbb{R}^m, \quad x \in \mathbb{R}^n. \tag{3.2}$$

Notice that particular cases of affine systems are bilinear systems considered in Subsubsections 3.2.2 and 3.2.3 ($a = b_1 = \cdots = b_m = 0$) and the classical *linear* systems

$$\dot{x} = Ax + \sum_{i=1}^{m} u_i b_i, \quad u = (u_1, \dots, u_m) \in U \subset \mathbb{R}^m, \quad x \in \mathbb{R}^n,$$

obtained in the case $a = 0$, $B_1 = \cdots = B_m = 0$.

Now Theorem 3.1 gives the following proposition.

Corollary 3.3. *Let G be a connected linear subgroup of* $\mathrm{Aff}(n;\mathbb{R})$ *that acts transitively on $\mathbb{R}^n$. If a right-invariant system* (3.1) *is controllable on G, then the induced affine system* (3.2) *is controllable on $\mathbb{R}^n$.*

3.4. **Notes.** Control systems on homogeneous spaces subordinated to a group action (in particular, bilinear and affine systems) were among the most important motivations for the study of right-invariant systems. The contents of this section is mainly due to Brockett [32]. The terminology used and the general approach were adopted by Jurdjevic and Kupka [73].

Boothby and Wilson [25, 27] found a complete list of linear groups that act transitively on $\mathbb{R}^n \setminus \{0\}$. Moreover, they presented an algorithm involving only rational matrix operations for verification whether a Lie group generated by given matrices belongs to this list.

Lie groups that act transitively on spheres are also listed, see Samelson [121], p. 26, Borel [28, 29], Montgomery and Samelson [98].

4. Lie saturate

An efficient method to obtain (sufficient) controllability conditions for right-invariant systems is the extension technique based on the computation of the tangent cone to the closure of the attainable set of a system at the group identity.

Definition 4.1. Right-invariant systems $\Gamma_1, \Gamma_2 \subset L$ are called *equivalent* one to another if $\mathrm{cl}(\mathbb{A}_{\Gamma_1}) = \mathrm{cl}(\mathbb{A}_{\Gamma_2})$.

Definition 4.2. Let $\Gamma \subset L$ be a right-invariant system. The Lie saturate of Γ, denoted by $\mathrm{LS}(\Gamma)$, is the largest subset of $\mathrm{Lie}(\Gamma)$ equivalent to Γ.

If systems Γ_1 and Γ_2 are equivalent to Γ, then their union $\Gamma_1 \cup \Gamma_2$ is obviously equivalent to Γ. That is why the Lie saturate of Γ always exists: it is the union of all systems in $\mathrm{Lie}(\Gamma)$ equivalent to Γ. The largest right-invariant system equivalent to Γ is $\{A \in L \mid \exp(tA) \in \mathrm{cl}(\mathbb{A}_\Gamma) \ \forall t \geq 0\}$, thus the Lie saturate may be described as follows.

Theorem 4.1. *For any system* $\Gamma \subset L$,

$$\mathrm{LS}(\Gamma) = \mathrm{Lie}(\Gamma) \cap \{A \in L \mid \exp(tA) \in \mathrm{cl}(\mathbb{A}_\Gamma) \ \forall t \geq 0\}.$$

Denote by $\mathrm{E}(\Gamma)$ the set $\{A \in \mathrm{LS}(\Gamma) \mid -A \in \mathrm{LS}(\Gamma)\}$. It is the largest vector subspace of L contained in $\mathrm{LS}(\Gamma)$.

The basic properties of Lie saturate are collected in the following proposition.

Theorem 4.2. (0) $\mathrm{LS} \circ \mathrm{LS} = \mathrm{LS}$,

(1) $\mathrm{LS}(\Gamma)$ *is a closed convex positive cone in* L, *i.e.,*

  (1a) $\mathrm{LS}(\Gamma)$ *is topologically closed:*

$$\mathrm{cl}(\mathrm{LS}(\Gamma)) = \mathrm{LS}(\Gamma),$$

  (1b) $\mathrm{LS}(\Gamma)$ *is convex:*

$$A, B \in \mathrm{LS}(\Gamma) \ \Rightarrow \ \alpha A + (1-\alpha)B \in \mathrm{LS}(\Gamma) \quad \forall\, \alpha \in [0,1],$$

  (1c) $\mathrm{LS}(\Gamma)$ *is a positive cone:*

$$A \in \mathrm{LS}(\Gamma) \ \Rightarrow \ \alpha A \in \mathrm{LS}(\Gamma) \quad \forall\, \alpha \geq 0.$$

Thus,

$$A, B \in \mathrm{LS}(\Gamma) \ \Rightarrow \ \alpha A + \beta B \in \mathrm{LS}(\Gamma) \quad \forall\, \alpha, \beta \geq 0.$$

(2) *For any* $A \in \mathrm{E}(\Gamma)$ *and any* $t \in \mathbb{R}$,

$$e^{t \,\mathrm{ad}\, A}\, \mathrm{LS}(\Gamma) \subset \mathrm{LS}(\Gamma).$$

That is,

$$\pm A, B \in \mathrm{LS}(\Gamma) \ \Rightarrow \ e^{t\,\mathrm{ad}\,A} B = B + (t \,\mathrm{ad}\, A)B + \frac{(t\,\mathrm{ad}\,A)^2}{2!}B + \ldots \in \mathrm{LS}(\Gamma) \quad \forall\, t \in \mathbb{R}.$$

(3) $\mathrm{E}(\Gamma)$ *is a subalgebra of* L. *In particular,*

$$\pm A, \pm B \in \mathrm{LS}(\Gamma) \ \Rightarrow \ \pm[A,B] \in \mathrm{LS}(\Gamma).$$

(4) *If* $A \in \mathrm{LS}(\Gamma)$ *and the one-parameter subgroup* $\{\exp(tA) \mid t \in \mathbb{R}\}$ *is periodic, then* $\mathbb{R}A \subset \mathrm{LS}(\Gamma)$.

Proof. (0) is obvious in view of the definition of the Lie saturate and Theorem 4.1.

(1) follows from the well-known properties of reachable sets: $\mathbb{A}_{\mathrm{cl}(\Gamma)} \subset \mathrm{cl}(\mathbb{A}_\Gamma)$, $\mathbb{A}_{\mathrm{co}(\Gamma)} \subset \mathrm{cl}(\mathbb{A}_\Gamma)$, and $\mathbb{A}_{\mathbb{R}_+\Gamma} = \mathbb{A}_\Gamma$.

To prove (2), assume that $\pm A, B \in \mathrm{LS}(\Gamma)$. Then

$$\exp(se^{t \,\mathrm{ad}\, A} B) = \exp(s \,\mathrm{Ad}_{\exp(tA)}\, B) = \exp(tA)\exp(sB)\exp(-tA) \in \mathrm{cl}(\mathbb{A}_\Gamma)$$

for any $s \geq 0$, $t \in \mathbb{R}$, thus $e^{t \,\mathrm{ad}\, A} B \in \mathrm{LS}(\Gamma)$ for all $t \in \mathbb{R}$.

Now (3) easily follows: if $\pm A, \pm B \in \mathrm{LS}(\Gamma)$, then $\pm e^{t \,\mathrm{ad}\, A} B, \pm B \in \mathrm{LS}(\Gamma)$, that is why

$$\pm[A, B] = \pm \lim_{t \to 0} \frac{e^{t \,\mathrm{ad}\, A} B - B}{t} \in \mathrm{LS}(\Gamma).$$

(4) follows from the chain

$$\{\exp(tA) \mid t \geq 0\} = \{\exp(tA) \mid t \in \mathbb{R}\} \subset \mathbb{A}_\Gamma$$

which is valid for all $A \in \mathrm{LS}(\Gamma)$ with periodic one-parameter group. □

The following theorem gives a general controllability test in terms of the Lie saturate.

Theorem 4.3. *A right-invariant system* $\Gamma \subset L$ *is controllable on a connected Lie group* G *if and only if* $\mathrm{LS}(\Gamma) = L$.

Proof. Necessity follows from the definition of the Lie saturate.

Sufficiency. Assume that $\mathrm{LS}(\Gamma) = L$. The connected Lie group G is generated as a semigroup by the one-parameter semigroups $\{\exp(tA) \mid A \in L,\ t \geq 0\}$, thus $\mathrm{cl}(\mathbb{A}) = G$. If in addition the rank condition $\mathrm{Lie}(\Gamma) = L$ is satisfied, then Γ is controllable by Theorem 2.8. □

Usually it is hard to construct the Lie saturate of a right-invariant system explicitly. That is why Theorem 4.3 is applied as a sufficient condition of controllability via the following procedure. Starting from a given system Γ, one constructs a completely ordered ascending family of extensions $\{\Gamma_\alpha\}$ of Γ, i.e.,

$$\Gamma_0 = \Gamma, \qquad \Gamma_\alpha \subset \Gamma_\beta \text{ if } \alpha < \beta.$$

The extension rules are provided by Theorem 4.2:

(1) Given Γ_α, one constructs $\Gamma_\beta = \mathrm{cl}(\mathrm{co}(\Gamma_\alpha))$.
(2) For $\pm A, B \in \Gamma_\alpha$, one constructs $\Gamma_\beta = \Gamma_\alpha \cup e^{\mathbb{R} \,\mathrm{ad}\, A} B$.
(3) For $\pm A, \pm B \in \Gamma_\alpha$, one constructs $\Gamma_\beta = \Gamma_\alpha \cup \mathbb{R}[A, B]$.
(4) Given $A \in \Gamma_\alpha$ with periodic one-parameter group, one constructs $\Gamma_\beta = \Gamma_\alpha \cup \mathbb{R}A$.

Theorem 4.2 guarantees that all extensions Γ_α belong to $\mathrm{LS}(\Gamma)$. If one obtains the equality $\Gamma_\alpha = L$ at some step α, then $\mathrm{LS}(\Gamma) = L$ and the system Γ is controllable.

4.1. **Notes.** The idea to consider the closure of attainable sets as an invariant of right-invariant systems important in controllability questions goes back to Jurdjevic and Sussmann [76]. The concept of Lie saturate and the extension technique were developed by Jurdjevic and Kupka [73, 74].

The Lie subsemigroup theory studies general subsemigroups of Lie groups, not necessarily appearing as reachability sets of right-invariant systems. There holds a generalization of Theorem 4.2 for this case.

A subset W of a Lie algebra L is called a *wedge* if W is a closed positive convex cone in L. The *edge* of a wedge W, denoted by $H(W)$, is the maximal vector subspace of L contained in W:

$$H(W) = W \cap -W.$$

A wedge W is called a *Lie wedge* if

$$e^{\operatorname{ad} A} W \subset W \quad \text{for all } A \in H(W).$$

For a closed subsemigroup S of a Lie group G that contains the identity element e, its tangent object

$$L(S) = \{A \in L \mid \exp(tA) \in S \ \forall t \geq 0\}$$

is a Lie wedge.

The basic results on the subsemigroup theory may be found in books by Hofmann and Lawson [60], Hilgert and Neeb [53], and Hilgert, Hofmann and Lawson [52].

5. HOMOGENEOUS SYSTEMS

5.1. **Controllability criterion.** A system $\Gamma \subset L$ is called *homogeneous* if, together with any element X, this system contains also the negative $-X$, i.e.,

$$\Gamma = -\Gamma.$$

Theorem 5.1. *Let Γ be a homogeneous right-invariant system on G. Then its reachable set $\mathbb{A}$ is a subgroup of G and coincides with the orbit $\mathcal{O}$.*

Proof. Apply Lemmas 2.1 and 2.2. □

Thus deciding controllability for Γ reduces to verifying the algebraic condition of coincidence of the connected Lie groups $\mathcal{O}$ and G.

Theorem 5.2. *A homogeneous right-invariant system $\Gamma \subset L$ is controllable on a connected Lie group if and only if* $\operatorname{Lie}(\Gamma) = L$.

Proof. By Lemma 2.1, the Lie algebra of the Lie group $\mathcal{O}$ is $\operatorname{Lie}(\Gamma)$. Then apply Theorem 5.1. □

5.2. **Control affine systems.** A control affine system

$$\Gamma = \left\{ A + \sum_{i=1}^{m} u_i B_i \mid u = (u_1, \dots, u_m) \in U \subset \mathbb{R}^m \right\}$$

is homogeneous if the drift term A is equal to zero and the control set U is symmetric with respect to the origin: $U = -U$. For this case Theorems 5.1, 5.2 specify as follows.

Theorem 5.3. *Assume that a control set $U \subset \mathbb{R}^m$ satisfies $U = -U$. Consider the homogeneous control affine system*

$$\Gamma = \left\{ \sum_{i=1}^{m} u_i B_i \mid u = (u_1, \dots, u_m) \in U \subset \mathbb{R}^m \right\} \subset L$$

on a Lie group G. Then:

(i) *The reachable set $\mathbb{A}$ coincides with the orbit $\mathcal{O}$, i.e., the connected Lie subgroup of G with the Lie algebra* $\operatorname{Lie}(\Gamma)$.

(ii) *If* $U = \mathbb{R}^m$*, then any point of* $\mathbb{A}$ *can be reached from the identity* e *in an arbitrary time:*

$$\mathbb{A}(e,T) = \mathbb{A} = \mathcal{O} \text{ for any } T > 0.$$

(iii) *If* G *is connected and* $U = \mathbb{R}^m$*, then the system* Γ *is controllable if and only if* $\mathrm{Lie}(B_1, \dots, B_m) = L$.

Proof. (i) and (iii) follow respectively from Theorems 5.1 and 5.2.

To prove (ii), choose any $T > 0$. Let a point x in G be reachable from e in some time $T_1 > 0$:

$$x = \exp(t_k X_k) \cdots \exp(t_1 X_1), \qquad \sum_{i=1}^{k} t_i = T_1,$$

where $t_1, \dots, t_k > 0$ and $X_1, \dots, X_k \in \Gamma$. Since the control set $U = \mathbb{R}^m$ is homothetic with respect to the origin, the vector fields $Y_i = \alpha X_i$, $i = 1, \dots, k$, belong to Γ for $\alpha = T_1/T > 0$. Thus x can be reached from e in the time T:

$$x = \exp(s_k Y_k) \cdots \exp(s_1 Y_1), \qquad \sum_{i=1}^{k} s_i = T,$$

where $s_i = t_i/\alpha$, $i = 1, \dots, k$. □

5.3. **Notes.** The controllability criterion for homogeneous matrix systems was given by Brockett [32]. In that work the criterion was also specified for the group of matrices with positive determinant $\mathrm{GL}_+(n;\mathbb{R})$, the group of unimodular matrices $\mathrm{SL}(n;\mathbb{R})$, the group of symplectic matrices $\mathrm{Sp}(n;\mathbb{R})$, and the group of orthogonal unimodular matrices $\mathrm{SO}(n;\mathbb{R})$.

The general controllability results for homogeneous right-invariant systems on Lie groups are due to Jurdjevic and Sussmann [76].

6. COMPACT LIE GROUPS

In this section we consider the case of a Lie group that is *compact* as a topological space.

6.1. Controllability conditions.

Theorem 6.1. *A right-invariant system* $\Gamma \subset L$ *is controllable on a compact connected Lie group* G *if and only if* $\mathrm{Lie}(\Gamma) = L$.

Proof. For any right-invariant vector field $A \in L$ on a compact Lie group G the negative and positive semitrajectories satisfy the inclusion

$$\mathrm{cl}\{\exp(-tA) \mid t \geq 0\} \subset \mathrm{cl}\{\exp(tA) \mid t \geq 0\}.$$

That is why any right-invariant system Γ on G is equivalent to the homogeneous system $\Gamma \cup -\Gamma$. But for homogeneous systems controllability is equivalent to the rank condition, see Theorem 5.2. □

Theorem 6.2. *Let a Lie group* G *be compact and connected, and let a right-invariant system* $\Gamma \subset L$ *be controllable on* G*. Then there exists* $T > 0$ *such that, for every* $g_0, g_1 \in G$*, there is a control that steers* g_0 *into* g_1 *in no more than* T *units of time.*

Proof. The interiors of the reachable sets $\mathbb{A}(e, \leq t)$, $t \geq 0$, form an open covering of the group G. By compactness of G, there is an instant $T_1 > 0$ such that

$$\operatorname{int} \mathbb{A}(e, \leq T_1) = G.$$

That is, the identity element e can be steered to any element $g_1 \in G$ in no more than T_1 units of time. A similar argument applied to $-\Gamma$ shows that there exists $T_2 > 0$ such that any element $g_0 \in G$ can be steered to e in no more than T_2 units of time. Then g_0 and g_1 can be connected by a trajectory of Γ in a time no more than $T = T_1 + T_2$. □

6.2. **Examples.**

6.2.1. *Special otrhogonal group in dimension* 3. Let $G = \mathrm{SO}(3;\mathbb{R})$, the set of all 3×3 real orthogonal matrices with positive determinant. The Lie group G is compact and connected. Its Lie algebra $L = \mathfrak{so}(3;\mathbb{R})$ is the set of all 3×3 real skew-symmetric matrices.

Take any linearly independent matrices $A_1, A_2 \in \mathfrak{so}(3;\mathbb{R})$ and consider the right-invariant system $\Gamma = \{A_1, A_2\}$. Notice that the matrices A_1, A_2, $[A_1, A_2]$ span the whole Lie algebra $\mathfrak{so}(3;\mathbb{R})$. By Theorem 6.1, the system Γ is controllable. That is, any rotation in $\mathrm{SO}(3;\mathbb{R})$ can be written as the product of exponentials

$$\exp(t_k A_{i_k}) \cdots \exp(t_1 A_{i_1}), \quad t_j \geq 0, \ i_j \in \{1, 2\}, \ k \in \mathbb{N}. \tag{6.1}$$

Moreover, by Theorem 6.2, there is $T > 0$ that gives a universal upper bound $\sum_{j=1}^{k} t_j \leq T$ for decomposition (6.1) of any rotation in $\mathrm{SO}(3;\mathbb{R})$.

The single-input right-invariant affine in control system

$$\dot{X} = (A_1 + uA_2)X, \quad u \in U \subset \mathbb{R}, \ X \in \mathrm{SO}(3;\mathbb{R}) \tag{6.2}$$

is also controllable (for any control set U containing more than one element). Moreover, there is $T > 0$ such that, given any two matrices $P, Q \in \mathrm{SO}(3;\mathbb{R})$, there is a piecewise constant control that steers P into Q in no more than T units of time. Notice that, in general, there may not exist arbitrarily small numbers T with the above property, even if the control is completely unrestricted, i.e., $U = \mathbb{R}$. Take, for instance, $A_1 = E_{12} - E_{21}$, $A_2 = E_{13} - E_{31}$. Write the solution of system (6.2) with the initial condition $X(0) = \mathrm{Id}$ as $X = (x_{ij})_{i,j=1,2,3}$. Then we have

$$\begin{aligned} \dot{x}_{12} &= x_{22} + ux_{32}, \\ \dot{x}_{32} &= -ux_{12}. \end{aligned}$$

Multiplying the first equation by x_{12}, the second equation by x_{32} and adding, we get

$$\frac{1}{2}\frac{d}{dt}(x_{12}^2 + x_{32}^2) = x_{22}x_{12}.$$

Since $x_{12}^2 + x_{32}^2$ vanishes at $t = 0$, we have

$$(x_{12}^2 + x_{32}^2)(t) = 2\int_0^t x_{22}(\tau)x_{12}(\tau)\, d\tau.$$

But $x_{22}(\tau)$ and $x_{12}(\tau)$ are entries of the orthogonal matrix $X(\tau)$, hence they are bounded in absolute value by 1. Therefore, we conclude that

$$(x_{12}^2 + x_{32}^2)(t) \leq 2t.$$

This shows that a matrix (a_{ij}) for which $a_{12}^2 + a_{32}^2 = 1$ cannot be reached from the identity in less than $\frac{1}{2}$ units of time.

6.2.2. *Special otrhogonal group in dimension n.* The previous considerations are generalized to the group $G = \mathrm{SO}(n;\mathbb{R})$ of rotations of the n-space. In this case the Lie algebra L of G is the set of all $n \times n$ skew-symmetric matrices $\mathfrak{so}(n;\mathbb{R})$.

Take the matrices $A_1 = \sum_{i=1}^{n-2}(E_{i,i+1} - E_{i+1,i})$ and $A_2 = E_{n-1,n} - E_{n,n-1}$. It is easy to show that $\mathrm{Lie}(A_1, A_2) = \mathfrak{so}(n;\mathbb{R})$. Thus, even though the group $\mathrm{SO}(n;\mathbb{R})$ is $\frac{1}{2}n(n-1)$-dimensional, the system

$$\dot{X} = (A_1 + uA_2)X, \quad X \in \mathrm{SO}(n;\mathbb{R}),\ u \in U \subset \mathbb{R},$$

in which only one control is involved, is controllable (if the control set U contains at least two distinct points).

Moreover, as before, we can find an upper bound for the time necessary to reach one point in $\mathrm{SO}(n;\mathbb{R})$ from another.

Notice that the set of pairs (A_1, A_2) such that $\mathrm{Lie}(A_1, A_2) = L$ is open and dense in $L \times L$ (this is valid for any semisimple Lie algebra L, see Theorem 8.1 below). Thus we can replace above the matrices A_1, A_2 by an "almost arbitrary" pair in $L \times L$.

6.2.3. *Serret-Frenet frames.* Let $x(t)$ denote any curve in a Euclidean space $\mathbb{R}^n$ whose derivatives $d^k x(t)/dt^k$, $k = 1, \dots, n$, span an n-dimensional vector space at each point along the curve. The Serret-Frenet frame along the curve x is described by an orthonormal matrix $R(t)$ in $\mathrm{SO}(n;\mathbb{R})$ that relates this frame to a standard orthonormal frame $e_1, e_2, \dots, e_n$ in $\mathbb{R}^n$ and that further satisfies the following differential equation in $\mathrm{SO}(n;\mathbb{R})$:

$$\frac{dR}{dt} = R(t)\begin{pmatrix} 0 & -k_1(t) & 0 & \cdots & 0 \\ k_1(t) & 0 & -k_2(t) & & \vdots \\ 0 & k_2(t) & 0 & \ddots & 0 \\ \vdots & & \ddots & \ddots & -k_{n-1}(t) \\ 0 & \cdots & 0 & k_{n-1}(t) & 0 \end{pmatrix}, \tag{6.3}$$

where $k_1(t), \dots, k_{n-1}(t)$ are called the curvature functions associated with the curve x. (For curves in $\mathbb{R}^3$, k_2 is called the torsion of x.) Notice that the curvatures $k_1, \dots, k_{n-2}$ are positive, while the last curvature k_{n-1} could be of either sign. The curve

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

and the rotation matrix $R(t)$ can be expressed as the curve

$$g(t) = \begin{pmatrix} 1 & 0 \\ x(t) & R(t) \end{pmatrix}$$

in the group $\mathrm{E}(n;\mathbb{R})$ of motions of the n-space realized as the closed subgroup of $\mathrm{GL}(n+1;\mathbb{R})$ of all matrices

$$\begin{pmatrix} 1 & 0 \\ x & R \end{pmatrix}, \qquad x \in \mathbb{R}^n,\ R \in \mathrm{SO}(n;\mathbb{R}).$$

Since the first vector in the Serret-Frenet frame coincides with the tangent vector dx/dt, it follows that $dx/dt = R(t)e_1$, where $e_1 = (1, 0, \dots, 0)^{\mathrm{T}}$. Combined with system (6.3) for the orientation matrix $R(t)$, this gives the following left-invariant control affine system in $\mathrm{E}(n;\mathbb{R})$:

$$\frac{dg}{dt} = g(t)\begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & -k_1(t) & 0 & \cdots & 0 \\ 0 & k_1(t) & 0 & -k_2(t) & & \vdots \\ 0 & 0 & k_2(t) & 0 & \ddots & 0 \\ \vdots & \vdots & & \ddots & \ddots & -k_{n-1}(t) \\ 0 & 0 & \cdots & 0 & k_{n-1}(t) & 0 \end{pmatrix}, \tag{6.4}$$

with $k_1, \dots, k_{n-1}$ playing the role of controls.

Consider the extreme case where all but one curvatures are constant. Then equation (6.3) can be written as the control affine system

$$\frac{dR}{dt} = R(t)(A + uB), \qquad R \in \mathrm{SO}(n;\mathbb{R}),\ u \geq 0, \tag{6.5}$$

where $u(t) = k_i(t)$ is the nonconstant curvature (we assume that $1 \leq i \leq n-2$; in the case $i = n-1$ the control should be unrestricted: $u \in \mathbb{R}$), and

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \tag{6.6}$$

$$A_1 = \begin{pmatrix} 0 & -k_1 & 0 & \cdots & 0 \\ k_1 & 0 & -k_2 & & \vdots \\ 0 & k_2 & 0 & \ddots & 0 \\ \vdots & & \ddots & \ddots & -k_{i-1} \\ 0 & \cdots & 0 & k_{i-1} & 0 \end{pmatrix}, \tag{6.7}$$

$$A_2 = \begin{pmatrix} 0 & -k_{i+1} & 0 & \cdots & 0 \\ k_{i+1} & 0 & -k_{i+2} & & \vdots \\ 0 & k_{i+2} & 0 & \ddots & 0 \\ \vdots & & \ddots & \ddots & -k_{n-1} \\ 0 & \cdots & 0 & k_{n-1} & 0 \end{pmatrix}, \tag{6.8}$$

and

$$B = \begin{pmatrix} 0 & \cdots & & & \cdots & 0 \\ \vdots & \ddots & & & & \vdots \\ & & 0 & -1 & & \\ & & 1 & 0 & & \\ \vdots & & & & \ddots & \vdots \\ 0 & \cdots & & & \cdots & 0 \end{pmatrix} = E_{i+1,i} - E_{i,i+1}. \tag{6.9}$$

Writing $h(t)$ for $R^{-1}(t)$ turns the left-invariant system (6.5) into the right-invariant system

$$\frac{dh}{dt} = -(A + uB)h(t), \qquad h \in \mathrm{SO}(n;\mathbb{R}),\ u \geq 0, \tag{6.10}$$

which will be called the *Serret-Frenet system.* It follows from Theorem 6.1 that system (6.10) is controllable if and only if the set $\Gamma = \{-A - uB \mid u \geq 0\}$ generates $\mathfrak{so}(n;\mathbb{R})$ as a Lie algebra, i.e., $\mathrm{Lie}(A,B) = \mathfrak{so}(n;\mathbb{R})$. A description of the Lie algebra $\mathrm{Lie}(A,B)$ is given in the following proposition.

Theorem 6.3. *Suppose that each fixed curvature k_j, $j \neq i$, in equalities* (6.7), (6.8) *is nonzero. The Lie algebra generated by the matrices A and B given by* (6.6)–(6.9) *is equal to $\mathfrak{so}(n;\mathbb{R})$ in all but one case. The exceptional case occurs when $n = 2m$, $i = m$, and $k_1 = \cdots = k_{m-1} = k_{m+1} = \cdots = k_{n-1}$. The Lie algebra in the exceptional case is equal to the Lie algebra of the unitary group* $\mathrm{U}(2m;\mathbb{R})$.

6.3. **Homogeneous spaces.**

6.3.1. *Sphere.* The $(n-1)$-sphere

$$S^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\}$$

is a homogeneous space of the group $\mathrm{SO}(n;\mathbb{R})$ of rotations of the n-space.

Let A, B_1, ..., B_m be $n \times n$ skew-symmetric matrices. The control affine right-invariant system

$$\dot{X} = (A + \sum_{i=1}^{m} u_i B_i)X, \quad X \in \mathrm{SO}(n;\mathbb{R}),\ u = (u_1, \dots, u_m) \in \mathbb{R}^m \tag{6.11}$$

induces the bilinear system

$$\dot{x} = (A + \sum_{i=1}^{m} u_i B_i)x, \quad x \in S^{n-1},\ u = (u_1, \dots, u_m) \in \mathbb{R}^m, \tag{6.12}$$

one can think of the unit vector $x \in S^{n-1}$ as the first column of the orthogonal matrix $X \in \mathrm{SO}(n;\mathbb{R})$. Theorems 6.1, 6.2 imply the following proposition.

Corollary 6.1. *Let matrices $A, B_1, \dots, B_m \in \mathfrak{so}(n;\mathbb{R})$ generate $\mathfrak{so}(n;\mathbb{R})$ as a Lie algebra. Then system* (6.12) *is globally controllable on the sphere S^{n-1}. Moreover, there exists $T > 0$ such that any points $x_0, x_1 \in S^{n-1}$ can be connected by a trajectory of* (6.12) *corresponding to a piecewise constant control in no more than T units of time.*

Remark. System (6.12) is globally controllable on the sphere S^{n-1} if and only if the reachable set $\mathbb{A}$ of system (6.11), which is always a subgroup of $\mathrm{SO}(n;\mathbb{R})$, acts transitively on S^{n-1}.

Each group $\mathrm{SO}(n;\mathbb{R})$ and $\mathrm{U}(2m;\mathbb{R})$, $2m = n$, acts linearly on $\mathbb{R}^n$ by left multiplication, and these actions are transitive on the spheres in $\mathbb{R}^n$. That is why Theorems 6.1, 6.3 yield the following.

Corollary 6.2. *Let matrices A and B be given by* (6.6)–(6.9). *If all curvatures k_j, $j \neq i$, are nonzero, then the bilinear system*

$$\dot{x} = Ax + uBx, \qquad x \in S^{n-1},\ u \geq 0$$

is controllable on the sphere S^{n-1}.

6.3.2. *Grassmann manifolds.* The Grassmann manifold $G(k,n)$ consists of all k-dimensional vector subspaces of $\mathbb{R}^n$. The manifold structure on $G(k,n)$ can be introduced by embedding it into the *orthogonal group*

$$\mathrm{O}(n;\mathbb{R}) = \{X \in \mathrm{M}(n;\mathbb{R}) \mid X^{\mathrm{T}} = X^{-1}\}.$$

Each k-dimensional subspace $S \in G(k,n)$ can be identified with the orthogonal reflection $P_S \in \mathrm{O}(n;\mathbb{R})$ given by $P_S(x) = x$ for $x \in S$ and $P_S(x) = -x$ for x in the orthogonal complement of S. The requirement that the correspondence $S \mapsto P_S$ is a homeomorphism turns $G(k,n)$ into a topological space. Since $\{P_S \mid S \in G(k,n)\}$ is a closed subset of the compact Lie group $\mathrm{O}(n;\mathbb{R})$, then $G(k,n)$ is compact.

The group $\mathrm{O}(n;\mathbb{R})$ acts on $G(k,n)$ in a natural way: for any $S \in G(k,n)$ and any $R \in \mathrm{O}(n;\mathbb{R})$, the subspace $RS = \{Rx \mid x \in S\}$ is an element of $G(k,n)$. This action is transitive. In terms of the correspondence $S \mapsto P_S$ it is expressed by $RS \mapsto RP_SR^{\mathrm{T}}$, with R^{T} equal to the transpose of R. The isotropy group is $H = \mathrm{O}(n-k;\mathbb{R}) \times \mathrm{O}(k;\mathbb{R})$, thus $\dim G(k,n) = \dim \mathrm{O}(n;\mathbb{R}) - \dim H = k(n-k)$.

For any skew-symmetric matrix A and each S in $G(k,n)$,

$$\left.\frac{d}{dt}\right|_{t=0} (\exp tA)P_S(\exp -tA) = AP_S - P_SA = [A, P_S].$$

That is, $X_A(S) = [A, P_S]$ is the infinitesimal generator of the one-parameter group of isomorphisms induced by A.

Let $\mathcal{P}$ denote the set of all vector fields X_A in $G(k,n)$ with A in $\mathfrak{so}(n;\mathbb{R})$. $\mathcal{P}$ is the family of vector fields in $G(k,n)$ subordinated to the group action of $\mathrm{O}(n;\mathbb{R})$ on $G(k,n)$. A description of the reachable set $\mathbb{A}_{\mathcal{F}}$ of any subfamily $\mathcal{F} \subset \mathcal{P}$ is obtained from Theorems 3.1, 6.1.

Theorem 6.4.

(a) *The reachable set $\mathbb{A}_{\mathcal{F}}(x)$ of $\mathcal{F}$ from any point $x \in G(k,n)$ is equal to the orbit of $\mathcal{F}$ through x.*

(b) *Let Γ denote the set of all matrices A such that X_A is in $\mathcal{F}$. Then $\mathbb{A}_{\mathcal{F}}(x) = Gx = \{gxg^{\mathrm{T}} \mid g \in G\}$, with G equal to $\mathbb{A}_\Gamma$, i.e., the subgroup of* $\mathrm{SO}(n;\mathbb{R})$ *generated by* $\{\exp tA \mid A \in \Gamma, t \in \mathbb{R}\}$.

(c) *$\mathcal{F}$ is controllable on $G(k,n)$ if and only if G acts transitively on $G(k,n)$.*

6.4. **Notes.** The controllability results in Subsection 6.1 and their application to the group of rotations in Subsubsections 6.2.1, 6.2.2 are due to Jurdjevic and Sussmann [76].

Serret-Frenet frames (Subsubsection 6.2.3) were studied by Jurdjevic [72]. The proof of Theorem 6.3 about the Lie algebra generated in the control problem on $\mathrm{SO}(n;\mathbb{R})$ with one fixed curvature may be found in [67].

The applications to Grassmann manifolds (Subsubsection 6.3.2) is also due to Jurdjevic [72].

By the argument of Subsubsection 6.2.1, Theorems 6.1 and 6.2 may be viewed as results describing generation of compact Lie groups. Related results on generation of both compact and noncompact classical Lie groups may be found in Subsection 8.6 and in works by Crouch and Silva Leite [38], Silva Leite [125, 126, 127, 128], Albuquerque and Silva Leite [5].

7. Semidirect products of Lie groups

In this section we consider the case of a Lie group G that is a semidirect product of a vector space V with a Lie group K. If K is compact, then complete controllability results are available; in particular, if the Lie group K has no nonzero fixed points in the space V, then the rank condition is equivalent to controllability. If K is not compact, then controllability conditions are obtained by considering compact subgroups of K.

Let K and V be Lie groups, and let K act on V. Consider the *semidirect product* $G = V \ltimes K$. The manifold G is the Cartesian product of V and K, and the group product in G is defined by

$$(v_1, k_1) \cdot (v_2, k_2) = (v_1 + k_1 v_2, k_1 k_2), \quad v_1, v_2 \in V,\ k_1, k_2 \in K.$$

The Lie algebra L of G is the *semidirect sum* $L(V) \rightthreetimes L(K)$, where $L(V)$ and $L(K)$ are Lie algebras of V and K respectively. The vector space L is a direct sum of the vector spaces $L(V)$ and $L(K)$, and the Lie algebra product in L is as follows:

$$[(a_1, b_1), (a_2, b_2)] = ([a_1, a_2] + b_1(a_2) - b_2(a_1), [b_1, b_2]), \ a_1, a_2 \in L(V),\ b_1, b_2 \in L(K).$$

Denote the projections from G onto the factors V and K respectively by τ and π:

$$\begin{aligned} \tau &: G \to V, \quad \tau(v, k) = v, \qquad v \in V,\ k \in K, \\ \pi &: G \to K, \quad \pi(v, k) = k, \qquad v \in V,\ k \in K. \end{aligned}$$

The projection π is a Lie group homomorphism. Denote by $L(K)$ the Lie algebra of K. The differential

$$\pi_* : L \to L(K)$$

is a Lie algebra homomorphism.

Throughout this section we assume that V is a vector Lie group, i.e., a finite-dimensional real vector space considered as an abelian Lie group. In addition, we suppose that the action of the Lie group K on the vector space V is linear.

Definition 7.1. We say that $v \in V$ is a *fixed point* under K if

$$Kv = \{gv \mid g \in K\} = \{v\}.$$

We write this $Kv = v$.

Notice that the origin $0 \in V$ is a fixed point for any linear action on V.

7.1. K compact and admits no nonzero fixed points in V.

In this subsection we prove the following result, which can be considered as a generalization of the controllability test for compact Lie groups (Theorem 6.1).

Theorem 7.1. *Let a compact connected Lie group K act linearly on a vector space V and let V admit no nonzero fixed points with respect to K. Then a right-invariant system $\Gamma \subset L$ is controllable on the Lie group $G = V \ltimes K$ if and only if* $\mathrm{Lie}(\Gamma) = L$.

7.1.1. *The proof of Theorem* 7.1 *in particular cases.* Before proving the theorem in its full generality, we give a shorter proof for the most interesting in applications cases $G = \mathrm{E}(n; \mathbb{R}) = \mathbb{R}^n \ltimes \mathrm{SO}(n; \mathbb{R})$ and $G = \mathbb{R}^{2m} \ltimes \mathrm{U}(2m; \mathbb{R})$.

Proof. The rank condition $\mathrm{Lie}(\Gamma) = L$ is necessary for controllability of Γ by Theorem 2.3.

Assume that $\mathrm{Lie}(\Gamma) = L$. Then the right-invariant system $\Gamma_K = \pi_*(\Gamma)$ on K is controllable since K is compact and connected, see Theorem 6.1. That is,

$$\pi(\mathbb{A}) = K. \tag{7.1}$$

It follows from Theorem 2.7 that it is sufficient to show that the identity $e = (0, \mathrm{Id}) \in G$ is contained in the interior of $\mathbb{A}$. Let (x, k) be a point in the interior of $\mathbb{A}$, which is nonempty by the rank condition. In view of (7.1), there exists $y \in V$ such that (y, k^{-1}) is contained in $\mathbb{A}$. Then $(x, k)(y, k^{-1}) = (x + ky, \mathrm{Id})$, and this product is in the interior of $\mathbb{A}$.

Denote $x + ky$ by v. Let Ω be a neighborhood of Id in K such that $(v, \Omega) \subset \operatorname{int} \mathbb{A}$.

For any $h \in \Omega$ and $n \in \mathbb{N}$, the element $(v, h)^n = (v + hv + \cdots + h^{n-1}v, h^n)$ is contained in the interior of $\mathbb{A}$. If $h^n = \mathrm{Id}$, and if $v = hw - w$ for some $w \in V$, then $v + hv + \cdots + h^{n-1}v = 0$, and $e = (0, \mathrm{Id})$ is contained in the interior of $\mathbb{A}$. To finish the proof, we have to show in either of the two cases ($K = \mathrm{SO}(n; \mathbb{R})$, $V = \mathbb{R}^n$, and $K = \mathrm{U}(2m; \mathbb{R})$, $V = \mathbb{R}^{2m}$) that for any $v \in V$ and any neighborhood Ω of Id in K there exists an element h in Ω such that $v \in \mathrm{Im}(h - \mathrm{Id})$ and $h^m = \mathrm{Id}$ for some $m \in \mathbb{N}$.

We outline a proof for the first case, for the second one it is analogous. Let P be a plane in $\mathbb{R}^n$, $n \geq 2$, that contains a given point $v \in \mathbb{R}^n$. Then for any neighborhood Ω of Id in $\mathrm{SO}(2; P)$ there exists a rotation $R \in \Omega$ such that $R - \mathrm{Id}$ is nonsingular and $R^m = \mathrm{Id}$ for some $m \in \mathbb{N}$. Then R can be extended to $\mathbb{R}^n$ by defining it equal to the identity on the orthogonal complement of P in $\mathbb{R}^n$. Hence $v \in \mathrm{Im}(R - \mathrm{Id})$ and $R^m = \mathrm{Id}$. ☐

7.1.2. *The proof of Theorem* 7.1 *in the general case.* We first obtain several auxiliary propositions under the condition that $\mathrm{Lie}(\Gamma) = L$.

Lemma 7.1. $\pi(\mathbb{A}) = K$.

Proof. The projected system $\Gamma_K = \pi_*(\Gamma)$ is a full-rank right-invariant system on the compact connected Lie group K, hence it is controllable on K, see Theorem 6.1. ☐

In the next three lemmas we study the following subset of G:

$$T = \{(v, \mathrm{Id}) \mid (v, \mathrm{Id}) \in \operatorname{int} \mathbb{A}\}. \tag{7.2}$$

Lemma 7.2. *T is nonempty.*

Proof. By Theorem 2.4, the system Γ is accessible, i.e., the interior of $\mathbb{A}$ is nonempty. Take any $(w, g) \in \operatorname{int} \mathbb{A}$. By virtue of Lemma 7.1, there exists $v \in V$ such that $(v, g^{-1}) \in \mathbb{A}$. Then

$$(w, g) \cdot (v, g^{-1}) = (w + gv, \mathrm{Id}) \in \operatorname{int} \mathbb{A}.$$

Hence, T is nonempty. ☐

Lemma 7.3. *For each $(v, \mathrm{Id}) \in T$ there exists an integer $N > 0$ such that $(\lambda v, \mathrm{Id}) \in T$ for all λ with $\lambda > N$.*

Proof. If $(v, \mathrm{Id}) \in \operatorname{int} \mathbb{A}$, then there exist $\varepsilon > 0$ such that $((1 + \lambda)v, \mathrm{Id}) \in \operatorname{int} \mathbb{A}$ for all λ with $|\lambda| < \varepsilon$. Hence $((1 + \lambda)v, \mathrm{Id})^n = (n(1 + \lambda)v, \mathrm{Id}) \in \operatorname{int} \mathbb{A}$ for each integer $n > 0$. Let N be any integer with $N(1 + \varepsilon) > 1$. Then the closed real interval $[N, N + 1]$ has the property that $(\lambda v, \mathrm{Id}) \in \operatorname{int} \mathbb{A}$ for all $\lambda \in [N, N + 1]$. But by the semigroup property of $\mathbb{A}$, then the whole real ray $\{\lambda \mid \lambda > N\}$ has such property. ☐

Lemma 7.4. *For each* $(v, \mathrm{Id}) \in T$ *and each* $g \in K$ *there exists an integer* $M > 0$ *such that* $(Mgv, \mathrm{Id}) \in T$.

Proof. For each $g \in K$, by Lemma 7.1 there exist vectors $v_g, v_{g^{-1}} \in V$ such that (v_g, g) and $(v_{g^{-1}}, g^{-1})$ belong to $\mathbb{A}$. Hence $(v_{g^{-1}}, g^{-1}) \cdot (v_g, g) = (g^{-1}v_g + v_{g^{-1}}, \mathrm{Id})$ belongs to $\mathbb{A}$.

If $(v, \mathrm{Id}) \in \operatorname{int} \mathbb{A}$, then let $M > 0$ be any integer such that

$$\left(v - M^{-1}\left(v_{g^{-1}} + g^{-1}v_g\right), \mathrm{Id}\right)$$

belongs to $\operatorname{int} \mathbb{A}$. Therefore,

$$\left(v - M^{-1}\left(v_{g^{-1}} + g^{-1}v_g\right), \mathrm{Id}\right)^M = \left(Mv - \left(v_{g^{-1}} + g^{-1}v_g\right), \mathrm{Id}\right) \in \operatorname{int} \mathbb{A}.$$

But then,

$$(v_g, g) \cdot \left(Mv - \left(v_{g^{-1}} + g^{-1}v_g\right), \mathrm{Id}\right) \cdot \left(v_{g^{-1}}, g^{-1}\right) = (Mgv, \mathrm{Id})$$

belongs to $\operatorname{int} \mathbb{A}$. □

Now we prove Theorem 7.1.

Proof. The rank condition $\mathrm{Lie}(\Gamma) = L$ is necessary for controllability by Theorem 2.3. In order to prove the sufficiency, assume that $\mathrm{Lie}(\Gamma) = L$.

By Lemma 7.2, there exists a vector $v \in V$ such that $(v, \mathrm{Id}) \in \operatorname{int} \mathbb{A}$. Let

$$\bar{v} = \int_K Kv \, d\mu,$$

where μ is the Haar measure on K such that $\mu(K) = 1$. Then $K\bar{v} = \bar{v}$, and by the hypothesis of the theorem, $\bar{v} = 0$.

On the other hand, the mean $\int_K Kv \, d\mu$ is contained in the convex cone generated by the set $\{gv \mid g \in K\}$, that is why

$$0 = \bar{v} = \sum_{j=1}^{p} \lambda_j g_j v \quad \text{for some } g_1, \dots, g_p \in K, \quad \lambda_1 > 0, \dots, \lambda_p > 0.$$

By Lemma 7.4, there exist integers $M_1 > 0, \dots, M_p > 0$ such that $(M_j \lambda_j g_j v, \mathrm{Id}) \in \operatorname{int} \mathbb{A}$ for each $j = 1, \dots, p$. Then for $M = \prod_{j=1}^{p} M_j$ we have $(M\lambda_j g_j v, \mathrm{Id}) \in \operatorname{int} \mathbb{A}$ for $j = 1, \dots, p$. Thus

$$e = (0, \mathrm{Id}) = (M\bar{v}, \mathrm{Id}) = \left(\sum_{j=1}^{p} M\lambda_j g_j v, \mathrm{Id}\right) = \prod_{j=1}^{p} (M\lambda_j g_j v, \mathrm{Id}) \in \operatorname{int} \mathbb{A}.$$

By Theorem 2.7, the system Γ is controllable on G. □

7.1.3. *The rank condition and irreducible actions.* A particular case covered by Theorem 7.1 is the case where K acts irreducibly on V. The following theorem deals with that case, and it gives a criterion which ensures that $\mathrm{Lie}(\Gamma) = L$ for a given subset Γ of L. For that purpose we consider the following construction.

Since the Lie group K acts linearly on the vector space V, then the group $G = V \ltimes K$ acts affinely on V:

$$(v, k)x = kx + v, \qquad (v, k) \in G, \ x \in V.$$

For each $M = (v, A) \in L$ and for each $x \in V$, $\{(\exp tM)x \mid t \in \mathbb{R}\}$ is a one-parameter group on V whose infinitesimal generator in the affine vector field $x \mapsto Ax + v$.

Definition 7.2. Given a subset $\Gamma \subset L$, then $\mathcal{F}(\Gamma)$ is the set of affine vector fields on V induced by Γ, i.e., $X \in \mathcal{F}(\Gamma)$ if and only if $X(x) = Ax + v$ for some $(v, A) \in \Gamma$.

We denote by $\mathcal{F}_x(\Gamma)$ the set $\{X(x) \mid X \in \mathcal{F}(\Gamma)\}$. Then we have the following proposition.

Theorem 7.2. *Suppose that K is a connected, compact, semisimple real Lie group which acts linearly and irreducibly on a vector space V. Let $G = V \ltimes K$ and let $\Gamma \subset L$. Then a necessary and sufficient condition for* $\mathrm{Lie}(\Gamma) = L$ *is that*

(i) $\mathrm{Lie}(\Gamma_K) = \mathrm{Lie}(\pi_*(\Gamma)) = L(K)$ *and*
(ii) $\mathcal{F}_x(\Gamma) \neq \{0\}$ *for all* $x \in V$.

Proof. Denote by $\mathcal{O}(\mathcal{F})(x)$ the orbit of $\mathcal{F}(\Gamma)$ through $x \in V$, i.e., the action of the group generated by $\{\exp tX \mid t \in \mathbb{R}, X \in \mathcal{F}(\Gamma)\}$. Let H denote the orbit $\mathcal{O}_\Gamma$, i.e., the subgroup of G generated by $\{\exp tA \mid t \in \mathbb{R}, A \in \Gamma\}$. Then $\mathcal{O}(\mathcal{F})(x) = Hx$.

If $\mathrm{Lie}(\Gamma) = L$, then $H = G$ since G is connected. Thus the orbits of $\mathcal{F}(\Gamma)$ through each point $x \in V$ are given by Gx. But $Gx \neq x$ for any $x \in V$, therefore for each $x \in V$ there exists $X \in \mathcal{F}(\Gamma)$ such that $X(x) \neq 0$. That is, condition (ii) holds. Since condition (i) is obviously satisfied, the necessity follows.

To prove the sufficiency, suppose that (i) and (ii) hold. Let π_Γ be the restriction of the projection π on $\mathrm{Lie}(\Gamma)$. Thus, $\pi_\Gamma : \mathrm{Lie}(\Gamma) \to L(K)$ is a Lie algebra homomorphism. By condition (i), π_Γ is onto. Since $\ker \pi_\Gamma$ is an ideal of $\mathrm{Lie}(\Gamma)$, and since π_Γ is onto, it follows that $\ker \pi_\Gamma$ is a linear subspace of V invariant under K. By the irreducibility assumption, either $\ker \pi_\Gamma = V$, or $\ker \pi_\Gamma = \{0\}$.

If $\ker \pi_\Gamma = V$, then obviously $\mathrm{Lie}(\Gamma) = L$. To complete the proof we show that the case $\ker \pi_\Gamma = \{0\}$ is impossible. If $\ker \pi_\Gamma = \{0\}$, then $\mathrm{Lie}(\Gamma)$ is isomorphic with $L(K)$. Since K is semisimple and compact, it follows that the integral group H of $\mathrm{Lie}(\Gamma)$ is compact. For any $x \in V$ the mean $\bar{x} = \int_H hx\, d\mu$ is a fixed point of H (μ is the normalized Haar measure on H). Then $\mathcal{F}_{\bar{x}} = 0$, which contradicts assumption (ii). Thus, $\ker \pi_\Gamma \neq \{0\}$ and the proof is complete. □

7.2. K compact and has nonzero fixed points in V.

If the linear action of a compact Lie group K has nonzero fixed points in V, then the rank condition is no longer sufficient for controllability.

Example 7.1. Let $K = \mathrm{SO}(1;\mathbb{R}) \times \mathrm{SO}(n;\mathbb{R})$, and let $V = \mathbb{R} \times \mathbb{R}^n$. The compact connected Lie group K acts naturally on the vector space V:

$$(1, g)(x, y) = (x, gy), \quad (1, g) \in K,\ (x, y) \in V.$$

For this action $Kv = v$ if and only if $v = (x, 0)$.

We take the Lie group $G = V \ltimes K$ and the right-invariant system on it:

$$\Gamma = \{(v, A) \mid v = (x, y),\ x > 0,\ A \in L(K)\}.$$

Then,

(i) $\mathrm{Lie}(\Gamma) = L$ and
(ii) $\mathbb{A} = \{(v, g) \mid v = (x, y),\ x > 0,\ g \in K\}$.

Hence, Γ is not controllable even though it has the full rank.

Now we obtain controllability conditions for the case when the action of a compact connected group K has nonzero fixed points in a vector space V. Denote by $\langle \cdot, \cdot \rangle$ the inner product on V invariant under K, and by d_V the corresponding metric on V. If d_K denotes the left- and right-invariant metric on K, we let d_G be the corresponding direct product metric on $G = V \ltimes K$:

$$d_G((v_1, g_1), (v_2, g_2)) = d_K(g_1, g_2) + d_V(v_1, v_2), \qquad (v_1, g_1), (v_2, g_2) \in G.$$

If $(w, h) \in G$, then

$$\begin{aligned} d_G((w,h)(v_1, g_1), (w,h)(v_2, g_2)) &= d_G((w + hv_1, hg_1), (w + hv_2, hg_2)) \\ = d_K(hg_1, hg_2) + d_V(hv_1, hv_2) &= d_K(g_1, g_2) + d_V(v_1, v_2). \end{aligned}$$

Thus, d_G is left-invariant.

We denote

$$\begin{aligned} V_1 &= \{v \in V \mid Kv = v\}, \\ V_2 &= V_1^{\perp}. \end{aligned}$$

It follows from the definitions of the subspace V_1 that for any $X \in L(K)$ and any $v \in V_1$ we have $Xv = 0$. Moreover, if $X \in L(K)$ and $w \in V_2$, then

$$\langle v, Xw \rangle = -\langle Xv, w \rangle = 0 \qquad \text{for all } v \in V_1.$$

Thus, both V_1 and V_2 are invariant under the elements of $L(K)$. Let P denote the orthogonal projection of V onto V_1. Recall that τ is the canonical projection of G onto V, thus τ_* is the projection of L onto V. Denote by Γ_V the projection $\tau_*(\Gamma)$ of a right-invariant system $\Gamma \subset L$. We now have the following

Theorem 7.3. *Let a compact connected Lie group K act linearly on a vector space V. Then a right-invariant system $\Gamma \subset L$ is controllable on the Lie group $G = V \ltimes K$ if and only if:*

(i) $\mathrm{Lie}(\Gamma) = L$ *and*
(ii) *the convex cone spanned by $P(\Gamma_V)$ is equal to V_1.*

Proof. We first prove the necessity. If $(a, A) \in \Gamma$, then $(a, A) = (a_1, 0) \oplus (a_2, A)$, where $a_1 = Pa$ and $a_2 = a - a_1$; the sign $\oplus$ means that the elements $(a_1, 0)$ and (a_2, A) commute. Hence

$$\exp t(a, A) = (a_1 t, \mathrm{Id})(a_2(t), \exp tA), \qquad \text{where } a_2(t) \in V_2 \text{ for all } t$$

since $AV_2 \subset V_2$.

It is now clear that if $Y = (b, B)$ is another element of Γ, then

$$\begin{aligned} &\exp t_2(b, B) \cdot \exp t_1(a_1, A) \\ &\quad = (a_1 t_1 + b_1 t_2, \mathrm{Id})(b_2(t_2) + (\exp t_2 B) a_2(t), \exp t_2 B \cdot \exp t_1 A). \end{aligned}$$

Thus, the projection of $\mathbb{A}$ onto V_1 is equal to the convex cone spanned by $P(\Gamma_V)$. If Γ is controllable, then such a cone must be equal to V_1.

To prove the sufficiency, assume that $\mathrm{Lie}(\Gamma) = L$ and $\mathrm{co}(P(\Gamma_V)) = V_1$.

Let, as before,

$$T_\Gamma = \{(v, \mathrm{Id}) \mid (v, \mathrm{Id}) \in \mathrm{int}\, \mathbb{A}\}.$$

By Lemma 7.2, the set T_Γ is nonempty. If $(z, \mathrm{Id}) \in T_\Gamma$, then let $w = \int_K Kz\, d\mu$ where μ is the normalized Haar measure on K. We have $Kw = w$ hence $w \in V_1$. If $w = 0$, then as in the proof of Theorem 7.1 it follows that $(0, \mathrm{Id}) \in T_\Gamma$ and $\mathbb{A} = G$.

If $w \neq 0$, then there exists a positive integer N such that $(v, \mathrm{Id}) \in T_\Gamma$ for $v = Nw$. Indeed, w belongs to the convex cone spanned by the orbit Kv. Thus, $w = \sum_{j=1}^{p} \lambda_j g_j v$ for some elements $g_1 \dots , g_p$ in K and positive numbers $\lambda_1, \dots ,$

λ_p. By Lemma 7.4, there exist integers M_j such that $(M_j\lambda_j g_j v, \mathrm{Id}) \in T_\Gamma$. The required integer N can then be taken to be equal to $\prod_{j=1}^{p} M_j$.

Now we show that there exists $\lambda > 0$ such that both λv and $-\lambda v$ belong to $\operatorname{int}\mathbb{A}_{\mathrm{co}(\Gamma)}$. Since $\mathrm{co}(P(\Gamma_V)) = V_1$, there exists an element of $\mathrm{co}(\Gamma)$ of the form $X = (-v+u, A)$ where $u \in V_2$ and $A \in L(K)$.

$$\exp tX = \exp t(-v+u, A) = (-vt + u(t), \exp tA), \qquad \text{where } u(t) \in V_2 \text{ for all } t.$$

Since $(v, \mathrm{Id}) \in T_\Gamma \subset \operatorname{int}\mathbb{A}$, it follows that some ball $B((v,\mathrm{Id}), \varepsilon)$ of radius ε centered at (v, Id) is contained in $\operatorname{int}\mathbb{A}$. From the left-invariance of the metric d_G it follows that $B((\exp tX)(v,\mathrm{Id}), \varepsilon)$ is contained in $\operatorname{int}\mathbb{A}_{\mathrm{co}(\Gamma)}$. Now, K is compact, hence there exists a time $t > 1$ such that $d_K(\exp tA, \mathrm{Id}) < \varepsilon$. Therefore,

$$d_G(((1-t)v + u(t), \mathrm{Id}), ((1-t)v + u(t), \exp tA)) < \varepsilon.$$

Thus,

$$(\exp tX)(v, \mathrm{Id}) = ((1-t)v + u(t), \mathrm{Id}) \in B((\exp tX)(v, \mathrm{Id}), \varepsilon)$$

and hence,

$$((1-t)v + u(t), \mathrm{Id}) \in T_{\mathrm{co}(\Gamma)}.$$

Now, $\int_K K((1-t)v + u(t))\, d\mu = (1-t)v$, and by a preceding argument it follows that $(M(1-t)v, \mathrm{Id}) \in T_{\mathrm{co}(\Gamma)}$ for some positive integer M. Since $M(1-t) < 0$, it follows, by Lemma 7.3, that there exists $\lambda > 0$ sufficiently large that both λv and $-\lambda v$ are in $T_{\mathrm{co}(\Gamma)}$. Since $T_{\mathrm{co}(\Gamma)}$ is a semigroup, it follows that $(0, \mathrm{Id}) = (\lambda v, \mathrm{Id}) \cdot (-\lambda v, \mathrm{Id})$ is in $T_{\mathrm{co}(\Gamma)}$. Thus $T_{\mathrm{co}(\Gamma)}$ contains the identity of G. This shows that $\mathbb{A}_{\mathrm{co}(\Gamma)} = \Gamma$. But $\mathbb{A}_{\mathrm{co}(\Gamma)} \subset \operatorname{cl}\mathbb{A}$, consequently, $\operatorname{cl}\mathbb{A} = G$. Together with the assumption $\mathrm{Lie}(\Gamma) = L$ this implies that the system Γ is controllable, see Theorem 2.8. ☐

7.3. **Semidirect product of a vector space with an arbitrary Lie group.**

Theorem 7.4. *Let H be a connected Lie group which acts linearly on a finite-dimensional real vector space V, and let $G = V \ltimes H$. Assume that H contains a compact group K which has no nonzero fixed points in V. Then, a necessary and sufficient condition that a right-invariant system $\Gamma \subset L$ be controllable on G is that*

(i) $\mathrm{Lie}(\Gamma) = L$ *and*
(ii) $\Gamma_H = \pi_*(\Gamma)$ *is controllable on H.*

Proof. The conditions of the theorem are obviously necessary. To prove the sufficiency, suppose that conditions (i) and (ii) hold. By Theorem 2.4, the full-rank system Γ is accessible, i.e., $\operatorname{int}\mathbb{A}$ is nonempty. If $(v, g) \in \operatorname{int}\mathbb{A}$, then by condition (ii) there exists $w \in V$ such that $(w, g^{-1}) \in \mathbb{A}$. Thus, $(v, g) \cdot (w, g^{-1}) = (v + gw, \mathrm{Id}) \in \operatorname{int}\mathbb{A}$. Hence, the set T defined by (7.2) is nonempty. If $(v, \mathrm{Id}) \in T$, then the element $w = \int_K Kv\, d\mu$ is invariant under K and hence $w = 0$. The rest of the proof follows as in the proof of Theorem 7.1. Hence, $e = (0, \mathrm{Id}) \in T \subset \operatorname{int}\mathbb{A}$, and thus $\mathbb{A} = G$ by Theorem 2.7. ☐

The following example shows that without any assumption on the compact subgroup K conditions (i) and (ii) do not in general guarantee controllability.

Example 7.2. Let $H = \mathrm{SO}_0(n, 1)$ be the connected component through the identity of the Lorentz group in $\mathbb{R}^n$. This group, as a subgroup of $\mathrm{GL}(n+1; \mathbb{R})$, acts linearly on $V = \mathbb{R}^{n+1}$. Consider the Lie group $G = V \ltimes H$. Let C be the light cone of H in V, and let $\Gamma = C \leftthreetimes L(H)$. Then conditions (i) and (ii) are satisfied, but the attainable set is $\mathbb{A} = C \ltimes H \neq G$. In this case, the maximal compact subgroup K of H is equal to $\mathrm{SO}(n; \mathbb{R}) \times \mathrm{SO}(1; \mathbb{R})$, which has many fixed points in V.

7.4. **Homogeneous spaces.**

7.4.1. *Serret-Frenet frames in the three-space.* The Serret-Frenet system associated with a curve $x(t)$ in $\mathbb{R}^3$ (see Subsubsection 6.2.3) is given by

$$\frac{dx}{dt} = R(t)e_1, \qquad \frac{dR}{dt} = R(t)\begin{pmatrix} 0 & -k & 0 \\ k & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix}.$$

If both the curvature k and the torsion τ are constant, then

$$\omega = \begin{pmatrix} \tau \\ 0 \\ k \end{pmatrix}$$

is the axis of rotation for

$$A = \begin{pmatrix} 0 & -k & 0 \\ k & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix}.$$

Then $\exp tA$ is the rotation about ω through the angle $t\sqrt{\tau^2 + k^2}$, and $x(t)$ is a helix along ω.

Suppose now that we consider curves whose curvature $k = \mathrm{const} \neq 0$ and whose torsion can take two distinct values: τ_1 and τ_2. Such curves are concatenations of helices along

$$\omega_1 = \begin{pmatrix} \tau_1 \\ 0 \\ k \end{pmatrix} \quad \text{and} \quad \omega_2 = \begin{pmatrix} \tau_2 \\ 0 \\ k \end{pmatrix}.$$

The corresponding family of left-invariant vector fields on the Euclidean group $G = \mathrm{E}(3;\mathbb{R}) = \mathbb{R}^3 \ltimes \mathrm{SO}(3;\mathbb{R})$ is $\Gamma = \{(e_1, A), (e_1, B)\} \subset \mathfrak{e}(3;\mathbb{R}) = \mathbb{R}^3 \leftthreetimes \mathfrak{so}(3;\mathbb{R})$ with

$$A = \begin{pmatrix} 0 & -k & 0 \\ k & 0 & -\tau_1 \\ 0 & \tau_1 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & -k & 0 \\ k & 0 & -\tau_2 \\ 0 & \tau_2 & 0 \end{pmatrix}.$$

It follows that $\mathrm{Lie}(\Gamma) = \mathbb{R}^3 \leftthreetimes \mathfrak{so}(3;\mathbb{R})$ because of the following calculations:

$$(e_1, A) - (e_1, B) = (\tau_1 - \tau_2)(0, A_1)$$

and

$$[(e_1, A), (e_1, B)] = (\tau_1 - \tau_2)(0, A_2),$$

where we denote

$$A_1 = E_{32} - E_{23}, \quad A_2 = E_{13} - E_{31}, \quad A_3 = E_{21} - E_{12}.$$

Then $[(0, A_1), (0, A_2)] = (0, A_3)$, and therefore $(0, \mathfrak{so}(3;\mathbb{R})) \subset \mathrm{Lie}(\Gamma)$. Hence $(e_1, 0) \in \mathrm{Lie}(\Gamma)$, and then $[(e_1, 0), (0, \mathfrak{so}(3;\mathbb{R}))] = (\mathbb{R}^3, 0) \subset \mathrm{Lie}(\Gamma)$. Thus $\mathrm{Lie}(\Gamma) = \mathbb{R}^3 \leftthreetimes \mathfrak{so}(3;\mathbb{R}) = \mathfrak{e}(3;\mathbb{R})$.

According to Theorem 7.1, any initial point $x_0 \in \mathbb{R}^3$ and any initial frame at x_0 can be connected to any terminal point $x_1 \in \mathbb{R}^3$ and any terminal frame at x_1 along the integral curves of the left-invariant family Γ in $G = \mathrm{E}(3;\mathbb{R})$.

7.4.2. *Serret-Frenet frames in the n-space.* Results of the previous subsubsection generalize for curves in $\mathbb{R}^n$ that have all but one curvature fixed, while the remaining free curvature can take any positive value. Indeed, according to Theorem 6.3, the matrices A and B that correspond to such case generate either $\mathfrak{so}(n;\mathbb{R})$ or $\mathfrak{u}(2m;\mathbb{R})$. The corresponding control system in G is given by

$$\Gamma = \{(e_1, A + uB) \mid u > 0\}.$$

We shall now show that $\mathrm{Lie}(\Gamma) = \mathbb{R}^n \leftthreetimes L(K)$, with $L(K)$ equal to the Lie algebra of either $K = \mathrm{SO}(n;\mathbb{R})$ or $K = \mathrm{U}(2m;\mathbb{R})$. By Theorem 6.3, the projection

$$\pi_\Gamma \,:\, \mathrm{Lie}(\Gamma) \to L(K), \qquad (a, A) \mapsto A,$$

is onto. On the other hand, $\ker \pi_\Gamma$ cannot be equal to zero, for then $\mathrm{Lie}(\Gamma)$ would be isomorphic to $L(K)$, which is impossible since for the system Γ, $\mathbb{A}$ is not contained in any compact subgroup of $G = \mathbb{R}^n \ltimes K$. Thus $(a, 0) \in \mathrm{Lie}(\Gamma)$ for some $a \in \mathbb{R}^n$, $a \neq 0$. The multiplication rule in $\mathbb{R}^n \leftthreetimes L(K)$

$$[(v, 0), (b, B)] = (-Bv, 0)$$

implies that $\ker \pi_\Gamma$ is an ideal in L. Taking into account that $\mathrm{Im}\,\pi_\Gamma = L(K)$ and $L(K)a = \mathbb{R}^n$, we obtain $\mathrm{Lie}(\Gamma) = L$.

Therefore, Theorem 7.1 is applicable, and the corresponding controllability conclusions for curves in $\mathbb{R}^n$ follow as in the previous subsubsection.

7.4.3. *Affine systems on $\mathbb{R}^n$.* Consider the single-input affine system

$$\dot{x} = Ax + a + u(Bx + b), \qquad x \in \mathbb{R}^n, \ u \in U \subset \mathbb{R}, \tag{7.3}$$

where A, B are real $n \times n$ matrices, and a, b are vectors in $\mathbb{R}^n$.

Equation (7.3) can be regarded as a part of a larger system defined as follows. Denote by H the orbit of the right-invariant system

$$\{A + uB \mid u \in U\} \subset \mathfrak{gl}(n;\mathbb{R}) \tag{7.4}$$

in $\mathrm{GL}(n;\mathbb{R})$. The elements $X = (a, A)$ and $Y = (b, B)$ belong to the Lie algebra $L = \mathbb{R}^n \leftthreetimes L(H) \subset \mathfrak{aff}(n;\mathbb{R})$ of the Lie group $G = \mathbb{R}^n \ltimes H \subset \mathrm{Aff}(n;\mathbb{R})$; we denote by $L(H)$ the Lie algebra of H, it is the subalgebra of $\mathfrak{gl}(n;\mathbb{R})$ generated by set (7.4). Thus

$$\Gamma = \{X + uY \mid u \in U\} \subset L$$

can be considered as a right-invariant control system on G. The affine system (7.3) is induced by the system Γ. Moreover, the affine action of the Lie group G is transitive on $\mathbb{R}^n$ since G contains all translations. By Corollary 3.3, if the right-invariant system Γ is controllable on the Lie group G, then the affine system (7.3) is controllable on $\mathbb{R}^n$.

By the construction, for the system $\Gamma_H = \pi_*(\Gamma)$ projected onto H we have $\mathrm{Lie}(\Gamma_H) = L(H)$. That is why by Theorem 7.4 the right-invariant system Γ (and, consequently, the affine system (7.3)) is controllable if:

(i) either H is compact, or Γ_H is controllable on H, and
(ii) $\mathrm{Lie}(\Gamma) = L$.

7.5. **Notes.** The results in Subsections 7.1–7.3 are due to Bonnard, Jurdjevic, Kupka, and Sallet [24].

The proof of Theorem 7.1 for the particular cases $G = \mathbb{R}^n \ltimes \mathrm{SO}(n;\mathbb{R})$ and $\mathbb{R}^{2m} \ltimes \mathrm{U}(2m;\mathbb{R})$ in Subsubsection 7.1.1 and the applications in Subsection 7.4 were developed by Jurdjevic [72].

One of the early results on controllability of right-invariant systems on the Euclidean group was obtained by Sallet [117]. This proposition obviously follows from Theorem 7.1.

Theorem 7.5. *Let* $X_1 = (a, A)$, $X_2 = (b, B) \in \mathbb{R}^n \leftthreetimes \mathfrak{so}(n;\mathbb{R})$ *be right-invariant vector fields on the Lie group* $G = \mathrm{E}(n;\mathbb{R})$. *Then a sufficient condition that the system* $\Gamma = \{X_1, X_2\}$ *is controllable on* G *is*

(i) $\mathrm{Lie}(X_1, X_2) = L$ *and*
(ii) $a \in \mathrm{Im}\, A$ *and* $b \in \mathrm{Im}\, B$.

8. Semisimple Lie groups

A Lie algebra L is called *semisimple* if it contains no nonzero solvable ideals. A Lie group G is called *semisimple* if its Lie algebra L is semisimple. A Lie algebra L is called *simple* if it contains no nontrivial (i.e., distinct from $\{0\}$ and L) ideals. A semisimple Lie algebra is a direct sum of its simple non-abelian ideals.

In this section we assume that L is a real finite-dimensional semisimple Lie algebra.

8.1. Preliminaries.

8.1.1. *Regular elements.* For any element $B \in L$ is defined the adjoint operator

$$\mathrm{ad}\, B : L \to L, \quad \mathrm{ad}\, B(C) = [B, C], \ C \in L.$$

A Lie algebra L is semisimple if and only if the *Killing form*

$$\mathrm{Kil} : L \times L \to \mathbb{R}, \quad \mathrm{Kil}\,(A, B) = \mathrm{tr}(\mathrm{ad}\, A \circ \mathrm{ad}\, B)$$

is nondegenerate.

The roots of the characteristic polynomial

$$P_B(t) = \det(\mathrm{ad}\, B - t\,\mathrm{Id}) = (-1)^n t^n + a_1(B) t^{n-1} + a_2(B) t^{n-2} + \cdots + a_n(B),$$
$$n = \dim L,$$

are eigenvalues of the operator $\mathrm{ad}\, B$, $B \in L$, and $a_1(B), \ldots, a_n(B)$ are forms on L. Since $\mathrm{ad}\, B(B) = 0$, then $a_n(B) \equiv 0$. The smallest number r such that

$$a_{n-r+1} \equiv 0, \ a_{n-r+2} \equiv 0, \ \ldots, \ a_n \equiv 0, \qquad \text{but} \qquad a_{n-r} \not\equiv 0,$$

is called the *rank* of the Lie algebra L and is denoted by $\mathrm{rk}\, L$. An element $B \in L$ is called *regular* if

$$a_{n-r}(B) \neq 0, \quad r = \mathrm{rk}\, L.$$

For a regular element B, the zero $0 \in \mathbb{C}$ is an eigenvalue of the adjoint operator $\mathrm{ad}\, B$ with the multiplicity r, thus

$$\dim(\ker \mathrm{ad}\, B) = \mathrm{rk}\, L.$$

The set of regular elements is open and dense in L.

8.1.2. *Weyl basis and normal real form.* Let $\mathcal{L}$ be a finite-dimensional semisimple Lie algebra over $\mathbb{C}$. Let L_0 be a *Cartan subalgebra* of $\mathcal{L}$, i.e., a nilpotent subalgebra that is its own normalizer in $\mathcal{L}$. Denote by R the set of nonzero roots of $\mathcal{L}$ with respect to L_0. Then there is a decomposition of $\mathcal{L}$ into the direct sum

$$\mathcal{L} = L_0 \oplus \sum\nolimits^{\oplus} \{L_\alpha \mid \alpha \in R\},$$

where L_α, $\alpha \in R$, are root spaces, which are one-dimensional.

For any $\alpha \in R$ there exists a unique element $H_\alpha \in L_0$ such that

$$\mathrm{Kil}\,(H, H_\alpha) = \alpha(H) \text{ for all } H \in L_0.$$

Define a subspace of L_0:

$$L(0) = \sum\nolimits^{\oplus} \{\mathbb{R} H_\alpha \mid \alpha \in R\}.$$

We have

$$L_0 = L(0) \oplus iL(0).$$

One can identify R with the dual space $L(0)^*$ of $L(0)$ and then introduce an ordering of roots in R induced by some vector space ordering of $L(0)^*$. A positive root is called fundamental if it cannot be written as a sum of two positive roots. Denote by Δ^+ the set of fundamental roots.

For any root $\alpha \in R$ there exists an element $E_\alpha \in L_\alpha$ such that $\mathrm{Kil}\,(E_\alpha, E_{-\alpha}) = 1$ and for all $\alpha, \beta \in R$

$$\begin{aligned} &[E_\alpha, E_{-\alpha}] = H_\alpha, \\ &[H, E_\alpha] = \alpha(H) E_\alpha \quad \text{for all } H \in L_0, \\ &[E_\alpha, E_\beta] = \begin{cases} 0 & \text{if } \alpha + \beta \notin R, \\ N_{\alpha\beta} E_{\alpha+\beta} & \text{if } \alpha + \beta \in R, \end{cases} \end{aligned}$$

where $N_{\alpha\beta}$ are some real constants. The system

$$H_\alpha, \ \alpha \in \Delta^+, \qquad E_\alpha, \ \alpha \in R$$

is called a *Weyl basis* of $\mathcal{L}$ with respect to L_0.

The subspace

$$L = L(0) \oplus \sum\nolimits^{\oplus} \{\mathbb{R} E_\alpha \mid \alpha \in R\} \tag{8.1}$$

is a *normal real form* of the complex Lie algebra $\mathcal{L}$, unique up to an isomorphism.

8.1.3. *Strongly regular elements.* Any element $A \in L$ admits a unique decomposition

$$A = A(0) + \sum \{k_\alpha E_\alpha \mid \alpha \in R\}, \tag{8.2}$$

where

$$A(0) \in L(0), \quad k_\alpha \in \mathbb{R}.$$

Remark. An element $B \in L(0)$ is regular if and only if the elements $\alpha(B)$ are nonzero for all $\alpha \in R$.

It turns out that the following two variations of the regularity property are relevant in controllability questions.

Definition 8.1. An element $B \in L$ is called *strongly regular* if:

(i) B is regular and
(ii) every nonzero eigenvalue of $\mathrm{ad}\, B$ is simple.

Definition 8.2. Given $A \in L$ with $A \notin L(0)$ and $B \in L(0)$, the element B is called *A-strongly regular* if the elements $\alpha(B)$ are nonzero and distinct for all $\alpha \in R$ such that $k_\alpha \neq 0$ in decomposition (8.2).

Remark. To compare strong regularity and A-strong regularity, notice that an element $B \in L(0)$ is strongly regular if and only if the elements $\alpha(B)$ are nonzero and distinct for *all* roots $\alpha \in R$.

8.1.4. *Root decompositions along eigenspaces of a strongly regular element.* Choose and fix a strongly regular element $B \in L$.

The complexification $L_c = L \otimes_{\mathbb{R}} \mathbb{C}$ is a complex semisimple Lie algebra. The adjoint operator in L_c is defined by

$$\operatorname{ad}_c B : L_c \to L_c, \quad \operatorname{ad}_c B(C) = [B, C], \ C \in L_c.$$

By the strongly regular property of B, the space

$$L_0 = \ker \operatorname{ad}_c B$$

is a Cartan subalgebra of L_c. Denote by $\operatorname{Sp}(B)$ the subset of $\mathbb{C}$ of all *nonzero* eigenvalues of $\operatorname{ad} B$. Notice that there is an isomorphism

$$R \to \operatorname{Sp}(B), \qquad \alpha \mapsto \alpha(B) \tag{8.3}$$

between $\operatorname{Sp}(B)$ and R — the set of roots of L_c with respect to the Cartan subalgebra L_0. That is why we can denote the root spaces

$$L_\alpha, \quad \alpha \in R,$$

as

$$L_a, \quad a = \alpha(B) \in \operatorname{Sp}(B).$$

Notice that $L_{\overline{a}} = \sigma(L_a)$, $a \in \operatorname{Sp}(B)$, where $\overline{a}$ is the complex conjugate to an eigenvalue a and σ is the conjugation in L_c with respect to L.

For $a \in \operatorname{Sp}(B)$ consider the real space

$$L(a) = (L_a + L_{\overline{a}}) \cap L.$$

Notice that

$$\dim L(a) = 1 \text{ if } a \in \mathbb{R},$$

in this case $L(a)$ is the eigenspace of $\operatorname{ad} B$ corresponding to the eigenvalue a, and

$$\dim L(a) = 2 \text{ if } a \notin \mathbb{R},$$

then $L(a)$ is an invariant subspace of $\operatorname{ad} B$. Thus we get the decompositions into direct sums of eigenspaces and invariant spaces:

$$L_c = \ker \operatorname{ad}_c B \oplus \sum\nolimits^{\oplus} \{L_a \mid a \in \operatorname{Sp}(B)\},$$

$$L = \ker \operatorname{ad} B \oplus \sum\nolimits^{\oplus} \{L(a) \mid a \in \operatorname{Sp}(B), \ \operatorname{Im} a \geq 0\}. \tag{8.4}$$

Then any element $A \in L_c$ has a complex decomposition

$$A = A_0 + \sum \{A_a \mid a \in \operatorname{Sp}(B)\}, \quad A_0 \in \ker \operatorname{ad}_c B, \ A_a \in L_a, \tag{8.5}$$

and any $A \in L$ has a real decomposition

(8.6)
$$A = A(0) + \sum \{A(a) \mid a \in \operatorname{Sp}(B), \ \operatorname{Im} a \geq 0\}, \quad A(0) \in \ker \operatorname{ad} B, \ A(a) \in L(a).$$

8.2. **Homogeneous systems.** Now we turn to controllability conditions for right-invariant systems on semisimple Lie groups.

First of all, in semisimple Lie algebras the rank condition is generically satisfied.

Theorem 8.1. *If L is semisimple, then the set S of all pairs (A, B) in $L \times L$ for which the Lie algebra generated by A, B is equal to L is an open and dense subset of $L \times L$.*

Proof. If $\mathrm{Lie}(A, B) = L$, then the homogeneous system $\{\pm A, \pm B\}$ is controllable on G. But controllable right-invariant systems remain controllable under small perturbations (see Theorem 2.10), thus S is open.

To show that S is dense, take any strongly regular element $B \in L$ and any element A in L for which $A(a) \neq 0$ for $a \in \mathrm{Sp}(B)$ in decomposition (8.6). Such pairs form a dense subset of $L \times L$, and each pair (A, B) belongs to S. □

In the semisimple case, homogeneous systems are naturally treated more easily, as well as in the general case (see Section 5).

Theorem 8.2. *Let G be a semisimple connected Lie group. Then for a generic pair of elements A, B in L, the system $\Gamma = \{\pm A, \pm B\}$ is controllable on G.*

Proof. For a generic pair of elements A, B in the Lie algebra of G, the elements A, B generate this Lie algebra, see Theorem 8.1. By Theorem 5.2, the homogeneous system Γ is controllable. □

8.3. **Multiple-input nonhomogeneous systems.** In the case of unbounded control range, the result for multiple-input nonhomogeneous systems is an easy consequence of the proposition from the previous subsection on homogeneous systems.

Theorem 8.3. *Let G be a semisimple connected Lie group. Then for generic elements A, B_1, ..., $B_m \in L$, the system $\Gamma = \{A + \sum_{i=1}^{m} u_i B_i \mid u_i \in \mathbb{R}\}$ is controllable on G.*

Proof. The vector space $\mathrm{span}(B_1, \dots, B_m)$ is contained in the Lie saturate $\mathrm{LS}(\Gamma)$. By Theorem 8.1, the set of all m-tuples of right-invariant vector fields $(B_1, \dots, B_m)$ that generate L is open and dense. Each system Γ with such $B_1, \dots, B_m$ is controllable independently of the drift vector field A. □

8.4. **Single-input nonhomogeneous systems.** Now we consider the much more complicated case of systems of the form $\Gamma = A + \mathbb{R}B$.

8.4.1. *Formulation of theorems.* We endow the complex plane $\mathbb{C}$ with the lexicographic ordering: $a > b$ if $\mathrm{Re}\, a > \mathrm{Re}\, b$ or $\mathrm{Re}\, a = \mathrm{Re}\, b$ and $\mathrm{Im}\, a > \mathrm{Im}\, b$.

Definition 8.3. An eigenvalue $a \in \mathrm{Sp}(B)$ is called *maximal* (resp. *minimal*) if for any $b \in \mathrm{Sp}(B)$, $b > 0$ (resp. $b < 0$), we have $[L_a, L_b] = \{0\}$.

($[L_a, L_b]$ is the vector space generated by the brackets $[X, Y]$, $X \in L_a$, $Y \in L_b$.)

Theorem 8.4. *Let G be a semisimple connected Lie group with finite center and Lie algebra L. Then for $A, B \in L$ the system $\Gamma = A + \mathbb{R}B$ is controllable on G if the following conditions are satisfied:*

(1) *B is strongly regular.*
(2) $\mathrm{Lie}(A, B) = L$.

(3) *Let* $A = A_0 + \sum\{A_a \mid a \in \mathrm{Sp}(B)\}$ *be the decomposition of* A *along the eigenspaces of* $\mathrm{ad}_c B$, *see* (8.5). *Then* $A_s \neq 0$ *if* s *is either maximal or minimal.*

(4) *If* $s \in \mathrm{Sp}(B)$ *is maximal and the real part* $r = \mathrm{Re}\, s$ *is a nonzero eigenvalue of* $\mathrm{ad}_c B$, *then* $\mathrm{Kil}\,(A_r, A_{-r}) < 0$ *provided that* L_r *and* L_s *belong to the same simple ideal of* L_c.

Remark. All the conditions (1)–(4) define semialgebraic subsets of $L \times L$. Moreover the subsets defined by (1)–(3) are open and dense in $L \times L$.

To approach the proof of Theorem 8.4, consider systems $\Gamma \subset L$ satisfying the following conditions:

A) Γ is a wedge, i.e., a closed convex positive cone.

B) The largest vector subspace $E(\Gamma)$ contained in Γ called the edge of Γ is a Lie subalgebra of L.

C) For any $X \in E(\Gamma)$ and any $t \in \mathbb{R}$, $\exp(t \,\mathrm{ad}\, X)$ maps Γ into itself.

D) $E(\Gamma)$ contains a strongly regular element B.

E) If $s \in \mathrm{Sp}(B)$ and s is maximal (resp. minimal), there exists a vector X_+ (resp. X_-) in Γ such that $X_+(s) \neq 0$ (resp. $X_-(s) \neq 0$).

F) If $r \in \mathrm{Sp}(B)$ is the real part of a maximal eigenvalue s and if L_r and L_s belong to the same simple ideal of L_c, then there exist $X_+, X_- \in \Gamma$ such that $\mathrm{Kil}\,(X_+(r), X_-(-r)) < 0$.

Remark. Hypotheses A), B), C) mean that Γ is a Lie wedge, see Section 4.

Theorem 8.5. *If a system* $\Gamma \subset L$ *satisfies conditions* A) *to* F) *and if* $\mathrm{Lie}(\Gamma) = L$, *then* Γ *is controllable on* G.

There holds the following more general result.

Theorem 8.6. *If a system* $\Gamma \subset L$ *satisfies conditions* D), E), F), *and if* $\mathrm{Lie}(\Gamma) = L$, *then* Γ *is controllable on* G.

Proof. The Lie saturate $\mathrm{LS}(\Gamma)$ satisfies the assumptions of Theorem 8.5. ☐

The main controllability result for semisimple Lie groups, Theorem 8.4, easily follows from Theorem 8.5.

Proof. The Lie saturate $\mathrm{LS}(\Gamma)$ satisfies all hypotheses of Theorem 8.5. For example, to show that $\pm B \in \mathrm{E}(\mathrm{LS}(\Gamma))$, consider the limits $\lim_{u \to \pm\infty} (A + uB)/|u| = \pm B$; by property A), these limits belong to $\mathrm{LS}(\Gamma)$. ☐

8.4.2. *Proof of Theorem* 8.5. Let $\mathrm{Sim}(L)$ (resp. $\mathrm{Sim}(L_c)$) denote the set of all simple ideals in L (resp. L_c). Then we have the direct Lie algebra decompositions:

$$L = \sum\nolimits^{\oplus} \{\Sigma \mid \Sigma \in \mathrm{Sim}(L)\}, \qquad L_c = \sum\nolimits^{\oplus} \{S \mid S \in \mathrm{Sim}(L_c)\}.$$

The conjugation $\sigma : L_c \to L_c$ permutes the elements of $\mathrm{Sim}(L_c)$. The connection between $\mathrm{Sim}(L)$ and $\mathrm{Sim}(L_c)$ is as follows:

(1) $\mathrm{Sim}(L) = \{(S + \sigma S) \cap L \mid S \in \mathrm{Sim}(L_c)\}$.

(2) If $\Sigma \in \mathrm{Sim}(L)$ and Σ_c denotes its complexification, then either $\Sigma_c \in \mathrm{Sim}(L_c)$ (inner case) or $\Sigma_c = S_1 \oplus S_2$, $S_1, S_2 \in \mathrm{Sim}(L_c)$ (outer case). In the outer case Σ induces a real Lie algebra isomorphism $S_1 \to S_2$, and Σ can be identified with the graph of σ in $S_1 \oplus S_2$, i.e., $\Sigma = \{(x, \sigma(x)) \mid x \in S_1\}$.

Let $B \in L$ be a strongly regular element in L. The Cartan subalgebra $L_0 = \ker \operatorname{ad}_c B$ of L_c splits:

$$L_0 = \sum\nolimits^{\oplus} \{L_0 \cap S \mid S \in \operatorname{Sim}(L_c)\}.$$

Each $L_0 \cap S$ is a Cartan algebra of S. The root system R of L_c is the union of the root systems R_S, $S \in \operatorname{Sim}(L_c)$.

In view of bijection (8.3), we write L_α and $L(\alpha)$ instead of $L_{\alpha(B)}$ and $L(\alpha(B))$. The conjugation σ acts on R as follows: $\sigma(\alpha)(X) = \overline{\alpha(\sigma(X))}$, where $X \in L_c$.

Definition 8.4. We endow $R \cup \{0\}$ with the total order structure pull back of the order structure on $\operatorname{Sp}(B) \cup \{0\}$ by the bijection $\alpha \in R \cup \{0\} \mapsto \alpha(B) \in \operatorname{Sp}(B) \cup \{0\}$.

Definition 8.5. A root $\alpha \in R$ is *maximal* (resp. *minimal*) if $\alpha(B) \in \mathbb{C}$ is maximal (resp. minimal) in the sense of Definition 8.3.

This is equivalent to the classical definition: α is maximal (resp. minimal) if $\alpha + \beta \notin R$ whenever $\beta \in R$ and $\beta > 0$ (resp. $\beta < 0$).

Let $S \in \operatorname{Sim}(L_c)$ be a simple component of L_c and s (resp. $-s$) its maximal (resp. minimal) root.

Definition 8.6. Denote by R'_S the set of roots $\alpha \in R_S$ such that $\alpha + s$ or $\alpha - s$ is a root. R''_S will denote the complement $R_S \setminus (R'_S \cup \{s, -s\})$.

Proposition 8.1. 1) *If $\alpha, \beta \in R'_S$ have the same sign and if $\alpha + \beta \in R_S$, then $\alpha + \beta \in \{s, -s\}$.*

2) *If $\alpha \in R'_S$, $\beta \in R''_S$ and $\alpha + \beta \in R_S$, then $\alpha + \beta \in R'_S$ and α, $\alpha + \beta$ have the same sign.*

3) *If $\alpha, \beta \in R''_S$ and $\alpha + \beta \in R_S$, then $\alpha + \beta \in R''_S$.*

Corollary 8.1. 1) *If $\alpha, \beta \in R'_S$ have the same sign, then for all $\gamma \in R''_S$ such that $\alpha + \beta + \gamma \in R_S$ and either $\alpha + \gamma$ or $\beta + \gamma$ is a root, $\alpha + \beta + \gamma \in \{s, -s\}$.*

2) *If $\alpha \in R'_S$, $\beta, \gamma \in R''_S$, $\alpha + \beta + \gamma \in R_S$, and at least one of the three linear forms $\alpha + \beta$, $\beta + \gamma$, $\alpha + \gamma$ is a root, then $\alpha + \beta + \gamma \in R'_S$ and α, $\alpha + \beta + \gamma$ have the same sign.*

Now we explain the main idea of the proof of Theorem 8.5 in the case of a simple Lie algebra L.

Denote by L' the Lie algebra generated by $\{L_\alpha \mid \alpha \in R'_S \cup \{\pm s\}\}$. Consider the Lie algebra I generated by elements X of L' such that $\mathbb{R}X \subset \Gamma$.

I is nonempty; if we show that it is an ideal of L, then we obtain $\Gamma = L$.

To prove that I is an ideal we check on the generators $X(\alpha)$ of I and $Y(\beta)$ of L that $[X(\alpha), Y(\beta)] \in I$. This is mainly obtained on the basis of the properties of the sets of roots R'_S and R''_S given by Proposition 8.1 and Corollary 8.1.

In the case of a semisimple Lie algebra L, the idea of the proof of Theorem 8.5 is analogous.

8.5. **The special linear group.** The Lie group $G = \operatorname{SL}(n; \mathbb{R})$ is simple and has a trivial center, thus the results of the previous subsection apply.

Take any $A, B \in \mathfrak{sl}(n; \mathbb{R})$ and consider the right-invariant system $\Gamma = A + \mathbb{R}B$ on the group $\operatorname{SL}(n; \mathbb{R})$. The eigenvalues of $\operatorname{ad} B$ are differences of eigenvalues of B. Let $\lambda_1, \dots, \lambda_n$ denote the (possibly complex) eigenvalues of B. Strongly regular elements in $\mathfrak{sl}(n; \mathbb{R})$ are characterized by the inequalities

$$\lambda_i - \lambda_j \neq \lambda_k - \lambda_l, \quad \{i, j\} \neq \{k, l\}, \ i \neq j.$$

The eigenspace L_a of $\operatorname{ad} B$ corresponding to a real eigenvalue $a = \lambda_i - \lambda_j \in \operatorname{Sp}(B)$ consists of all matrices of the form $b_i \otimes c_j$, where b_i and c_j are respectively eigenvalues of B and its transpose B^{T}, i.e., $Bb_i = \lambda_i b_i$ and $B^{\mathrm{T}} c_j = \lambda_j c_j$. If an eigenvalue $a \in \operatorname{Sp}(B)$ is complex, then $\overline{a}$ is also an eigenvalue, and L_a is the two-dimensional vector space spanned by $\operatorname{Re}(b_i \otimes c_j)$ and $\operatorname{Im}(b_i \otimes c_j)$. The eigenspace L_0 that corresponds to a zero eigenvalue consists of all matrices that commute with B.

8.5.1. *The special linear group in dimension* 2. Consider two examples, which are generic for the group $\mathrm{SL}(2;\mathbb{R})$.

Example 8.1. Let

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The eigenvalues of B are $\lambda_1 = i$ and $\lambda_2 = -i$, hence the nonzero eigenvalues of the adjoint operator $\operatorname{ad} B$ are $\operatorname{Sp}(B) = \{\pm i\}$. The unit eigenvectors of B are $b_1 = (e_1 + ie_2)/\sqrt{2}$ and $b_2 = \overline{b}_1 = (e_1 - ie_2)/\sqrt{2}$, where $\{e_1, e_2\}$ is the canonical basis in $\mathbb{R}^2$. Since $B^{\mathrm{T}} = -B$, it follows that $c_1 = b_2$ and $c_2 = b_1$. Thus

$$b_1 \otimes c_2 = \frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} \text{ and } b_2 \otimes c_1 = \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}.$$

The linear hull of the matrices

$$\operatorname{Re}(b_1 \otimes c_2) = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } \operatorname{Im}(b_1 \otimes c_2) = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is the two-dimensional vector space of 2×2 symmetric matrices with zero trace. Hence, decomposition (8.4) induced by B is the classic decomposition of matrices into a symmetric part and an antisymmetric part.

Now we check conditions of Theorem 8.4. The matrix B is a strongly regular element of the Lie algebra $\mathfrak{sl}(2;\mathbb{R})$. Condition (3) means that the matrix A has a nonzero symmetric part. Under this assumption A and B generate $\mathfrak{sl}(2;\mathbb{R})$ as a Lie algebra. Finally, condition (4) is void. Consequently, by Theorem 8.4, the system $\Gamma = A + \mathbb{R}B$ is controllable on $\mathrm{SL}(2;\mathbb{R})$ if the matrix A is not skew-symmetric. This condition is also necessary for controllability: if $A^{\mathrm{T}} = -A$, then the rank condition for Γ is violated.

Example 8.2. Now consider the case

$$B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The eigenvalues of B are ± 1 and $\operatorname{Sp}(B) = \{\pm 2\}$. The corresponding eigenspaces of $\operatorname{ad} B$ are one-dimensional and are spanned by

$$e_1 \otimes e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } e_2 \otimes e_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

We use B, $e_1 \otimes e_2$, and $e_2 \otimes e_1$ as the basis for decomposition (8.4) and write any matrix $A = (a_{ij})$ as $A = a_{11}B + a_{12}e_1 \otimes e_2 + a_{21}e_2 \otimes e_1$.

By Theorem 8.4, the system $\Gamma = A + \mathbb{R}B$ is controllable on $\mathrm{SL}(2;\mathbb{R})$ if $a_{12}a_{21} < 0$. On the other hand, if $a_{12}a_{21} \geq 0$, then Γ is not controllable since in this case the bilinear system induced by Γ has invariant quadrants in $\mathbb{R}^2$.

Remark. The preceding two examples are exhaustive for $\mathrm{SL}(2;\mathbb{R})$ since any matrix $B \in \mathfrak{sl}(2;\mathbb{R})$ with a nonzero spectrum is similar to one of the matrices

$$\begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}, \quad \begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix}, \ b \in \mathbb{R} \setminus \{0\}.$$

Using the control scaling $u \mapsto u/b$, any system $\Gamma = A + \mathbb{R}B$ on $\mathrm{SL}(2;\mathbb{R})$ with $\det B \neq 0$ can be reduced to the systems considered in the previous two examples.

Now we return to the general case in $\mathrm{SL}(n;\mathbb{R})$. If a strongly regular element B has a real spectrum, then B can be diagonalized. The eigenspaces of $\operatorname{ad} B$ are then the one-dimensional spaces spanned by the matrices $e_i \otimes e_j = E_{ij}$, where $e_1, \ldots, e_n$ is the standard basis in $\mathbb{R}^n$. The maximal eigenvalue of $\operatorname{ad} B$ is the largest difference between the diagonal elements of B, and the minimal eigenvalue is its negative. Rearranging the base vectors e_i, we can put the diagonal elements of B in the ascending order $\lambda_1 < \lambda_2 < \cdots < \lambda_n$. Then condition (4) in Theorem 8.4 takes the form $a_{1n}a_{n1} < 0$, where a_{ij} is the general entry of the drift matrix A.

This argument leads to the following result.

Theorem 8.7. *Let $n \times n$ real skew-symmetric matrices $A = (a_{ij})$ and B satisfy the conditions:*

(i) $a_{1n}a_{n1} < 0$,
(ii) $B = \operatorname{diag}(b_1, \ldots, b_n)$,
(iii) $b_1 < b_2 < \cdots < b_n$,
(iv) $b_i - b_j \neq b_k - b_m$ *for* $(i,j) \neq (k,m)$.

Then the system $\Gamma = A + \mathbb{R}B$ is controllable on the group $\mathrm{SL}(n;\mathbb{R})$ if and only if the matrix A is permutation-irreducible.

Recall that an $n \times n$ matrix A is called *permutation-reducible* if there exists a permutation matrix P such that

$$P^{-1}AP = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix},$$

where A_3 is a $k \times k$ matrix with $0 < k < n$. An $n \times n$ matrix is called *permutation-irreducible* if it is not permutation-reducible. Permutation-irreducible matrices are matrices having no nontrivial invariant coordinate subspaces.

8.5.2. *The conjecture of Jurdjevic-Kupka.* In the case of strongly regular elements B with real eigenvalues, Theorem 8.4 covers the case of a skew-symmetric drift term A. From this point of view, the case where both A and B are symmetric is at the other end of the spectrum created by Theorem 8.4.

Conjecture 1. If matrices $A, B \in \mathfrak{sl}(n;\mathbb{R})$ are symmetric, then the right-invariant system $\Gamma = A + \mathbb{R}B$ is not controllable, neither on $\mathrm{SL}(n;\mathbb{R})$ nor on $\mathbb{R}^n \setminus \{0\}$.

In dimensions $n = 2, 3$ this conjecture is easily proved by constructing invariant quadrants or octants for the induced bilinear system

$$\dot{x} = Ax + uBx, \quad x \in \mathbb{R}^n \setminus \{0\}, \quad u \in \mathbb{R}. \tag{8.7}$$

For $n > 3$ the question remains open. A partial confirmation of the preceding conjecture in arbitrary dimensions under some additional assumptions is given by computing all invariant orthants of bilinear systems in $\mathbb{R}^n$.

8.5.3. *Invariant orthants of bilinear systems.* Let $A, B_1, \dots, B_m$ be arbitrary real $n \times n$ matrices. In this subsubsection we present a criterion for a bilinear system

$$\dot{x} = Ax + \sum_{i=1}^{m} u_i B_i x, \quad x \in \mathbb{R}^n \setminus \{0\}, \quad u_i \in \mathbb{R} \tag{8.8}$$

to have invariant orthants. This result implies a partial confirmation of Conjecture 1.

First we give the necessary notation and definitions. The set of indices

$$\Sigma_n = \{\, \sigma = (\sigma_1, \dots, \sigma_n) \mid \sigma_i = \pm 1 \ \forall i = 1, \dots, n \,\}$$

will be used for parametrization of *orthants*, i.e., sets of the form

$$\mathbb{R}^n_\sigma = \{\, x = (x_1, \dots, x_n) \in \mathbb{R}^n \mid x_i \sigma_i \geq 0 \ \forall i = 1, \dots, n \,\}.$$

A subset of the state space is called *positive* (*negative*) *invariant* for a vector field or a control system if all trajectories of the field or the system starting in this set (resp., its complement) do not leave it (resp., its complement) for all positive moments of time.

Remark. A system is globally controllable iff it has neither positive nor negative invariant sets (except the trivial ones — the whole state space and the empty set). Thus conditions of existence of nontrivial invariant sets are sufficient conditions of global noncontrollability.

Definition 8.7. An $n \times n$ matrix $A = (a_{ij})$ is called *sign-symmetric* if $a_{ij} a_{ji} \geq 0$ for all $i, j = 1, \dots, n$.

Construction 8.1. For any sign-symmetric $n \times n$ matrix A we construct the *graph* $H(A)$ by the following rule. The graph $H(A)$ has n vertices $1, 2, \dots, n$. Its vertices i, j, $i \neq j$, are connected by the edge (i, j) if and only if at least one of the numbers a_{ij}, a_{ji} is nonzero. We take into account only edges that connect *distinct* vertices of the graph $H(A)$; self loops are thus explicitly excluded from consideration. Every edge (i, j) is marked by the sign $+$ or $-$: if $a_{ij} \geq 0$ and $a_{ji} \geq 0$, then the sign $+$ is applied, and if $a_{ij} \leq 0$ and $a_{ji} \leq 0$, then we apply $-$ (there can be no other combinations of signs by virtue of sign-symmetry of A). The marked edges are called *positive* or *negative* depending on the sign $+$ or $-$. For the graph $H(A)$ we define the following function $s(i, j)$, $i, j = 1, \dots, n$, $i \neq j$: $s(i, j) = 0$ if the vertices i, j are not connected by an edge in $H(A)$, $s(i, j) = 1$ for the positive, and $s(i, j) = -1$ for the negative edge (i, j) in the graph $H(A)$. A loop (i.e., a closed path composed of edges) of a graph is called *even* (*odd*) if it contains an even (resp. odd) number of negative edges. A graph is said to *satisfy the even-loop property* if all its loops are even.

Remark. If a graph H satisfies the even-loop property, then there is a subset V of the set of its vertices such that:

a) any negative edge of the graph H has exactly one vertex in V;
b) any positive edge of the graph H has either 0 or 2 vertices in V.

In other words, such a graph H is *bichromatic*: its vertices can be colored in two colors, the first color corresponds to the set V and the second one to its complement.

Construction 8.2. Assume that a graph H satisfies the even-loop property and V is any subset of the set of its vertices that satisfies the previous conditions a), b). Then the *index of the graph* H corresponding to the set V is the set $\sigma = (\sigma_1, \dots, \sigma_n) \in \Sigma_n$ defined as follows: $\sigma_i = +1$ if $i \notin V$ and $\sigma_i = -1$ if $i \in V$.

Theorem 8.8. *Let $A, B_1, \dots, B_m$ be $n \times n$ matrices. The bilinear system* (8.8) *has positive* (*negative*) *invariant orthants if and only if the following conditions are satisfied:*

1. *the matrix A is sign-symmetric;*
2. *the matrices $B_1, \dots, B_m$ are diagonal;*
3. *the graph $H(A)$* (*resp. $H(-A)$*) *satisfies the even-loop property.*

Then positive (*negative*) *invariant orthants are $\mathbb{R}^n_\sigma$, where σ is any index of the graph $H(A)$* (*resp. $H(-A)$*)*, and their number is equal to 2^c, where c is the number of connected components of the graph $H(A)$.*

If system (8.8) *has no invariant coordinate subspaces* (*in particular, if this system satisfies the rank condition everywhere in $\mathbb{R}^n \setminus \{0\}$*)*, then it has either* 0 *or* 2 *invariant orthants.*

Proof. The outline of the proof of Theorem 8.8 is as follows. System (8.8) has invariant orthants if and only if the matrices B_i, $i = 1, \dots, m$, are diagonal and the linear vector field Ax has invariant orthants. The search of these orthants is based upon two ideas. First, it is common knowledge that the positive orthant

$$\mathbb{R}^n_+ = \{\, x = (x_1, \dots, x_n) \in \mathbb{R}^n \mid x_i \geq 0 \ \forall i = 1, \dots, n \,\}$$

is positive invariant for the field Ax if and only if all off-diagonal entries of the matrix A are nonnegative. Second, if the field Ax has an invariant orthant, then successive changes of coordinates $(x_1, \dots, x_i, \dots, x_n) \mapsto (x_1, \dots, -x_i, \dots, x_n)$ in $\mathbb{R}^n$ map this orthant onto $\mathbb{R}^n_+$. During this process we can keep track of signs of entries a_{ij} of the matrix A and obtain conditions of existence of invariant orthants in terms of sign combinations of a_{ij}. These conditions are conveniently expressed in terms of the graph $H(A)$ that corresponds to the matrix A as given in Theorem 8.8. □

Theorem 8.8 is related to Conjecture 1 since there exists an orthogonal transformation of $\mathbb{R}^n$ that diagonalizes a symmetric matrix B; then a symmetric matrix A turns into a symmetric one. That is why we can assume that B is diagonal and A is symmetric.

Theorem 8.8 implies that Conjecture 1 holds in dimensions 2 and 3: in fact, for these dimensions if A is sign-symmetric and B is diagonal, then system (8.7) has a positive or negative invariant orthant. Even for $n = 4$ there are symmetric matrices A for which the field Ax and system (8.7) have no invariant orthants (see the example below). Here the question of global controllability, i.e., of absence of *any* invariant sets, is left open. But for symmetric matrices A with at least one of the graphs $H(A)$, $H(-A)$ satisfying the even-loop property Conjecture 1 is valid. However, in these cases not the symmetry but the sign-symmetry of A is essential.

Example 8.3. Let $A = (a_{ij})$ be any 4×4 matrix of the form

$$\begin{pmatrix} * & + & 0 & + \\ + & * & + & 0 \\ 0 & + & * & - \\ + & 0 & - & * \end{pmatrix},$$

i.e., $a_{12}, a_{21}, a_{14}, a_{41}, a_{23}, a_{32} > 0$, $a_{34}, a_{43} < 0$, $a_{13} = a_{31} = a_{24} = a_{42} = 0$, and diagonal entries are arbitrary. The corresponding graph $H(A)$ is given at Fig. 1.

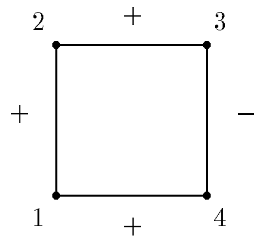

FIGURE 1. The graph $H(A)$.

The only loop $(1,2,3,4)$ is negative in both graphs $H(A)$ and $H(-A)$.

That is why, for any 4×4 diagonal matrix B, system (8.7) has no invariant orthants. But the question of global controllability of this system (if it has the full rank) seems to be open. These statements remain stable under small perturbations of the matrix A.

8.6. **Classical Lie groups.**

8.6.1. *Complex simple Lie algebras.* Let $\mathrm{M}(n;\mathbb{C})$ denote the set of all $n\times n$ complex matrices and $\mathfrak{d}(n;\mathbb{C})$ the subset of all diagonal matrices in $\mathrm{M}(n;\mathbb{C})$.

The Lie algebra $\mathfrak{gl}(n;\mathbb{C})$ is the vector space $\mathrm{M}(n;\mathbb{C})$ endowed with the matrix commutator $[A,B]=AB-BA$ as a Lie bracket.

The Lie algebra $\mathfrak{sl}(n;\mathbb{C})$ is the subalgebra of $\mathfrak{gl}(n;\mathbb{C})$ consisting of all traceless matrices:

$$\mathfrak{sl}(n;\mathbb{C})=\{A\in\mathfrak{gl}(n;\mathbb{C})\mid \operatorname{tr}A=0\}.$$

The subalgebra

$$\mathfrak{sl}(n;\mathbb{C})\cap\mathfrak{d}(n;\mathbb{C})=\{\operatorname{diag}(a_1,\dots,a_n)\mid a_1+\dots+a_n=0\}$$

is a Cartan subalgebra of $\mathfrak{sl}(n;\mathbb{C})$. The algebra $\mathfrak{sl}(n;\mathbb{C})$ is called algebra of type A_l, $l=n-1$.

Let $(\cdot,\cdot)$ be a nonsingular symmetric bilinear form on $\mathbb{C}^n$. The Lie algebra $\mathfrak{so}(n;\mathbb{C})$ consists of all endomorphisms A of $\mathbb{C}^n$ such that

$$(Ax,y)+(x,Ay)=0,\qquad x,y\in\mathbb{C}^n.$$

Assume that the form $(\cdot,\cdot)$ is defined by the $n\times n$ matrix I given by

$$\begin{pmatrix} 0 & \mathrm{Id}_l \\ \mathrm{Id}_l & 0\end{pmatrix}\quad \text{if } n=2l \quad\text{or}\quad \begin{pmatrix} 0 & \mathrm{Id}_l & 0 \\ \mathrm{Id}_l & 0 & 0 \\ 0 & 0 & 1\end{pmatrix}\quad \text{if } n=2l+1,$$

where Id_l is the identity $l\times l$ matrix. Then

$$\mathfrak{so}(n;\mathbb{C})=\{A\in\mathfrak{gl}(n;\mathbb{C})\mid IA^{\mathrm{T}}+AI=0\}.$$

In this case the Cartan subalgebra may be chosen as the subspace of $\mathfrak{so}(n;\mathbb{C})$ consisting of matrices of the form

$$\begin{aligned}&\operatorname{diag}(a_1,\dots,a_l,-a_1,\dots,-a_l) && \text{if } n=2l,\\ &\operatorname{diag}(a_1,\dots,a_l,-a_1,\dots,-a_l,0) && \text{if } n=2l+1.\end{aligned}$$

The algebra $\mathfrak{so}(n;\mathbb{C})$ is called algebra of type B_l for $n = 2l + 1$ and algebra of type D_l for $n = 2l$.

Now let $(\cdot, \cdot)$ be a nonsingular skew-symmetric bilinear form on $\mathbb{C}^n$. The Lie algebra $\mathfrak{sp}(n;\mathbb{C})$ is defined as the set of all endomorphisms A of $\mathbb{C}^n$ such that

$$(Ax, y) + (x, Ay) = 0, \qquad x, y \in \mathbb{C}^n.$$

If the form $(\cdot, \cdot)$ is defined by the matrix

$$J = \begin{pmatrix} 0 & \mathrm{Id}_l \\ -\mathrm{Id}_l & 0 \end{pmatrix},$$

then

$$\mathfrak{sp}(n;\mathbb{C}) = \{A \in \mathfrak{gl}(n;\mathbb{C}) \mid JA^{\mathrm{T}} + AJ = 0\}.$$

The set of all matrices in $\mathfrak{sp}(n;\mathbb{C})$ of the form

$$\operatorname{diag}(a_1, \dots, a_l, -a_1, \dots, -a_l)$$

is a Cartan subalgebra of $\mathfrak{sp}(n;\mathbb{C})$. The algebra $\mathfrak{sp}(n;\mathbb{C})$ is called algebra of type C_n.

In Subsubsections 8.6.2, 8.6.3, and 8.6.4 we assume that Cartan subalgebras L_0 in Lie algebras A_l, B_l, C_l, and D_l are chosen as above. In particular, the condition $B \in L(0)$ will imply that the matrix B is diagonal.

The algebras A_l, B_l, C_l, and D_l are known as *classical complex Lie algebras.* In addition to them, there exist five exceptional Lie algebras denoted as G_2, F_4, E_6, E_7, and E_8.

The classical classification result on complex simple Lie algebras states that all Lie algebras

$$A_l,\ l \geq 1, \quad B_l,\ l \geq 2, \quad C_l,\ l \geq 3, \quad D_l,\ l \geq 4,$$

and

$$G_2, \quad F_4, \quad E_6, \quad E_7, \quad E_8$$

are simple and any simple Lie algebra over $\mathbb{C}$ is isomorphic to exactly one of these.

8.6.2. *Generation of classical Lie algebras.* The central role for the controllability results in this subsection is played by the following proposition, which describes pairs of elements that generate classical Lie algebras.

Theorem 8.9. *Let L be the normal real form of a complex Lie algebra of type A_l, B_l, C_l, or D_l. Let elements $A, B \in L$ be such that $B \in L(0)$ is A-strongly regular. Then* $\operatorname{Lie}(A, B) = L$ *if and only if the matrix A is permutation-irreducible.*

Remark. The nontrivial part in this theorem is sufficiency: if B is diagonal and A permutation-reducible, then $\operatorname{Lie}(A, B)$ consists of permutation-reducible matrices.

8.6.3. *Homogeneous systems.*

Theorem 8.10. *Let G be a connected Lie group with Lie algebra L that is a normal real form of a complex Lie algebra of type A_l, B_l, C_l, or D_l. Let elements $A, B \in L$ be such that $B \in L(0)$ is A-strongly regular. Then the system $\Gamma = \mathbb{R}A + \mathbb{R}B$ is controllable on G if and only if the matrix A is permutation-irreducible.*

Proof. For homogeneous systems controllability is equivalent to the rank condition. Thus the statement follows from Theorem 8.9. □

8.6.4. *Nonhomogeneous systems.*

Theorem 8.11. *Let G be a connected Lie group with Lie algebra L that is a normal real form of a complex Lie algebra $\mathcal{L}$ of type A_l or D_l. Let elements $A, B \in L$ be such that:*

- (i) *$B \in L(0)$ is A-strongly regular.*
- (ii) *Let $A = A(0) + \sum\{A(\alpha) \mid \alpha \in R\}$ be the decomposition of A along the root spaces of $\mathcal{L}$ relative to L_0, see (8.2). Then $\operatorname{Kil}(A(s), A(-s)) < 0$ for the maximal root s.*

Then the system $\Gamma = A + \mathbb{R}B$ is controllable on G if and only if the matrix A is permutation-irreducible.

This theorem is proved by an argument analogous to the one used in the proof of Theorem 8.5 in Subsubsection 8.4.2. The only essential difference is that the rank condition follows from Theorem 8.9.

8.7. **Notes.** The controllability results in Subsection 8.4 were obtained by El Assoudi, Gauthier, and Kupka [9]. They are a culmination of the series of works on controllability on semisimple Lie groups by Jurdjevic and Kupka [73, 74], Gauthier and Bornard [42], Gauthier, Kupka and Sallet [43], El Assoudi and Gauthier [7, 8], Silva Leite and Crouch [129], El Assoudi [6]. In particular, Theorem 8.7 was obtained by Gauthier and Bornard [42]. This work contains also an easy procedure of verification whether a square matrix is permutation-irreducible.

Proposition 8.1 is due to Joseph [66].

Conjecture 1 on noncontrollability of a single-input bilinear system with symmetric matrices in the right-hand side was suggested by Jurdjevic and Kupka [73].

Invariant orthants of bilinear systems were described by Sachkov [114] via application of bichromatic graphs to the study of invariant sets of dynamical systems; this idea is due to Hirsch [54].

Results in Subsection 8.6 are due to Silva Leite and Crouch [129]. In addition to Theorem 8.9, there are other results on generation of classical Lie algebras and Lie groups, see Crouch and Silva Leite [38], Silva Leite [125, 126, 127, 128], Albuquerque and Silva Leite [5].

Results related to subsemigroups of semisimple Lie groups may be found in works by San Martin [122] and San Martin and Tonelli [123].

9. NILPOTENT LIE GROUPS

A Lie algebra L is called *nilpotent* if its descending central series

$$L_{(1)} = [L, L],\ L_{(2)} = [L, L_{(1)}],\ \dots,\ L_{(i)} = [L, L_{(i-1)}],\ \dots, \quad i \in \mathbb{N},$$

stabilizes at zero:

$$L \supset L_{(1)} \supset L_{(2)} \supset \cdots \supset L_{(N)} = \{0\}$$

for some $N \in \mathbb{N}$. Any nilpotent Lie algebra is solvable since $L_{(i)} \supset L^{(i)}$, $i \in \mathbb{N}$, where $L^{(i)}$ denotes an element of the derived series

$$L^{(1)} = [L, L],\ L^{(2)} = [L^{(1)}, L^{(1)}],\ \dots,\ L^{(i)} = [L^{(i-1)}, L^{(i-1)}],\ \dots, \quad i \in \mathbb{N}.$$

Another equivalent characterization of nilpotency of L is that all adjoint operators $\operatorname{ad} x$, $x \in L$, are nilpotent and thus have zero spectrum.

9.1. **Arbitrary systems.** Controllability of a right-invariant system $\Gamma \subset L$ on a nilpotent Lie group G is completely characterized in terms of the wedge, i.e., the topologically closed convex positive cone generated by Γ:

$$W(\Gamma) = \operatorname{cl}(\operatorname{co}(\Gamma)) \subset L.$$

Since $\Gamma \subset W(\Gamma) \subset \mathrm{LS}(\Gamma)$, it is obvious that Γ and $W(\Gamma)$ are controllable or noncontrollable simultaneously.

Theorem 9.1. *Let G be a nilpotent connected Lie group with Lie algebra L and let $\Gamma \subset L$ be a right-invariant system on G which generates L as a Lie algebra. Then Γ is controllable on G if and only if one of the following conditions holds:*

(i) $(\operatorname{int}_{W-W} W) \cap L^{(1)} \neq \emptyset$ *or*
(ii) $\operatorname{int} \operatorname{cl}(L^{(1)} + W) \cap \exp^{-1}(e) \neq \emptyset$,

where $W = W(\Gamma)$ is the wedge generated by Γ.

Remark. Here $\operatorname{int}_{W-W} W$ is the interior of the wedge W relative to the vector space $W - W$ generated by W, and e is the identity of the Lie group G.

The sufficiency in Theorem 9.1 follows from the description of maximal open subsemigroups S of nilpotent Lie groups in terms of their tangent objects

$$L(S) = \{\, x \in L \mid \exp(\mathbb{R}_+ x) \subset \operatorname{cl}(S) \,\}.$$

An open subsemigroup S of a Lie group G is proper, i.e., $S \neq G$ if and only if $e \notin S$. Hence the set of open subsemigroups in G is inductive, and any proper subsemigroup is contained in a *maximal* one.

Theorem 9.2. *Let G be a nilpotent connected Lie group and let S be a maximal open proper subsemigroup of G. Then $L(S)$ is a halfspace bounded by a codimension one subalgebra in $L(G)$.*

For necessity in Theorem 9.1, the Hahn-Banach theorem gives that $W + L^{(1)}$ is contained in a halfspace in L; then $\exp\left(\operatorname{int}\left(W + L^{(1)}\right)\right)$ is a proper open semigroup of G, which implies that $\exp(W)$ is contained in a proper subsemigroup of G.

The controllability test of Theorem 9.1 is essentially nilpotent. This result is false for the group $\mathrm{SL}(2;\mathbb{R})$. It also fails in the following solvable non-nilpotent example.

Example 9.1. Let G be the (unique) two-dimensional connected simply connected non-abelian Lie group, which is represented by the matrices

$$\left\{ \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \mid x > 0,\ y \in \mathbb{R} \right\}.$$

The Lie group G is solvable but not nilpotent. Its Lie algebra has the form

$$L = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in \mathbb{R} \right\}.$$

Consider the following wedge in L:

$$W = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a \in \mathbb{R},\ b \geq 0 \right\}.$$

Direct computations show that

$$\exp(\mathbb{R}_+ W) = \left\{ \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \mid x > 0,\ y \geq 0 \right\},$$

which is a proper subsemigroup of G, thus W is not controllable on G. On the other hand, it is easy to see that both conditions (i), (ii) of Theorem 9.1 hold for the wedge W in this example.

9.2. **Abelian groups.** Let G be a (connected) *abelian* Lie group. Then $G = \mathbb{R}^{n-k} \times T^k$, for some $k \leq n$, where $n = \dim G$ and $T^k = S^1 \times \cdots \times S^1$ is the k-dimensional torus.

For such Lie groups, Theorem 9.1 implies the following.

Corollary 9.1. *Let G be an abelian Lie group with Lie algebra L and let $\Gamma \subset L$ be a right-invariant system on G. Then Γ is controllable on G if and only if*

$$\operatorname{int}(\operatorname{cl}(\operatorname{co}(\Gamma))) \cap \exp^{-1}(e) \neq \emptyset.$$

If in addition G is simply connected, then Γ is controllable if and only if

$$\operatorname{int}(\operatorname{cl}(\operatorname{co}(\Gamma))) \ni e.$$

9.3. **Quotient systems.** Let G be an arbitrary Lie group with Lie algebra L. Let h be an ideal of L and H the corresponding connected subgroup of G. Suppose that H is closed, and so G/H is a Lie group. Denote the projection from G onto G/H by π and its differential by π_*. The projection of the system Γ onto G/H is correctly defined:

$$\pi_*(\Gamma) = \{\, \pi_* v \mid v \in \Gamma \,\} \subset L/h.$$

Notice that controllability of the system Γ on G implies controllability of its projection $\pi_*(\Gamma)$ on G/H.

The derived subalgebra $L^{(1)}$ is an ideal of L and for simply connected G its derived subgroup $G^{(1)} = [G, G]$ is closed. Moreover, the quotient group $G/G^{(1)}$ is abelian. Thus the above construction and Corollary 9.1 allow us to give the following general necessary controllability condition for control affine systems

$$\Gamma = \left\{ A + \sum_{i=1}^{m} u_i B_i \mid u_i \in \mathbb{R},\ i = 1, \dots, m \right\} \subset L \tag{9.1}$$

on simply connected Lie groups.

Introduce the notation for the Lie algebra generated by the vector fields in Γ near controls:

$$L_0 = \operatorname{Lie}(B_1, \dots, B_m),$$

Theorem 9.3. *Let a connected Lie group G be simply connected. If a control affine right-invariant system* (9.1) *is controllable, then*

(1) $\pi_*(L_0) = L/L^{(1)}$,
(2) $m \geq \dim L - \dim L^{(1)}$,

where $L_0 = \operatorname{Lie}(B_1, \dots, B_m)$.

Proof. (1) If Γ is controllable on G, then $\pi_*(\Gamma)$ is controllable on the abelian simply connected Lie group $G/G^{(1)}$. Then it follows from Corollary 9.1 that $\pi_*(L_0) = L/L^{(1)}$.

(2) The Lie algebra $\pi_*(L_0)$ is abelian and is spanned by the vectors $\pi_* B_1, \dots,$ $\pi_* B_m$. Therefore

$$m \geq \dim(\pi_*(L_0)) = \dim(L/L^{(1)}) = \dim L - \dim L^{(1)}.$$

□

Remark. This theorem implies that right-invariant systems on a simply connected Lie group G with nontrivial $G/G^{(1)}$ essentially differ from right-invariant systems on semisimple Lie groups (notice that if G is semisimple, then $G^{(1)} = G$). In semisimple Lie groups, $m = 2$ is sufficient for controllability of a generic control affine right-invariant system, see Theorem 8.3. But Theorem 9.3 gives a lower bound

$$m \geq \dim G/G^{(1)}$$

for the number of controlled vector fields $B_1, \dots, B_m$ necessary to achieve controllability on simply connected G.

9.4. **Control affine systems.** For control affine right-invariant systems (9.1) on simply connected nilpotent Lie groups there is a simple controllability criterion in terms of the Lie subalgebra L_0 of L generated by the non-drift vector fields and the connected subgroup G_0 of G with the Lie algebra L_0.

Theorem 9.4. *Let G be a connected, simply connected nilpotent Lie group. Then system* (9.1) *is controllable on G if and only if $L_0 = L$.*

Sufficiency of the condition $L_0 = L$ for controllability of Γ is valid for arbitrary Lie groups G: it follows from the inclusion $L_0 \subset \mathrm{LS}(\Gamma)$. So the essential part is necessity. Here the key role is played by the necessary controllability conditions in terms of the notion of a *symplectic vector*.

Consider the co-adjoint representation ρ^* of the group G in the dual space L^* of L. For any covector $\lambda \in L^*$, the co-adjoint orbit θ_λ of λ by the ρ^* action $\theta_\lambda = \rho^*_G(\lambda)$ is a smooth submanifold of L^* diffeomorphic to the homogeneous space G/E_λ, where E_λ is the isotropy subgroup of λ, $E_\lambda = \{\, g \in G \mid \rho^*_g(\lambda) = \lambda \,\}$. Further, the system Γ can be projected from G onto the homogeneous space $G/E_\lambda \simeq \theta_\lambda$, and controllability of Γ on G obviously implies controllability of its projection Γ_λ on G/E_λ. This leads to necessary controllability conditions in terms of the co-adjoint representation.

Definition 9.1. $\lambda \in L^*$ is called a *symplectic vector* for $w \in L$ if the co-adjoint orbit θ_λ is not trivial and $\langle w, \beta \rangle > 0$ for all $\beta \in \theta_\lambda$.

(We denote by $\langle \cdot, \cdot \rangle$ the pairing of a vector and covector.)

Theorem 9.5. *If there is a vector field $\xi \in L$ belonging to the centralizer of the subalgebra L_0 such that the nonzero vector field $[A, \xi]$ has a symplectic vector, then system* (9.1) *cannot be controllable on G.*

In fact, the existence of such vector field $\xi \in L$ yields that the function

$$f_\xi \ : \ \theta_\lambda \to \mathbb{R}, \qquad \beta \mapsto f_\xi(\beta) = -\langle \xi, \beta \rangle$$

is strictly increasing on the trajectories of projection of Γ onto the co-adjoint orbit θ_λ. Indeed, the solution of the Cauchy problem $\dot{g}(t) = A(g(t))$, $g(0) = g_0$ is given by $g(t) = \exp(tA)g_0$. Further, the function

$$h(t) = \mathrm{Ad}\left(g(t)^{-1}\right) = \mathrm{Ad}\left(g_0^{-1}\right) \circ \exp(-t \,\mathrm{ad}\, A)$$

has the derivative

$$\dot{h}(t) = \mathrm{Ad}\left(g_0^{-1}\right) \circ \exp(-t \,\mathrm{ad}\, A) \circ (-\,\mathrm{ad}\, A) = -\,\mathrm{Ad}\left(g(t)^{-1}\right) \circ \mathrm{ad}\, A.$$

Now for $\lambda \in L^*$ the co-adjoint action ρ^* of the element $g \in G$ is determined by

$$\rho^*_g(\lambda) = \mathrm{Ad}^*\left(g^{-1}\right) \lambda,$$

consequently, for any $\xi \in L$, $\lambda \in L^*$

$$\begin{aligned}\frac{d}{dt} f_\xi(\rho^*_{g(t)}(\lambda)) &= -\frac{d}{dt}\langle \xi, \rho^*_{g(t)}(\lambda)\rangle = -\frac{d}{dt}\langle \xi, \mathrm{Ad}^*\left(g(t)^{-1}\right)\lambda\rangle \\ &= -\frac{d}{dt}\langle \mathrm{Ad}\left(g(t)^{-1}\right)\xi, \lambda\rangle = \langle \mathrm{Ad}\left(g(t)^{-1}\right)\circ \mathrm{ad}\, A(\xi), \lambda\rangle \\ &= \langle [A,\xi], \rho^*_{g(t)}(\lambda)\rangle.\end{aligned}$$

Thus if λ is a symplectic vector for $[A,\xi]$, then f_ξ is increasing along co-adjoint orbits of trajectories of the field A. If in addition $\mathrm{ad}\,\xi$ vanishes on the subalgebra L_0, then the same holds for trajectories of the whole system Γ, which is impossible for a controllable system.

Another important fact for necessity in Theorem 9.4 is the following proposition related to *hypersurface* systems, i.e., control affine systems (9.1) with L_0 a codimension one subalgebra of L.

Theorem 9.6. *Let $\Gamma \subset L$ be a control affine system* (9.1) *on a connected Lie group G such that L_0 is a codimension one* ideal *of L.*

1. *If G_0 is closed in G, then Γ is controllable if and only if $A \notin L_0$ and $G/G_0 \simeq S^1$.*
2. *If G_0 is not closed in G, then Γ is controllable if and only if $A \notin L_0$.*

Remark. The previous theorem holds without the assumption that L_0 is an ideal; this is important for a generalization of Theorem 9.4 to a subclass of solvable Lie groups including nilpotent ones (see Section 13 below).

Now we outline the scheme of proof of necessity in Theorem 9.4. Suppose the system Γ is controllable on the group G. Then the theory of symplectic vectors implies that the subalgebra L_0 is an ideal of L. The rank condition for Γ is satisfied: $\mathrm{Lie}(\Gamma) = \mathrm{Lie}(A, L_0) = L$, hence L_0 has codimension 0 or 1 in L. But the codimension one case is impossible since then Theorem 9.6 yields $G/G_0 \simeq S^1$, which contradicts simple connectedness of G. Thus $L_0 = L$, and necessity in Theorem 9.4 follows.

Example 9.2. Let G be the Heisenberg group of dimension $2p+1$. It may be represented as a subgroup of $\mathrm{GL}(p+2;\mathbb{R})$ generated by the matrices

$$\mathrm{Id} + X_i, \ \mathrm{Id} + Y_i, \ Z, \quad i = 1, \dots, p,$$

where

$$X_i = E_{1,i+1}, \ Y_i = E_{i+1,p+2}, \quad i = 1, \dots, p.$$

The Lie algebra L of G is spanned by the matrices

$$X_i, \ Y_i, \ Z, \quad i = 1, \dots, p,$$

with the nonzero brackets

$$[X_i, Y_i] = Z, \quad i = 1, \dots, p.$$

The Heisenberg group G is simply connected and nilpotent, hence Theorem 9.4 describes all controllable systems on G.

9.5. **Notes.** The controllability test for arbitrary right-invariant systems and the description of maximal subsemigroups in nilpotent Lie groups (Subsection 9.1) is due to Hilgert, Hofmann, and Lawson [51].

The result on quotient systems (Subsection 9.3) was obtained by Sachkov [111].

The criterion for control affine systems (Subsection 9.4) was given by Ayala [11], and the notion of a symplectic vector is due to Ayala and Vergara [12].

Controllability of projections of right-invariant systems onto nilpotent and solvable manifolds might be studied via application of the theory of flows on these manifolds, see e.g. a book by Auslender, Green, and Hahn [10]. This may be important for studying local controllability of nonlinear systems via nilpotent approximations (Crouch and Byrnes [39]).

10. PRODUCTS OF LIE GROUPS

In this section we present controllability conditions on products of vector groups with nilpotent Lie groups. The results obtained for such Lie groups may be viewed as a generalization of results for nilpotent Lie groups, see Subsection 9.1.

Denote by $\mathrm{cl}(N)$ the topological closure of a subgroup N of a Lie group G.

Theorem 10.1. *Let G be a connected Lie group, C a connected compact subgroup of G, and N a nilpotent normal subgroup of G such that $G = C\,\mathrm{cl}(N)$. If W is a wedge in L which generates L as a Lie algebra, then W is controllable if and only if*

$$\mathrm{int}_{W-W}(W) \cap (L(C) + L^{(1)}) \neq \emptyset.$$

The previous controllability result is proved via the following description of all maximal open semigroups in products of compact and nilpotent groups.

Definition 10.1. A subsemigroup M of G is called a *maximal subsemigroup* of G if the only subsemigroups containing M are M and G, and M is not a subgroup.

Theorem 10.2. *Let G be a connected Lie group, C a connected compact subgroup of G, and N a nilpotent normal subgroup of G such that $G = C\,\mathrm{cl}(N)$. If S is a maximal open subsemigroup of G, then its tangent wedge*

$$L(S) = \{X \in L \mid \exp(tX) \in \mathrm{cl}(S)\ \forall t \geq 0\}$$

is a halfspace bounded by an ideal in L.

10.1. **Notes.** The results in this section are due to Hilgert [49].

Another important (and more general) result on maximal semigroups related to controllability is the characterization of maximal subsemigroups in Lie groups with cocompact radical by Lawson, see Section 11.

11. LIE GROUPS WITH COCOMPACT RADICAL

Denote by $\operatorname{Rad} G$ the *radical* of a Lie group G, i.e., the maximal solvable normal subgroup of G. In this section we suppose that a Lie group G has *cocompact radical*, that is, the quotient group $K = G/\operatorname{Rad} G$ is compact. This wide class of Lie groups contains:

- solvable Lie groups ($K = \{e\}$),
- compact Lie groups,
- semidirect products of a vector space V with a compact Lie group ($V \subset \operatorname{Rad} G$).

11.1. **Controllability conditions and maximal subsemigroups.**

The next theorem gives a Lie-algebraic description of controllability on Lie groups with cocompact radical, complete in the simply connected case.

Theorem 11.1. *Suppose that* $G/\operatorname{Rad} G$ *is compact, and let* $\Gamma \subset L$ *be a right-invariant system that satisfies the rank condition* $\operatorname{Lie}(\Gamma) = L$*. If* Γ *is not contained in any half-space of* L *with boundary a subalgebra, then* Γ *is controllable on the connected Lie group* G*. The converse holds if* G *is simply connected.*

This result is a consequence of the following classification of maximal subsemigroups of Lie groups with cocompact radical.

Theorem 11.2. *The maximal subsemigroups* M *with non-empty interior of a connected, simply connected Lie group* G *with* $G/\operatorname{Rad} G$ *compact are in one-to-one correspondence with their tangent objects*

$$L(M) = \{A \in L \mid \exp(tA) \in \operatorname{cl}(M) \ \forall t \geq 0\},$$

and the latter are precisely the closed half-spaces with boundary a subalgebra. Furthermore, M *is the semigroup generated by* $\exp(L(M))$*.*

Theorem 11.1 follows from Theorem 11.2 since the attainable set of any noncontrollable right-invariant system $\Gamma \subset L$, $\operatorname{Lie}(\Gamma) = L$, is a proper subsemigroup of G contained in some maximal subsemigroup with non-empty interior.

11.2. **Reductive Lie groups.**

A Lie algebra L is called *reductive* if its radical, i.e., the maximal solvable ideal, coincides with its center. L is reductive if and only if the derived subalgebra $L^{(1)}$ is semisimple. In this case L is the direct sum of its center and $L^{(1)}$. A connected Lie group is called *reductive* if it has reductive Lie algebra.

In this subsection we present a characterization of controllable systems Γ on a reductive group G under the assumptions that Γ is a closed convex cone in L, Γ is pointed, i.e., it has the zero edge: $\Gamma \cap -\Gamma = \{0\}$, and is invariant under the adjoint action of the group K, where NAK is an Iwasawa decomposition of G.

Recall that (see Section 4):

1. a right-invariant system Γ is controllable if and only if the closed convex cone $\operatorname{cl}(\operatorname{co}(\Gamma))$ generated by Γ is controllable;
2. a system Γ is controllable, i.e., $\mathbb{A} = G$ if and only if the closure $\operatorname{cl}(\mathbb{A})$ coincides with G, provided that Γ satisfies the rank condition $\operatorname{Lie}(\Gamma) = L$.

Let $G = NAK$ be an Iwasawa decomposition of a Lie group G, where N, A, and K are respectively a maximal nilpotent subgroup, a main vector subgroup, and a maximal compact subgroup of G. Denote by $L(N)$ and $L(K)$ the Lie algebras of the Lie groups N and K respectively, and by $L^{(1)}(K)$ the derived subalgebra of K.

Theorem 11.3. *Let* G *be a connected, simply connected reductive Lie group with Iwasawa decomposition* NAK*. Let* Γ *be an* $\operatorname{Ad}(K)$*-invariant convex cone in* L *satisfying* $\Gamma \cap -\Gamma = \{0\}$ *and* $\operatorname{int}\Gamma \neq \emptyset$*. Then the following assertions hold.*

- (i) *If* $(\operatorname{int}\Gamma) \cap (L(N) + L^{(1)}(K)) \neq \emptyset$*, then* Γ *is controllable.*
- (ii) *If* $\Gamma \cap (L(N) + L^{(1)}(K)) = \{0\}$*, then*

$$\Gamma = \operatorname{LS}(\Gamma) = \{X \in L \mid \exp(\mathbb{R}_+ X) \subset \operatorname{cl}(\mathbb{A})\}$$

  and Γ *is not controllable.*
- (iii) *If* $\emptyset \neq (\Gamma \cap (L(N) + L^{(1)}(K))) \setminus \{0\} \subset \partial\Gamma$*, then* Γ *is not controllable.*

The main idea in the proof of this theorem is to describe a reductive group as a homogeneous space of a group with cocompact radical and after that use the characterization of maximal subsemigroups in such groups given by Theorem 11.2.

11.3. **Notes.** The description of maximal subsemigroups in Lie groups with cocompact radical and conditions of controllability on such Lie groups in Subsection 11.1 are due to Lawson [87].

The controllability result for reductive Lie groups in Subsection 11.2 was obtained by Hilgert [50].

11.3.1. *The rank condition and hypersurface principle.* The customary procedure to show noncontrollability is either to show the violation of the rank controllability condition, see Theorem 2.3, or to construct a (not necessarily smooth) hypersurface in the state space of a system intersected by all trajectories of the system in one direction only, see e.g. the hypersurface principle in Theorem 12.2. By Lawson's Theorem 11.1, for right-invariant systems on simply connected Lie groups with cocompact radical such hypersurface can always be found among codimension one subgroups. An interesting question is whether any full-rank noncontrollable right-invariant system have such codimension one subgroup? A positive answer will give a new method of obtaining sufficient controllability conditions, and a negative one will give an example of a complex obstacle to controllability.

12. HYPERSURFACE SYSTEMS

The class of control systems in an n-state with $(n-1)$ independent controls has specific features that simplify their study, especially in the case of unbounded controls. The more so this is true for right-invariant systems.

Definition 12.1. A control affine right-invariant system

$$\Gamma = \left\{ A + \sum_{i=1}^{m} u_i B_i \mid u_i \in \mathbb{R},\ i = 1, \dots, m \right\} \subset L \tag{12.1}$$

on a Lie group G is called *hypersurface* if the Lie algebra L_0 generated by the vector fields $B_1, \dots, B_m$ is a codimension one subalgebra of L:

$$\dim L_0 = \dim \mathrm{Lie}(B_1, \dots, B_m) = \dim L - 1.$$

Denote by G_0 the connected subgroup of G corresponding to the subalgebra L_0.

Controllability of hypersurface right-invariant systems is completely characterized by the following proposition.

Theorem 12.1. *Let* Γ *be a hypersurface control affine system* (12.1) *on a connected Lie group* G*.*

(1) *If* G_0 *is closed in* G*, then* Γ *is controllable if and only if* $A \notin L_0$ *and* $G/G_0 \simeq S^1$*.*

(2) *If* G_0 *is not closed in* G*, then* Γ *is controllable if and only if* $A \notin L_0$*.*

Proof. The condition $A \notin L_0$ is necessary for controllability in both cases (1), (2) since it is equivalent to the rank condition $\mathrm{Lie}(\Gamma) = \mathrm{Lie}(A, L_0) = L$. Notice also that $L_0 \subset \mathrm{LS}(\Gamma)$, that is why Γ is controllable if and only if the extended system $\tilde{\Gamma} = \mathrm{cl}(\mathrm{co}(\Gamma)) = \mathbb{R}_+ A + L_0$ is controllable.

(1) If $\mathrm{cl}(G_0) = G_0$, then the right coset space G/G_0 is a smooth one-dimensional manifold, i.e., the line $\mathbb{R}$ or the circle S^1. Since for any $x \in G$ any point of

the right coset G_0x is reachable from x for the system $\tilde{\Gamma}$, we can project Γ' onto G/G_0. It is easy to see that the projected system is controllable if $G/G_0 = S^1$ and noncontrollable if $G/G_0 = \mathbb{R}$.

(2) If the codimension one subgroup G_0 is not closed in G, then it is dense in G, thus the reachable set $\mathbb{A}$ is also dense in G. If $A \notin L_0$, then the system Γ has the full rank, thus it is controllable by Theorem 2.8. □

Remark. Theorem 12.1 generalizes the analogous criterion of Theorem 9.6 with the additional assumption that L_0 is an ideal of L.

Corollary 12.1. *A hypersurface system cannot be controllable on a simply connected Lie group.*

Proof. If G is simply connected, then its codimension one subgroup G_0 is closed. Furthermore, G is simply connected, that is why G/G_0 is simply connected as well. Thus $G/G_0 = \mathbb{R}$, and it follows from Theorem 12.1 that Γ is not controllable. □

The previous propositions imply the following *hypersurface principle* — a general necessary controllability condition for simply connected Lie groups.

Theorem 12.2. *Let* $\Gamma \subset L$ *be a control affine system* (12.1) *on a connected, simply connected Lie group* G*. Suppose that there exists a codimension one subalgebra* l *of the Lie algebra* L *containing* L_0*. Then* Γ *is not controllable.*

Proof. The system Γ can be extended to an affine system of the form

$$\Gamma_1 = \left\{ A + \sum_{i=1}^{m} u_i B_i + \sum_{i=m+1}^{k} u_i B_i \mid u_i \in \mathbb{R},\ i = 1, \dots, m \right\},$$

where $B_{m+1}, \dots, B_k$ complement $B_1, \dots, B_m$ to a basis of the subalgebra l. By Corollary 12.1, the system Γ' is not controllable, and therefore Γ_1 is not controllable too. □

The sense of this proposition is that if the codimension one subalgebra $l \supset L_0$ exists, then attainable set of Γ lies "to one side" of the connected codimension one subgroup of G corresponding to l: by simple connectedness of G, this codimension one subgroup separates G into two disjoint parts.

12.1. **Notes.** General hypersurface nonlinear systems were studied bu Hunt [61, 62].

The results of this section are due to Sachkov [111].

The hypersurface principle given by Theorem 12.2, is a necessary controllability condition for an arbitrary simply connected Lie group. By Lawson's Theorem 11.1, if a simply connected Lie group has cocompact radical, then this principle is also sufficient for controllability. It would be interesting to extend the class of simply connected Lie groups with cocompact radical so that the hypersurface principle remain a criterion of controllability.

13. Completely solvable Lie groups

In this section we suppose that Γ is a control affine system (12.1) and give controllability conditions for right-invariant systems on a subclass of solvable Lie groups.

Definition 13.1. A solvable Lie algebra L is called *completely solvable* if all adjoint operators $\operatorname{ad} x$, $x \in L$, have real spectra. A Lie group is *completely solvable* if it has completely solvable Lie algebra.

The triangular group $\mathrm{T}(n;\mathbb{R})$ (see Example 13.1 below) is completely solvable, as well as any of its subgroups. Nilpotent Lie groups are completely solvable since adjoint operators in nilpotent Lie algebras have zero spectrum. On the other hand, e.g., the group of motions of the plane $\mathrm{E}(2;\mathbb{R})$ is solvable but not completely solvable (the group $\mathrm{E}(2;\mathbb{R})$ and its simply connected covering $\tilde{\mathrm{E}}(2;\mathbb{R})$ are treated in Section 15).

Completely solvable Lie algebras have many codimension one subalgebras (this is crucial for the controllability test for completely solvable Lie groups).

Lemma 13.1. *If L is a completely solvable Lie algebra, then for any subalgebra $l_1 \subset L$, $l_1 \neq L$, there exists a subalgebra $l_2 \subset L$ such that $l_1 \subset l_2$ and $\dim l_2 = \dim l_1 + 1$.*

It turns out that the controllability criterion for systems affine in control on nilpotent Lie groups (Theorem 9.4) is valid for completely solvable Lie groups as well.

Theorem 13.1. *Let G be a connected, simply connected completely solvable Lie group. Then system* (12.1) *is controllable on G if and only if $L_0 = L$.*

Proof. Sufficiency. If $L_0 = L$, then $\mathrm{LS}(\Gamma) \supset \mathrm{LS}(L_0) = L$. By Theorem 4.3, the system Γ is controllable.

Necessity follows from Theorem 12.2 and Lemma 13.1. □

Example 13.1. Let $G = \mathrm{T}(n;\mathbb{R})$ be the group of all real $n \times n$ upper triangular matrices with positive diagonal entries. $\mathrm{T}(n;\mathbb{R})$ is a connected, simply connected, and solvable Lie group. Its Lie algebra $L = \mathfrak{t}(n;\mathbb{R})$ consists of all $n \times n$ upper triangular matrices. The derived subalgebra $L^{(1)}$ consists of all strictly upper triangular matrices, and $L/L^{(1)}$ is the n-dimensional abelian Lie algebra of all diagonal $n \times n$ matrices.

By Theorem 13.1, a control affine system Γ is controllable on $\mathrm{T}(n;\mathbb{R})$ if and only if $L_0 = L$.

By Theorem 9.3, controllability of Γ on $\mathrm{T}(n;\mathbb{R})$ can be achieved with not less than $n = \dim L/L^{(1)}$ controlled vector fields. This lower estimate is exact. For example, the system $\Gamma = \{\, A + \sum_{i=1}^{n} u_i B_i \mid u_i \in \mathbb{R} \,\}$ with $B_i = E_{ii} + E_{i,i+1}$ for $i = 1, \dots, n-1$ and $B_n = E_{nn}$ is controllable on $\mathrm{T}(n;\mathbb{R})$. Really, it is easy to see that $\mathrm{Lie}(B_1, \dots, B_n) = \mathfrak{t}(n;\mathbb{R})$.

Example 13.2. Let $G = \mathrm{E}(2;\mathbb{R})$ be the Euclidean group of motions of the two-dimensional plane $\mathbb{R}^2$. $\mathrm{E}(2;\mathbb{R})$ is a connected but not simply connected Lie group. It can be represented by 3×3 matrices of the form

$$\begin{pmatrix} c_{11} & c_{12} & b_1 \\ c_{21} & c_{22} & b_2 \\ 0 & 0 & 1 \end{pmatrix}, \quad C = (c_{ij}) \in \mathrm{SO}(2;\mathbb{R}), \quad b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \in \mathbb{R}^2,$$

where C is a rotation matrix and b is a translation vector. The corresponding matrix Lie algebra $L = \mathfrak{e}(2;\mathbb{R})$ is spanned by the matrices $A_1 = E_{13}$, $A_2 = E_{23}$, and $A_3 = E_{21} - E_{12}$. We have $L^{(1)} = \mathrm{span}(A_1, A_2)$ and $L^{(2)} = \{0\}$; therefore, L is solvable.

Consider the right-invariant system $\Gamma = \{ A_1 + uA_3 \mid u \in \mathbb{R} \}$. We use the Lie saturation technique and show that the system Γ is controllable on $\mathrm{E}(2;\mathbb{R})$.

We have $A_1, \pm A_3 \in \mathrm{LS}(\Gamma)$. That is why $\exp(s \operatorname{ad} A_3)A_1 \in \mathrm{LS}(\Gamma)$ for any $s \in \mathbb{R}$. But $\exp(s \operatorname{ad} A_3)A_1 = (\cos s)A_1 + (\sin s)A_2$. Consequently, $\operatorname{span}(A_1, A_2) \subset \mathrm{LS}(\Gamma)$; therefore, $\mathrm{LS}(\Gamma) = L$. Thus Γ is controllable on $\mathrm{E}(2;\mathbb{R})$.

Obviously, Γ can also be considered as a right-invariant system on the simply connected covering group $\tilde{\mathrm{E}}(2;\mathbb{R})$ of $\mathrm{E}(2;\mathbb{R})$. The above proof of controllability of Γ on $\mathrm{E}(2;\mathbb{R})$ is purely algebraic; i.e., it does not use any global geometric properties of $\mathrm{E}(2;\mathbb{R})$. That is why Γ is controllable on $\tilde{\mathrm{E}}(2;\mathbb{R})$ as well.

The spectrum of the operator $\operatorname{ad} A_3$ consists of $\pm i$ and 0. Therefore this example shows that the assumption of complete solvability of L, i.e., of reality of spectrum of adjoint operators in Theorem 13.1 is essential. Detailed controllability conditions for right-invariant systems on the group $\mathrm{E}(2;\mathbb{R})$ and its simply connected covering $\tilde{\mathrm{E}}(2;\mathbb{R})$ are given in Example 15.2.

13.1. **Notes.**

The results of this section were obtained by Sachkov [111].

Completely solvable Lie algebras and Lie groups are also called *triangular over* $\mathbb{R}$ or *algebras*, resp. *groups*, *of type* (R), see e.g. the survey by Vinberg, Gorbatcevich, and Onishchik [139].

13.1.1. *Lie algebras hard to contol.* For any Lie group G and any system affine in control

$$\Gamma = \left\{ A + \sum_{i=1}^{m} u_i B_i \mid u_i \in \mathbb{R} \right\}$$

on G controllability of the homogeneous part

$$\Gamma_0 = \left\{ \sum_{i=1}^{m} u_i B_i \mid u_i \in \mathbb{R} \right\}$$

is sufficient for controllability of Γ on G. We call a Lie algebra L *hard to control* if any affine in control system $\Gamma \subset L$ and its homogeneous part Γ_0 are simultaneously controllable or noncontrollable (on the connected simply connected Lie group G corresponding to L). In Lie algebras L hard to control the drift term A in an affine system $\Gamma \subset L$ does not help in control, which is not the case for general Lie algebras.

There is the expanding chain of classes of Lie algebras hard to control:

$$\text{abelian} \subset \text{nilpotent} \subset \text{completely solvable}. \tag{13.1}$$

The abelian case is Corollary 9.1, the nilpotent one is Theorem 9.4, and the completely solvable one is Theorem 13.1.

On the other hand, the Lie algebra of the group $\mathrm{E}(2;\mathbb{R})$ of motions of the plane is solvable, not completely solvable and not hard to control, see Example 15.2.

By the hypersurface principle (Theorem 12.2), all Lie algebras satisfying the following property:

$$\left\{ \begin{array}{l} \text{any subalgebra } l \subset L,\ l \neq L, \text{ is contained} \\ \text{in a codimension one subalgebra of } L \end{array} \right\} \tag{13.2}$$

are hard to control. The author does not know, whether this inclusion is strict. By Lemma 13.1, completely solvable Lie algebras satisfy property (13.2). The natural question is, whether there are Lie algebras hard to control not contained in chain (13.1)? If yes, can this chain be continued by any reasonable class of

Lie algebras? The theory on codimension one subalgebras of Lie algebras of Hofmann [56, 57, 59] may be important for this question.

13.1.2. *Codimension one and two subalgebras.* The solution of the controllability problem for completely solvable Lie groups (see Section 13) is based upon the following fact: any proper subalgebra of a real *completely solvable* Lie algebra is contained in a codimension one subalgebra. On the other hand, any proper subalgebra of a real *solvable* Lie algebra is included in some subalgebra of codimension one or two.

This suggests the following approach to controllability on solvable Lie groups. Project a system along the connected subgroup corresponding to the indicated codimension one or two subalgebra. Then: 1) if this group is closed and normal, we obtain a right-invariant system on a one- or two-dimensional Lie group (such systems are transparent); 2) if this subgroup is closed, we obtain a nonlinear system on a one- or two-dimensional smooth manifold (such systems are tractable by the nonlinear controllability theory); 3) and if this subgroup is not closed, then try to apply and develop the theory of control systems on foliations.

14. Lie groups differing from their derived subgroups

Lie algebras L which satisfy the condition $L \neq L^{(1)}$ make up a wide class that contains the class of solvable Lie algebras (since for solvable Lie algebras $L \neq L^{(1)}$) but does not coincide with it: for example, $\mathfrak{gl}(n;\mathbb{R})$ is not solvable and has the derived subalgebra $\mathfrak{sl}(n;\mathbb{R})$. On the other hand, if a Lie algebra L is semisimple, then $L = L^{(1)}$. The reverse is not true: the Lie algebra of infinitesimal motions of the three-space $\mathfrak{e}(2;\mathbb{R}) = \mathbb{R}^3 \leftthreetimes \mathfrak{so}(3;\mathbb{R})$ is not semisimple although it coincides with its derived subalgebra.

In this section we present controllability conditions for single-input systems

$$\Gamma = \{\, A + uB \mid u \in \mathbb{R} \,\} = A + \mathbb{R}B \subset L \tag{14.1}$$

on a Lie group G that does not coincide with its derived subgroup $G^{(1)}$. Consequently, we assume that $L \neq L^{(1)}$.

14.1. **Notation and definitions.** First we introduce the notation connected with eigenvalues and eigenspaces of the adjoint operator $\operatorname{ad} B$ in L.
The derived subalgebra and the second derived subalgebra:

$$L^{(1)} = [L, L], \qquad L^{(2)} = [L^{(1)}, L^{(1)}],$$

the complexifications of L and $L^{(i)}$, $i = 1, 2$:

$$L_c = L \otimes_{\mathbb{R}} \mathbb{C}, \qquad L_c^{(i)} = L^{(i)} \otimes_{\mathbb{R}} \mathbb{C},$$

the adjoint representations and operators:

$$\operatorname{ad} : L \to \operatorname{End}(L), \qquad (\operatorname{ad} B)X = [B, X] \quad \forall X \in L,$$

$$\operatorname{ad}_c : L_c \to \operatorname{End}(L_c), \qquad (\operatorname{ad}_c B)X = [B, X] \quad \forall X \in L_c,$$

spectra of the operators $\operatorname{ad} B|_{L^{(i)}}$, $i = 1, 2$:

$$\operatorname{Sp}^{(i)} = \left\{ a \in \mathbb{C} \mid \operatorname{Ker}\left(\operatorname{ad}_c B|_{L_c^{(i)}} - a \operatorname{Id}\right) \neq \{0\} \right\},$$

real and complex spectra of the operators $\operatorname{ad} B|_{L^{(i)}}$, $i = 1, 2$:

$$\operatorname{Sp}_r^{(i)} = \operatorname{Sp}^{(i)} \cap \mathbb{R}, \qquad \operatorname{Sp}_c^{(i)} = \operatorname{Sp}^{(i)} \setminus \mathbb{R},$$

complex eigenspaces of $\operatorname{ad}_c B|_{L_c^{(1)}}$:

$$L_c(a) = \operatorname{Ker}\left(\operatorname{ad}_c B|_{L_c^{(1)}} - a\operatorname{Id}\right),$$

real invariant subspaces of $\operatorname{ad} B|_{L^{(1)}}$, which are one-dimensional for real $a \in \operatorname{Sp}_r^{(i)}$ and two-dimensional for complex $a \in \operatorname{Sp}_c^{(i)}$:

$$L(a) = (L_c(a) + L_c(\overline{a})) \cap L,$$

complex root subspaces of $\operatorname{ad}_c B|_{L_c^{(i)}}$, $i = 1, 2$:

$$L_c^{(i)}(a) = \bigcup_{N=1}^{\infty} \operatorname{Ker}\left(\operatorname{ad}_c B|_{L_c^{(i)}} - a\operatorname{Id}\right)^N,$$

real invariant subspaces of $\operatorname{ad} B|_{L^{(i)}}$, $i = 1, 2$, realification of the complex root subspaces:

$$L^{(i)}(a) = \left(L_c^{(i)}(a) + L_c^{(i)}(\overline{a})\right) \bigcap L,$$

components of $L^{(i)}$ corresponding to the real eigenvalues of $\operatorname{ad} B|_{L^{(i)}}$, $i = 1, 2$:

$$L_r^{(i)} = \sum\nolimits^{\oplus} \left\{ L^{(i)}(a) \mid a \in \operatorname{Sp}_r^{(i)} \right\}.$$

The subalgebras $L^{(1)}$ and $L^{(2)}$ are ideals of L, so they are $(\operatorname{ad} B)$-invariant, and the restrictions $\operatorname{ad} B|_{L^{(1)}}$ and $\operatorname{ad} B|_{L^{(2)}}$ are well-defined.

In the following lemma we collect several simple statements about decomposition of the subalgebras $L^{(1)}$ and $L^{(2)}$ into sums of invariant subspaces of the adjoint operator $\operatorname{ad} B$.

Lemma 14.1. *(1)* $L^{(i)} = \sum\nolimits^{\oplus} \left\{ L^{(i)}(a) \mid a \in \operatorname{Sp}^{(i)},\ \operatorname{Im} a \geq 0 \right\}$, $i = 1, 2$,

(2) $\operatorname{Sp}^{(2)} \subset \operatorname{Sp}^{(1)}$, $\operatorname{Sp}_r^{(2)} \subset \operatorname{Sp}_r^{(1)}$,

(3) $L^{(2)}(a) \subset L^{(1)}(a)$ *for any* $a \in \operatorname{Sp}^{(2)}$,

(4) $L_r^{(2)} \subset L_r^{(1)}$,

(5) $\operatorname{Sp}^{(2)} \subset \operatorname{Sp}^{(1)} + \operatorname{Sp}^{(1)}$.

Proof. It is obtained by the standard linear-algebraic arguments. In addition, Jacobi's identity is applied in item (5). □

Consider the quotient operator

$$\widetilde{\operatorname{ad}} B \ : \ L^{(1)}/L^{(2)} \to L^{(1)}/L^{(2)}$$

defined as follows:

$$\left(\widetilde{\operatorname{ad}} B\right)\left(X + L^{(2)}\right) = (\operatorname{ad} B)X + L^{(2)} \quad \forall X \in L^{(1)}.$$

Analogously for $a \in \operatorname{Sp}^{(1)}$ we define the quotient operator in the quotient root space:

$$\widetilde{\operatorname{ad}} B(a) \ : \ L^{(1)}(a)/L^{(2)}(a) \to L^{(1)}(a)/L^{(2)}(a),$$

$$\left(\widetilde{\operatorname{ad}} B(a)\right)\left(X + L^{(2)}(a)\right) = (\operatorname{ad} B)X + L^{(2)}(a) \quad \forall X \in L^{(1)}(a),$$

and its complexification:

$$\widetilde{\operatorname{ad}_c} B(a) \ : \ L_c^{(1)}(a)/L_c^{(2)}(a) \to L_c^{(1)}(a)/L_c^{(2)}(a),$$

$$\left(\widetilde{\operatorname{ad}_c} B(a)\right)\left(X + L_c^{(2)}(a)\right) = (\operatorname{ad}_c B)\,X + L_c^{(2)}(a) \quad \forall X \in L_c^{(1)}(a).$$

Definition 14.1. Let $a \in \mathrm{Sp}^{(1)}$. Denote by $\mathrm{j}(a)$ the geometric multiplicity of the eigenvalue a of the operator $\widetilde{\mathrm{ad}_c}\, B(a)$ in the vector space $L_c^{(1)}(a)/L_c^{(2)}(a)$.

Remarks.

(a) For $a \in \mathrm{Sp}^{(1)}$ the number $\mathrm{j}(a)$ is equal to the number of Jordan blocks of the operator $\widetilde{\mathrm{ad}}\, B(a)$ in the space $L^{(1)}(a)/L^{(2)}(a)$.

(b) If an eigenvalue $a \in \mathrm{Sp}^{(1)}$ is simple, then $\mathrm{j}(a) = 0$ for $a \in \mathrm{Sp}^{(2)}$ and $\mathrm{j}(a) = 1$ for $a \in \mathrm{Sp}^{(1)} \setminus \mathrm{Sp}^{(2)}$.

Suppose that $L = L^{(1)} \oplus \mathbb{R}B$ for some B in L (this assumption will be justified by Theorem 14.1 below). Then by Lemma 14.1

$$L = \mathbb{R}B \oplus L^{(1)} = \mathbb{R}B \oplus \sum\nolimits^{\oplus} \left\{ L^{(1)}(a) \mid a \in \mathrm{Sp}^{(1)}, \, \mathrm{Im}\, a \geq 0 \right\}, \tag{14.2}$$

that is why any element $X \in L$ can uniquely be decomposed as follows:

$$X = X_B + \sum \left\{ X(a) \mid a \in \mathrm{Sp}^{(1)}, \, \mathrm{Im}\, a \geq 0 \right\}, \quad X_B \in \mathbb{R}B, \; X(a) \in L^{(1)}(a).$$

We will consider such decomposition for the uncontrolled vector field A of the system Γ:

$$A = A_B + \sum \left\{ A(a) \mid a \in \mathrm{Sp}^{(1)}, \, \mathrm{Im}\, a \geq 0 \right\}.$$

Denote by $\widetilde{A}(a)$ the canonical projection of the vector $A(a) \in L^{(1)}(a)$ onto the quotient space $L^{(1)}(a)/L^{(2)}(a)$.

Definition 14.2. Let $L = L^{(1)} \oplus \mathbb{R}B$, $a \in \mathrm{Sp}^{(1)}$, and $\mathrm{j}(a) = 1$. We say that a vector $A \in L$ *has the zero a-top* if

$$\widetilde{A}(a) \in \left(\widetilde{\mathrm{ad}}\, B(a) - a \,\mathrm{Id} \right) \left(L^{(1)}(a)/L^{(2)}(a) \right).$$

In the opposite case we say that A *has a nonzero a-top*. We use the corresponding notations: $\mathrm{top}\,(A, a) = 0$ or $\mathrm{top}\,(A, a) \neq 0$.

Remark. Geometrically, if a vector A has a nonzero a-top, then the vector $\widetilde{A}(a)$ has a nonzero component corresponding to the highest adjoint vector in the (single) Jordan chain of the operator $\widetilde{\mathrm{ad}}\, B(a)$. Due to nonuniqueness of the Jordan base, this component is nonuniquely determined, but its property to be zero is basis-independent.

Definition 14.3. A pair of complex numbers (α, β), $\mathrm{Re}\, \alpha \leq \mathrm{Re}\, \beta$, is called an *$N$-pair of eigenvalues of the operator* $\mathrm{ad}\, B$ if the following conditions hold:

(1) $\alpha, \beta \in \mathrm{Sp}^{(1)}$,

(2) $L^{(2)}(\alpha) \not\subset \sum \left\{ [L^{(1)}(a), L^{(1)}(b)] \mid a, b \in \mathrm{Sp}^{(1)}, \;\; \mathrm{Re}\, a, \; \mathrm{Re}\, b \notin [\mathrm{Re}\, \alpha, \mathrm{Re}\, \beta] \right\}$,

(3) $L^{(2)}(\beta) \not\subset \sum \left\{ [L^{(1)}(a), L^{(1)}(b)] \mid a, b \in \mathrm{Sp}^{(1)}, \;\; \mathrm{Re}\, a, \; \mathrm{Re}\, b \notin [\mathrm{Re}\, \alpha, \mathrm{Re}\, \beta] \right\}$.

Remark. In other words, to generate both root spaces $L^{(2)}(\alpha)$ and $L^{(2)}(\beta)$ for an N-pair (α, β), we need at least one root space $L^{(1)}(\gamma)$ with $\mathrm{Re}\, \gamma \in [\alpha, \beta]$. In Theorem 14.2 below, N-pairs are the strongest obstacle to controllability under the necessary conditions of Theorem 14.1. In some generic cases the property of absence of real N-pairs can be verified by Lemma 14.5.

14.2. **Necessary controllability conditions.**

14.2.1. *Formulation of results.* It turns out that controllability on simply connected Lie groups G with $G \neq G^{(1)}$ is a very strong property: it imposes essential restrictions both on the group G and on the system Γ.

Theorem 14.1. *Let a connected Lie group G be simply connected and its Lie algebra L satisfy the condition $L \neq L^{(1)}$. If a right-invariant system $\Gamma \subset L$ is controllable, then*:

1. $\dim L^{(1)} = \dim L - 1$,
2. $B \notin L^{(1)}$,
3. $L_r^{(2)} = L_r^{(1)}$,
4. $\mathrm{Sp}_r^{(2)} = \mathrm{Sp}_r^{(1)}$,
5. $\mathrm{Sp}_r^{(1)} \subset \mathrm{Sp}^{(1)} + \mathrm{Sp}^{(1)}$,
6. $\mathrm{j}(a) \leq 1$ *for all* $a \in \mathrm{Sp}^{(1)}$,
7. $\mathrm{top}\,(A, a) \neq 0$ *for all* $a \in \mathrm{Sp}^{(1)}$ *for which* $\mathrm{j}(a) = 1$.

The notations $\mathrm{j}(a)$ and $\mathrm{top}\,(A, a)$ used in Theorem 14.1 are introduced in Definitions 14.1 and 14.2.

Remarks.

(a) The first condition is a characterization of the state space G but not of the system Γ. It means that no single-input system $\Gamma = \{\, A + uB \,\}$ can be controllable on a simply connected Lie group G with $\dim G^{(1)} < \dim G - 1$. That is, to control on such a group, one has to increase the number of inputs. This agrees with the general lower estimate $m > \dim G - \dim G^{(1)}$ for the number of the controlled vectorfields $B_1, \dots, B_m$ necessary for controllability of the multi-input system $\Gamma = \{\, A + \sum_{i=1}^{m} u_i B_i \,\}$ on a simply connected group G, see Theorem 9.3.

(b) Conditions (3)–(7) are nontrivial only for Lie algebras L with $L^{(2)} \neq L^{(1)}$ (in particular, for solvable noncommutative L). If $L^{(2)} = L^{(1)}$, then these conditions are obviously satisfied.

(c) The third condition means that $\mathrm{j}(a) = 0$ for all $a \in \mathrm{Sp}_r^{(1)}$, that is why condition (6) is nontrivial only for $a \in \mathrm{Sp}_c^{(1)}$.

(d) By the same reason, in condition (7) the inclusion $a \in \mathrm{Sp}^{(1)}$ can be replaced by $a \in \mathrm{Sp}_c^{(1)}$. Note that if $\mathrm{j}(a) = 0$, then by the formal Definition 14.2 the vector A has the zero a-top.

(e) The fourth and fifth conditions are implied by the third one but are easier to verify. The simple (and strong) "arithmetic" necessary controllability condition (5) can be verified by a single glance at spectrum of the operator $\mathrm{ad}\, B|_{L^{(1)}}$.

(f) For solvable L under conditions (1), (2) the spectrum $\mathrm{Sp}^{(1)} = \mathrm{Sp}(\mathrm{ad}\, B|_{L^{(1)}})$ is the same for all $B \notin L^{(1)}$ modulo homotheties. Then conditions (4), (5) depend on L but not on B.

(g) For the case of simple spectrum of the operator $\mathrm{ad}\, B|_{L^{(1)}}$ the necessary controllability conditions take respectively the more simple form:

Corollary 14.1. *Let a Lie group G be simply connected and its Lie algebra L satisfy the condition $L \neq L^{(1)}$. Suppose that the spectrum $\mathrm{Sp}^{(1)}$ is simple. If a right-invariant system $\Gamma \subset L$ is controllable, then*:

1. $\dim L^{(1)} = \dim L - 1$,

(2) $B \notin L^{(1)}$,
(3) $\mathrm{Sp}_r^{(2)} = \mathrm{Sp}_r^{(1)}$,
(4) $\mathrm{Sp}_r^{(1)} \subset \mathrm{Sp}^{(1)} + \mathrm{Sp}^{(1)}$,
(5) $A(a) \neq 0$ *for all* $a \in \mathrm{Sp}^{(1)} \setminus \mathrm{Sp}^{(2)}$.

14.2.2. *Outline of the proof of Theorem* 14.1. The main tools to obtain the necessary controllability conditions given in Theorem 14.1 are the rank controllability condition (Theorem 2.3) and the hypersurface principle (Theorem 12.2).

First the following auxiliary propositions are proved.

Lemma 14.2. *Let L be a Lie algebra such that $L \neq L^{(1)}$ and let $B \in L$. Suppose that*:

(1) $\dim L^{(1)} < \dim L - 1$ *or*
(2) $B \in L^{(1)}$ *or*
(3) $L^{(1)} \oplus \mathbb{R}B = L$ *and* $L_r^{(2)} \neq L_r^{(1)}$.

Then there exists a codimension one subalgebra of L containing B.

Lemma 14.3. *Let L be a Lie algebra and $A, B \in L$. Let $L = \mathbb{R}B \oplus L^{(1)}$. Suppose that there exists an eigenvalue $a \in \mathrm{Sp}^{(1)}$ such that*:

(1) $\mathrm{j}(a) > 1$ *or*
(2) $\mathrm{j}(a) = 1$ *and* $\mathrm{top}\,(A, a) = 0$.

Then $\mathrm{Lie}(A, B) \neq L$.

Now Theorem 14.1 follows. If one of its conditions (1)–(5) is violated, then by Lemma 14.2 and Theorem 12.2 the reachable set $\mathbb{A}$ in contained in the closed semigroup of the Lie group G bounded by a codimension one subgroup of G. And if one of conditions (6), (7) does not hold, then by Lemma 14.3 and Theorem 2.3 the set $\mathbb{A}$ lies in the proper connected subgroup of G with the Lie algebra $\mathrm{Lie}(\Gamma)$.

14.3. **Sufficient controllability conditions.**

14.3.1. *Formulation of results.* Under the necessary assumptions of Theorem 14.1, there exist wide sufficient controllability conditions. Notice that the assumption of simple connectedness can now be removed. So the below sufficient conditions are completely Lie-algebraic, i.e., local; this is in contrast with the global assumption (the finiteness of center of G) essential for sufficient controllability conditions for semisimple Lie groups G (see Section 8).

Theorem 14.2. *Let $\Gamma \subset L$ be a right-invariant system on a connected Lie group G. Suppose that the following conditions are satisfied*:

(1) $\dim L^{(1)} = \dim L - 1$,
(2) $B \notin L^{(1)}$,
(3) $L_r^{(2)} = L_r^{(1)}$,
(4) $\dim L_c(a) = 1$ *for all* $a \in \mathrm{Sp}_c^{(1)}$,
(5) $\mathrm{top}\,(A, a) \neq 0$ *for all* $a \in \mathrm{Sp}_c^{(1)}$,
(6) *the operator* $\mathrm{ad}\, B|_{L^{(1)}}$ *has no N-pairs of real eigenvalues.*

Then the system Γ is controllable on the Lie group G.

The notation $\mathrm{top}\,(A, a)$ and the notion of N-pair used in Theorem 14.2 are introduced in Definitions 14.2 and 14.3.

Remarks.

(a) Conditions (1)–(3) are necessary for controllability in the case of a simply connected $G \neq G^{(1)}$, see Theorem 14.1.
(b) Conditions (4) and (5) are close to the necessary conditions (6) and (7) of Theorem 14.1 respectively. Notice that the fourth condition means that all complex eigenvalues of $\operatorname{ad} B|_{L^{(1)}}$ are geometrically simple.
(c) Conditions (2) and (5) are open, i.e., they are preserved under small perturbations of A and B.
(d) The most restrictive of conditions (1)–(6) is the last one. It can be shown that the smallest dimension of $L^{(1)}$ in which this condition is satisfied and preserved under small perturbations of spectrum of $\operatorname{ad} B|_{L^{(1)}}$ for solvable L is 6. This can be used to obtain a classification of controllable systems Γ on small-dimensional simply connected solvable Lie groups G, see Section 16.
(e) The technically complicated condition (6) can be changed by more simple and more restrictive one, and sufficient conditions can be given as in Corollary 14.2 below.
(f) Under the additional assumption of simplicity of the spectrum $\operatorname{Sp}^{(1)}$ the sufficient controllability conditions take the even more simple form presented in Corollary 14.3 below.

Corollary 14.2. *Suppose that the following conditions are satisfied for a system* $\Gamma \subset L$ *on a Lie group* G:
(1) $\dim L^{(1)} = \dim L - 1$,
(2) $B \notin L^{(1)}$,
(3) $L_r^{(2)} = L_r^{(1)}$,
(4) $\dim L_c(a) = 1$ *for all* $a \in \operatorname{Sp}_c^{(1)}$,
(5) $\operatorname{top}(A, a) \neq 0$ *for all* $a \in \operatorname{Sp}_c^{(1)}$,
(6) $\operatorname{Sp}_r^{(1)} = \emptyset$ *or* $\operatorname{Sp}^{(1)} \subset \{\operatorname{Re} z > 0\}$ *or* $\operatorname{Sp}^{(1)} \subset \{\operatorname{Re} z < 0\}$.

Then the system Γ *is controllable on* G.

Corollary 14.3. *Suppose that the following conditions are satisfied for a system* $\Gamma \subset L$ *on a Lie group* G:
(1) $\dim L^{(1)} = \dim L - 1$,
(2) $B \notin L^{(1)}$,
(3) *the spectrum* $\operatorname{Sp}^{(1)}$ *is simple,*
(4) $\operatorname{Sp}_r^{(2)} = \operatorname{Sp}_r^{(1)}$,
(5) $A(a) \neq 0$ *for all* $a \in \operatorname{Sp}_c^{(1)}$,
(6) $\operatorname{Sp}_r^{(1)} = \emptyset$ *or* $\operatorname{Sp}^{(1)} \subset \{\operatorname{Re} z > 0\}$ *or* $\operatorname{Sp}^{(1)} \subset \{\operatorname{Re} z < 0\}$.

Then the system Γ *is controllable on* G.

14.3.2. *Outline of the proof of Theorem* 14.2. This theorem is obtained via the Lie saturation technique, see Section 4: a sequence of increasing lower bounds of the tangent cone $\operatorname{LS}(\Gamma)$ to the closure of the attainable set $\mathbb{A}$ at the identity e is shown to stabilize at the whole Lie algebra L.

The central role in the proof is played by the following proposition.

Lemma 14.4. *Let* $C \in \operatorname{LS}(\Gamma) \cap L^{(1)}$. *Suppose that for any* $a \in \operatorname{Sp}_c^{(1)}$ *the following conditions hold*:
(1) $\dim L_c(a) = 1$ *and*
(2) $\operatorname{top}(C, a) \neq 0$ *or* $L^{(1)}(a) \subset \operatorname{LS}(\Gamma)$.

Suppose additionally that for the number

$$r = \max\{\, \operatorname{Re} a \mid a \in \operatorname{Sp}^{(1)},\, C(a) \neq 0 \,\}$$

$$(\textit{or } r = \min\{\, \operatorname{Re} a \mid a \in \operatorname{Sp}^{(1)},\, C(a) \neq 0 \,\})$$

we have $r \notin \operatorname{Sp}^{(1)}$ *or* $C(r) = 0$*. Then*

$$\operatorname{LS}(\Gamma) \supset \sum \left\{ L^{(1)}(a) \mid a \in \operatorname{Sp}^{(1)},\, \operatorname{Re} a = r,\, a \neq r \right\}.$$

Now the idea of the proof of Theorem 14.2 can be outlined as follows. In view of (14.2), the Lie algebra L splits into the direct sum of the line $\mathbb{R}B$ and the root spaces $L^{(1)}(a)$, $a \in \operatorname{Sp}^{(1)}$. We show that the Lie saturate $\operatorname{LS}(\Gamma)$ coincides with L. First of all, it easy to see that $\mathbb{R}B \subset \operatorname{LS}(\Gamma)$. Then we prove by contradiction that $L^{(1)}(a) \subset \operatorname{LS}(\Gamma)$ for all $a \in \operatorname{Sp}^{(1)}$. Indeed, suppose that there exist numbers

$$n = \min \left\{ \operatorname{Re} a \mid a \in \operatorname{Sp}^{(1)},\, L^{(1)}(a) \not\subset \operatorname{LS}(\Gamma) \right\},$$

$$m = \max \left\{ \operatorname{Re} a \mid a \in \operatorname{Sp}^{(1)},\, L^{(1)}(a) \not\subset \operatorname{LS}(\Gamma) \right\}$$

and consider the interval $[n, m] \subset \mathbb{R}$. Then Lemma 14.4 implies that n, m is an N-pair of real eigenvalues of the operator $\operatorname{ad} B|_{L^{(1)}}$, which contradicts hypothesis (6) of Theorem 14.2. The theorem is proved.

Corollaries 14.2 and 14.3 follow from Theorem 14.2 and the next proposition, which gives simple conditions that guarantee nonexistence of real N-pairs of eigenvalues.

Lemma 14.5. *Suppose that* $B \notin L^{(1)}$ *and* $L_r^{(1)} = L_r^{(2)}$*. Then any one of the following conditions is sufficient for the operator* $\operatorname{ad} B|_{L^{(1)}}$ *not to have real N-pairs of eigenvalues*:

(1) $\operatorname{Sp}_r^{(1)} = \emptyset$ *or*
(2) $\operatorname{Sp}^{(1)} \subset \{\operatorname{Re} z > 0\}$ *or*
(3) $\operatorname{Sp}^{(1)} \subset \{\operatorname{Re} z < 0\}$.

The controllability conditions of Theorems 14.1 and 14.2 for Lie groups $G \neq G^{(1)}$ yield complete description of controllability for several particular classes of Lie groups: metabelian ones, some subgroups of the group of affine transformations of the n-space, and small-dimensional simply connected solvable Lie groups. These results are presented in Sections 15 and 16.

14.4. **Notes.** The results of this section are due to Sachkov [113].

The results of Hofmann [58] on compact elements in solvable Lie algebras might be helpful in order to understand controllability on solvable Lie groups without the assumption of simple connectedness essential for necessary controllability conditions in this section.

15. Metabelian groups

Lie algebras L having derived series of length 2:

$$L \supset L^{(1)} \supset L^{(2)} = \{0\},$$

are called *metabelian*. A Lie group with a metabelian Lie algebra is also called *metabelian*.

A metabelian Lie algebra is obviously solvable. Thus results of the previous subsection yield controllability conditions for metabelian Lie groups.

Theorem 15.1. *Let G be a metabelian connected Lie group. Then the following conditions are sufficient for controllability of a system $\Gamma = A + \mathbb{R}B \subset L$ on G:*

(1) $\dim L^{(1)} = \dim L - 1$,
(2) $B \notin L^{(1)}$,
(3) $\mathrm{Sp}_r^{(1)} = \emptyset$,
(4) $\dim L_c(a) = 1$ *for all* $a \in \mathrm{Sp}_c^{(1)}$,
(5) $\mathrm{top}\,(A, a) \neq 0$ *for all* $a \in \mathrm{Sp}_c^{(1)}$.

If the group G is simply connected, then conditions (1)–(5) are also necessary for controllability of the system Γ on G.

The notation $\mathrm{top}\,(A, a)$ was introduced in Definition 14.2.

Proof. Sufficiency follows from Corollary 14.2.

In order to prove necessity for the simply connected G suppose that Γ is controllable. Then (1) and (2) follow from items (1) and (2) of Theorem 14.1.

Condition (3) follows from item (3) of Theorem 14.1 and from the metabelian property of G:

$$L_r^{(1)} = L_r^{(2)} \subset L^{(2)} = \{0\}.$$

Condition (4). For any $a \in \mathrm{Sp}_c^{(1)}$ we have $L^{(2)}(a) = \{0\}$, that is why $\mathrm{j}(a)$ is equal to geometric multiplicity of the eigenvalue a of the operator $\mathrm{ad}\,B|_{L^{(1)}(a)}$, i.e., to $\dim L_c(a)$. By item (6) of Theorem 14.1, we have $\mathrm{j}(a) = 1$, that is why $\dim L_c(a) = 1$.

Condition (5). For any $a \in \mathrm{Sp}_c^{(1)}$ we have $\mathrm{j}(a) = 1$, then by item (7) of Theorem 14.1, $\mathrm{top}\,(A, a) \neq 0$. □

Example 15.1. Let l be a finite-dimensional real Lie algebra acting linearly on a finite-dimensional real vector space V. Consider their semidirect sum $L = V \leftthreetimes l$. It is a subalgebra of the Lie algebra of affine transformations of the space V since $L \subset V \leftthreetimes \mathfrak{gl}(V)$. If l is abelian, then L is metabelian:

$$L^{(1)} = lV \leftthreetimes \{0\}, \quad L^{(2)} = \{0\}.$$

In the following subsection we study in detail a particular case when l is one-dimensional.

15.1. **Semidirect products.** Let V be a real finite-dimensional vector space, $\dim V = n$, and M a nonzero linear operator in V. Consider the metabelian Lie algebra $L(M)$ which is the semidirect sum of the abelian Lie algebra V with the one-dimensional Lie algebra $\mathbb{R}M$. This Lie algebra can be represented by $(n+1) \times (n+1)$ matrices:

$$L(M) = \left\{ \begin{pmatrix} Mt & b \\ 0 & 0 \end{pmatrix} \mid t \in \mathbb{R},\ b \in \mathbb{R}^n \right\} \subset \mathfrak{gl}(n+1; \mathbb{R}). \tag{15.1}$$

Denote by $G(M)$ the connected Lie subgroup of $\mathrm{GL}(n+1;\mathbb{R})$ corresponding to $L(M)$. It is the semidirect product of the vector Lie group $\mathbb{R}^n$ with the one-dimensional Lie group $G_1 = \{\exp(Mt) \mid t \in \mathbb{R}\}$. Elements of the group $G(M)$ are the matrices

$$\begin{pmatrix} \exp(Mt) & p \\ 0 & 1 \end{pmatrix}, \quad t \in \mathbb{R},\ p \in \mathbb{R}^n,$$

thus $G(M)$ may be viewed as a subgroup of the group $\mathrm{Aff}(n;\mathbb{R})$ of affine transformations of the n-space generated by the one-parameter group of automorphisms G_1 and all translations $p \in \mathbb{R}^n$. The group $G(M)$ is not simply connected iff the one-parameter subgroup G_1 is periodic, which obviously occurs iff

$$(15.2) \qquad \begin{cases} \text{the matrix } M \text{ is semisimple,} \\ \mathrm{Sp}(M) = ir \cdot (k_1, \dots, k_n) \quad \text{for some } r \in \mathbb{R},\ (k_1, \dots, k_n) \in \mathbb{Z}^n. \end{cases}$$

Remark. If conditions (15.2) hold, then by Theorem 7.1 on semidirect products of vector spaces with compact Lie groups a system $\Gamma \subset L(M)$ is controllable on $G(M)$ if and only if it has the full rank: $\mathrm{Lie}(\Gamma) = L(M)$.

On the other hand, the controllability test for simply connected metabelian Lie groups (Theorem 15.1) implies the following controllability conditions for the universal covering $\tilde{G}(M)$ and for the group $G(M)$ itself.

Theorem 15.2. *Let M be a nonzero $n \times n$ matrix, $G = \tilde{G}(M)$, $L = L(M)$. A system $\Gamma = A + \mathbb{R}B \subset L$ is controllable on G if and only if the following conditions hold*:

(1) *the matrix M has a purely complex spectrum,*
(2) $B \notin L^{(1)}$,
(3) $\mathrm{span}(B, A, (\mathrm{ad}\, B)A, \dots, (\mathrm{ad}\, B)^{n-1}A) = L$.

For the group $G(M)$ conditions (1)–(3) *are sufficient for controllability. If conditions* (15.2) *are violated, then* (1)–(3) *are equivalent to controllability on $G(M)$.*

Example 15.2. Let $G = \mathrm{E}(2;\mathbb{R})$ be the Euclidean group of motions of the plane $\mathbb{R}^2$. Its Lie algebra $L = \mathfrak{e}(2;\mathbb{R})$ is spanned by the matrices $A_1 = E_{13}$, $A_2 = E_{23}$, $A_3 = E_{21} - E_{12}$ and has form (15.1):

$$L = L(M), \qquad M = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

It is solvable (in fact, metabelian):

$$L^{(1)} = \mathrm{span}(A_1, A_2) \supset L^{(2)} = \{0\},$$

but not completely solvable:

$$\mathrm{Sp}(\mathrm{ad}\, A_3) = \{\pm i,\ 0\}.$$

The Lie group $\mathrm{E}(2;\mathbb{R}) = G(M)$ is connected but not simply connected, compare with (15.2).

Consider the system $\Gamma = A + \mathbb{R}B \subset \mathfrak{e}(2;\mathbb{R})$ on $\tilde{\mathrm{E}}(2;\mathbb{R})$ — the simply connected covering of $\mathrm{E}(2;\mathbb{R})$. A complete characterization of controllability of Γ on $\tilde{\mathrm{E}}(2;\mathbb{R})$ is derived from Theorem 15.2.

Theorem 15.3. *A system $\Gamma = A + \mathbb{R}B \subset \mathfrak{e}(2;\mathbb{R})$ is controllable on $\tilde{\mathrm{E}}(2;\mathbb{R})$ if and only if the vectors A, B are linearly independent and $B \notin \mathrm{span}(A_1, A_2)$.*

Compare the controllability conditions for $\tilde{\mathrm{E}}(2;\mathbb{R})$ with the following conditions for $\mathrm{E}(2;\mathbb{R})$ derived from Theorem 7.1.

Theorem 15.4. *A system $\Gamma = A + \mathbb{R}B \subset \mathfrak{e}(2;\mathbb{R})$ is controllable on $\mathrm{E}(2;\mathbb{R})$ if and only if the vectors A, B are linearly independent and $\{A, B\} \not\subset \mathrm{span}(A_1, A_2)$.*

15.2. **Affine systems.** Given any matrix $A \in \mathrm{M}(n;\mathbb{R})$ and any vector $b \in \mathbb{R}^n$, consider the affine system

$$\dot{x} = uAx + b, \qquad x \in \mathbb{R}^n, \quad u \in \mathbb{R}. \tag{Σ}$$

According to Subsection 3.3, such system is subordinated to the linear action of the group $G(A) \subset \mathrm{Aff}(n;\mathbb{R})$ described in the previous subsection.

This observation in combination with the controllability results for right-invariant systems on Lie groups of the form $G(A)$ lead to complete controllability conditions for affine systems Σ.

Theorem 15.5. *The system Σ is globally controllable on $\mathbb{R}^n$ if and only if the following conditions hold*:

1. *the matrix A has a purely complex spectrum and*
2. $\mathrm{span}(b, Ab, \dots, A^{n-1}b) = \mathbb{R}^n$.

Proof. Sufficiency. Consider the right-invariant system $\Gamma = \overline{A} + \mathbb{R}\overline{B} \subset L(A)$ on the Lie group $G(A)$, where the matrices $\overline{A}, \overline{B} \in L(A)$ are given by

$$\overline{A} = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}, \qquad \overline{B} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}.$$

The affine system Σ is induced by the right-invariant system Γ. On the other hand, the group of affine transformations $G(A) \subset \mathrm{Aff}(n;\mathbb{R})$ acts transitively on $\mathbb{R}^n$ since it contains all translations. By Corollary 3.3, if the right-invariant system Γ is controllable on $G(A)$, then the affine system Σ is controllable on $\mathbb{R}^n$. By Theorem 15.2, the system Γ is controllable on $G(A)$, thus sufficiency follows.

Necessity. If one of conditions (1), (2) of Theorem 15.5 is violated, then the system Σ has codimension 1 or 2 invariant subspaces in $\mathbb{R}^n$. □

15.3. **Notes.** The results of this section were obtained by Sachkov [113].

16. SMALL-DIMENSIONAL SIMPLY CONNECTED SOLVABLE LIE GROUPS

Given a Lie algebra L, there is the "largest" connected Lie group G having Lie algebra L — the simply connected one. All other connected Lie groups with Lie algebra L are "smaller" than G in the sense that they are quotients G/C, where C is a discrete subgroup of center of G. A right-invariant system $\Gamma \subset L$ may thus be considered on any of these groups, and the simply connected group G is the hardest among them to control. Hence given a right-invariant system Γ on a Lie group (or a homogeneous space of a Lie group) H, it is natural first to study its controllability on the simply connected covering $\widetilde{H}$ of H. If Γ is controllable on $\widetilde{H}$, then it is obviously controllable on H (and on all its homogeneous spaces); in the opposite case one should use particular geometric properties of H (e.g., existence of periodic one-parameter subgroups) to verify controllability of Γ on H. It is obvious and remarkable that controllability conditions on a simply connected Lie group G should have a completely Lie-algebraic form: they are fully determined by the Lie algebra L and its subset Γ (see, e.g., Theorems 9.4, 13.1, 14.1, 15.1, 15.2).

This motivates the following definition.

Definition 16.1. A system $\Gamma \subset L$ is called *controllable* if it is controllable on the (unique) connected simply connected Lie group with Lie algebra L.

And the next definition makes sense at least for solvable Lie algebras in small dimensions.

Definition 16.2. A Lie algebra L is called *controllable* if there exist $A, B \in L$ such that the system $\Gamma = A + \mathbb{R}B$ is controllable.

Indeed, it turns out that controllability conditions on solvable Lie groups (Sections 11 and 14) imply that for solvable small-dimensional Lie algebras L:

- existence of a controllable single-input system $\Gamma \subset L$, i.e., controllability of L, is a strong restriction on L;
- if L is controllable, then almost all pairs $(A, B) \in L \times L$ give rise to controllable systems $\Gamma = A + \mathbb{R}B$;
- controllability of a system $\Gamma \subset L$ depends primarily on L but not on Γ.

Moreover, these results yield a complete description of controllability in small-dimensional solvable Lie algebras presented in the following subsections.

Up to dimension 6 inclusive we describe all solvable Lie algebras L that are controllable, and give controllability tests for single-input systems $\Gamma = A + \mathbb{R}B \subset L$ (the only gap in this picture is the class $L_{6,IV}$ of six-dimensional Lie algebras not completely studied).

The general "bird's-eye view" of controllable small-dimensional solvable Lie algebras is as follows:

$\dim L = 1$: the (unique) Lie algebra is controllable;
$\dim L = 2$: the two Lie algebras are noncontrollable;
$\dim L = 3$: there is one family of controllable Lie algebras $L_3(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$;
$\dim L = 4$: there is one family of controllable Lie algebras $L_4(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$;
$\dim L = 5$: there are two families of controllable Lie algebras:

1. $L_{5,I}(\lambda, \mu)$, $\lambda, \mu \in \mathbb{C} \setminus \mathbb{R}$, $\lambda \neq \mu, \bar{\mu}$,
2. $L_{5,II}(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$;

$\dim L = 6$: there are five families of controllable Lie algebras:

1. $L_{6,I}(\lambda, \mu)$, $\lambda, \mu \in \mathbb{C} \setminus \mathbb{R}$, $\lambda \neq \mu, \bar{\mu}$,
2. $L_{6,II}(\lambda, \mu, k)$, $\lambda, \mu \in \mathbb{C} \setminus \mathbb{R}$, $\operatorname{Re}\lambda = \operatorname{Re}\mu$, $\lambda \neq \mu, \bar{\mu}$, $k \in \mathbb{R} \setminus \{0\}$,
3. $L_{6,III}(\lambda, k, l)$, $\lambda \in \mathbb{C} \setminus (\mathbb{R} \cup i\mathbb{R})$, $k, l \in \mathbb{R}$, $k^2 + l^2 \neq 0$,
4. $L_{6,IV}(\lambda, k, l)$, $\lambda \in \mathbb{C} \setminus (\mathbb{R} \cup i\mathbb{R})$, $k, l \in \mathbb{R}$, $k^2 + l^2 \neq 0$,
5. $L_{6,V}(\lambda, k, l)$, $\lambda \in \mathbb{C} \setminus (\mathbb{R} \cup i\mathbb{R})$, $k, l \in \mathbb{R}$, $k^2 + l^2 \neq 0$,

and one exceptional class $L_{6,IV}(bi)$, $b \in \mathbb{R} \setminus \{0\}$, containing both controllable and noncontrollable Lie algebras.

All controllable Lie algebras L are presented by a scheme in the complex plane $\mathbb{C}$ containing eigenvalues of the adjoint operator $\operatorname{ad} B|_{L^{(1)}}$, $B \in L \setminus L^{(1)}$, and arrows between these eigenvalues describing Lie brackets between eigenvectors of the operator $\operatorname{ad} B|_{L^{(1)}}$ (these schemes are given at the very end of this section). Notice that for solvable Lie algebras L with codimension one subalgebras $L^{(1)}$ (and only such solvable Lie algebras may be controllable, see condition 1 of Theorem 14.1), spectra of all adjoint operators $\operatorname{ad} B|_{L^{(1)}}$, $B \in L \setminus L^{(1)}$, are homothetic with respect to the origin $0 \in \mathbb{C}$, and the homothety equivalence class of spectra of $\operatorname{ad} B|_{L^{(1)}}$, $B \in L \setminus L^{(1)}$, is determined not by $B \in L \setminus L^{(1)}$ but by L itself (in fact, by the isomorphism class of L).

Now we present the classification of controllability in small-dimensional solvable Lie algebras. These results are obtained by virtue of controllability conditions of Sections 11, 12, and 14. The proofs are outlined up to the first nontrivial dimension 3: for dimensions 4–6 the idea of proofs is analogous to the 3-dimensional case but the argument is much longer.

16.1. **One-dimensional Lie groups.** The unique one-dimensional Lie algebra is abelian and isomorphic to $\mathbb{R}$.

Theorem 16.1. *The one-dimensional Lie algebra $\mathbb{R}$ is controllable.*
A system $\Gamma = A + \mathbb{R}B \subset \mathbb{R}$ is controllable if and only if $B \neq 0$.

Proof. Apply Corollary 9.1. □

16.2. **Two-dimensional Lie groups.** There are two nonisomorphic two-dimensional Lie algebras: abelian $\mathbb{R}^2$, and solvable non-abelian $S_2 = \operatorname{span}(x, y)$, $[x, y] = y$.

Theorem 16.2. *Both two-dimensional Lie algebras $\mathbb{R}^2$ and S_2 are not controllable.*

Proof. Both $\mathbb{R}^2$ and S_2 are completely solvable, thus Theorem 13.1 applies. □

16.3. **Three-dimensional Lie groups.**

Construction 16.1. The Lie algebra $L_3(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, Fig. 1.

$$L_3(\lambda) = \operatorname{span}(x, y, z),$$
$$\operatorname{ad} x|_{\operatorname{span}(y,z)} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \quad \lambda = a + bi.$$

The Lie algebra $L_3(\lambda)$ is schematically represented in Fig. 1 by the eigenvalues $\lambda, \bar{\lambda} \in \mathbb{C}$ and realifications of the eigenvectors $y, z \in L_3(\lambda)$ of the adjoint operator $\operatorname{ad} x|_{\operatorname{span}(y,z)}$.

Theorem 16.3. *Let $L = L_3(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, and let $A, B \in L$. The system $\Gamma = A + \mathbb{R}B \subset L$ is controllable if and only if the following conditions are satisfied:*

(1) *$B \notin L^{(1)}$,*
(2) *the vectors A and B are linearly independent.*

Proof. Sufficiency. We show that all hypotheses of Corollary 14.3 hold.

Conditions (1) and (2) are obviously satisfied.

Condition (3). Consider the decomposition $B = B_x x + B_y y + B_z z$. We have

$$\operatorname{Sp}^{(1)} = \operatorname{Sp}(\operatorname{ad} B|_{L^{(1)}}) = B_x \cdot \operatorname{Sp}(\operatorname{ad} x|_{L^{(1)}}) = B_x \cdot \{\lambda, \bar{\lambda}\}.$$

$B \notin L^{(1)}$ is equivalent to $B_x \neq 0$, thus the spectrum $\operatorname{Sp}^{(1)}$ is simple.

Condition (4): $\operatorname{Sp}_r^{(2)} = \operatorname{Sp}_r^{(1)} = \emptyset$.

Condition (5), $A(a) \neq 0$ for all $a \in \operatorname{Sp}_c^{(1)}$, means that the vector A has a nonzero projection onto $L^{(1)}$ along the line $\mathbb{R}B$, i.e., that A and B are linearly independent.

Condition (6): $\operatorname{Sp}_r^{(1)} = \emptyset$.

Now it follows from Corollary 14.3 that the system Γ is controllable.

Necessity follows from Corollary 14.1. □

Theorem 16.4. *A three-dimensional solvable Lie algebra is controllable if and only if it is isomorphic to $L_3(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$.*

Proof. Sufficiency. The set of systems Γ that satisfy conditions (1) and (2) of Theorem 16.3 is nonempty.

Necessity. Let $\Gamma = A + \mathbb{R}B \subset L$ be a controllable system. By Theorem 14.1, $\dim L^{(1)} = 2$ and $B \notin L^{(1)}$. The derived subalgebra $L^{(1)}$ is nilpotent and two-dimensional, thus it is abelian. Consequently, $\operatorname{Sp}_r^{(2)} \subset \operatorname{Sp}^{(2)} = \emptyset$. Thus

$$\operatorname{Sp}^{(1)} = \operatorname{Sp}(\operatorname{ad} B|_{L^{(1)}}) = \{\lambda, \bar{\lambda}\}, \quad \lambda = \alpha + i\beta \in \mathbb{C} \setminus \mathbb{R}.$$

Then there exist a basis y, z of the plane $L^{(1)}$ such that

$$[B, y] = \alpha y + \beta z, \quad [B, z] = -\beta y + \alpha z.$$

Taking into account that $L^{(1)}$ is abelian, we obtain that $L \simeq L_3(\lambda)$. □

16.4. **Four-dimensional Lie groups.**

Construction 16.2. The Lie algebra $L_4(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, Fig. 2.

$$\begin{aligned} & L_4(\lambda) = \operatorname{span}(x, y, z, w), \\ & \operatorname{ad} x|_{\operatorname{span}(y,z,w)} = \begin{pmatrix} a & -b & 0 \\ b & a & 0 \\ 0 & 0 & 2a \end{pmatrix}, \quad \lambda = a + bi, \\ & [y, z] = w. \end{aligned}$$

The arrows in the schematic representation of the Lie algebra $L_4(\lambda)$ in Fig. 2 mean that Lie bracket of the vectors y and z gives the vector w.

Theorem 16.5. *Let $L = L_4(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, and let $A, B \in L$. The system $\Gamma = A + \mathbb{R}B \subset L$ is controllable if and only if the following conditions are satisfied:*

1. *$B \notin L^{(1)}$,*
2. *$A(\lambda) \neq 0$.*

Theorem 16.6. *A four-dimensional solvable Lie algebra is controllable if and only if it is isomorphic to $L_4(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$.*

16.5. **Five-dimensional Lie groups.**

Construction 16.3. The Lie algebra $L_{5,I}(\lambda, \mu)$, $\lambda, \mu \in \mathbb{C} \setminus \mathbb{R}$, Fig. 3.

$$\begin{aligned} & L_{5,I}(\lambda, \mu) = \operatorname{span}(x, y, z, u, v), \\ & \operatorname{ad} x|_{\operatorname{span}(y,z,u,v)} = \begin{pmatrix} a & -b & 0 & 0 \\ b & a & 0 & 0 \\ 0 & 0 & c & -d \\ 0 & 0 & d & c \end{pmatrix}, \quad \lambda = a + bi, \ \mu = + di. \end{aligned}$$

Construction 16.4. The Lie algebra $L_{5,II}(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, Fig. 4.

$$\begin{aligned} & L_{5,II}(\lambda) = \operatorname{span}(x, y, z, u, v), \\ & \operatorname{ad} x|_{\operatorname{span}(y,z,u,v)} = \begin{pmatrix} a & -b & 0 & 0 \\ b & a & 0 & 0 \\ 1 & 0 & a & -b \\ 0 & 1 & b & a \end{pmatrix}, \quad \lambda = a + bi. \end{aligned}$$

The circles around the eigenvalues λ, $\bar{\lambda}$ in Fig. 4 mean that they have double algebraic multiplicity. (Notice that according to the previous matrix their geometric multiplicity is single.)

Theorem 16.7. *Let $L = L_{5,I}(\lambda, \mu)$, $\lambda, \mu \in \mathbb{C} \setminus \mathbb{R}$, $\lambda \neq \mu, \bar{\mu}$, and let $A, B \in L$. The system $\Gamma = A + \mathbb{R}B \subset L$ is controllable if and only if the following conditions are satisfied:*

1. *$B \notin L^{(1)}$,*
2. *$A(\lambda) \neq 0$ and $A(\mu) \neq 0$.*

Theorem 16.8. *Let $L = L_{5,II}(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, and let $A, B \in L$. The system $\Gamma = A + \mathbb{R}B \subset L$ is controllable if and only if the following conditions are satisfied:*

1. $B \notin L^{(1)}$,
2. $\operatorname{top}(A, \lambda) \neq 0$.

Remark. The notation $\operatorname{top}(A, \lambda) \neq 0$ in Theorem 16.8 (and in Theorem 16.14 below) means that the vector A has a nonzero component corresponding to the higher order root space of the operator $\operatorname{ad} B|_{L^{(1)}}$ corresponding to its eigenvalue λ.

Theorem 16.9. *A five-dimensional solvable Lie algebra is controllable if and only if it is isomorphic to $L_{5,I}(\lambda, \mu)$, $\lambda, \mu \in \mathbb{C} \setminus \mathbb{R}$, $\lambda \neq \mu, \bar{\mu}$, or $L_{5,II}(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$.*

16.6. Six-dimensional Lie groups.

Construction 16.5. The Lie algebra $L_{6,I}(\lambda, \mu)$, $\lambda, \mu \in \mathbb{C} \setminus \mathbb{R}$, Fig. 5.

$$L_{6,I}(\lambda, \mu) = \operatorname{span}(x, y, z, u, v, w),$$

$$\operatorname{ad} x|_{\operatorname{span}(y,z,u,v,w)} = \begin{pmatrix} a & -b & 0 & 0 & 0 \\ b & a & 0 & 0 & 0 \\ 0 & 0 & c & -d & 0 \\ 0 & 0 & d & c & 0 \\ 0 & 0 & 0 & 0 & 2a \end{pmatrix}, \quad \lambda = a + bi, \ \mu = c + di,$$

$$[y, z] = w.$$

Construction 16.6. The Lie algebra $L_{6,II}(\lambda, \mu, k)$, $\lambda, \mu \in \mathbb{C} \setminus \mathbb{R}$, $\operatorname{Re}\lambda = \operatorname{Re}\mu$, $k \in \mathbb{R} \setminus \{0\}$, Fig. 6.

$$L_{6,II}(\lambda, \mu, k) = \operatorname{span}(x, y, z, u, v, w),$$

$$\operatorname{ad} x|_{\operatorname{span}(y,z,u,v,w)} = \begin{pmatrix} a & -b & 0 & 0 & 0 \\ b & a & 0 & 0 & 0 \\ 0 & 0 & a & -d & 0 \\ 0 & 0 & d & a & 0 \\ 0 & 0 & 0 & 0 & 2a \end{pmatrix}, \quad \lambda = a + bi, \ \mu = a + di,$$

$$[y, z] = w, \quad [u, v] = kw.$$

Construction 16.7. The Lie algebra $L_{6,III}(\lambda, k, l)$, $\lambda \in \mathbb{C} \setminus (\mathbb{R} \cup i\mathbb{R})$, $k, l \in \mathbb{R}$, $k^2 + l^2 \neq 0$, Fig. 7.

$$L_{6,III}(\lambda, k, l) = \operatorname{span}(x, y, z, u, v, w),$$

$$\operatorname{ad} x|_{\operatorname{span}(y,z,u,v,w)} = \begin{pmatrix} a & -b & 0 & 0 & 0 \\ b & a & 0 & 0 & 0 \\ 0 & 0 & 3a & -b & 0 \\ 0 & 0 & b & 3a & 0 \\ 0 & 0 & 0 & 0 & 2a \end{pmatrix}, \quad \lambda = a + bi,$$

$$[w, y] = ku + lv, \quad [w, z] = -lu + kz.$$

Construction 16.8. The Lie algebra $L_{6,IV}(\lambda, k, l)$, $\lambda \in \mathbb{C} \setminus (\mathbb{R} \cup i\mathbb{R})$, $k, l \in \mathbb{R}$, $k^2 + l^2 \neq 0$, Fig. 8.

$$L_{6,IV}(\lambda, k, l) = \operatorname{span}(x, y, z, u, v, w),$$

$$\operatorname{ad} x|_{\operatorname{span}(y,z,u,v,w)} = \begin{pmatrix} a & -b & 0 & 0 & 0 \\ b & a & 0 & 0 & 0 \\ 0 & 0 & -a & -b & 0 \\ 0 & 0 & b & -a & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \lambda = a + bi,$$

$$[y, v] = -[z, u] = kw, \quad [y, u] = [z, v] = lw.$$

Construction 16.9. The Lie algebra $L_{6,V}(\lambda,k,l)$, $\lambda\in\mathbb{C}\setminus(\mathbb{R}\cup i\mathbb{R})$, $k,l\in\mathbb{R}$, $k^2+l^2\neq 0$, Fig. 9.

$$L_{6,V}(\lambda,k,l)=\operatorname{span}(x,y,z,u,v,w),$$

$$\operatorname{ad}x|_{\operatorname{span}(y,z,u,v,w)}=\begin{pmatrix} a & -b & 0 & 0 & 0\\ b & a & 0 & 0 & 0\\ 1 & 0 & a & -b & 0\\ 0 & 1 & b & a & 0\\ 0 & 0 & 0 & 0 & 2a\end{pmatrix},\quad \lambda=a+bi,$$

$$[y,z]=kw,\quad [y,u]=[z,v]=lw.$$

Construction 16.10. The class of Lie algebras $L_{6,VI}(bi)$, $b\in\mathbb{R}\setminus\{0\}$, Fig. 10.

A Lie algebra L belongs to the class $L_{6,VI}(bi)$ if:

$L=\operatorname{span}(x,y,z,u,v,w)$,
$L^{(1)}=\operatorname{span}(y,z,u,v,w)$,
$\operatorname{Sp}(\operatorname{ad}x|_{L^{(1)}})=\{\pm bi,0\}$,
both eigenvalues $\pm bi$ have double algebraic multiplicity,
$w\in L^{(2)}$.

The class $L_{6,VI}$ contains a lot of nonisomorphic Lie algebras in which multiplication can not be described in detail as in Lie algebras $L_{6,I}$–$L_{6,V}$.

Theorem 16.10. *Let* $L=L_{6,I}(\lambda,\mu)$, $\lambda,\mu\in\mathbb{C}\setminus\mathbb{R}$, $\lambda\neq\mu,\bar{\mu}$, *and let* $A,B\in L$. *The system* $\Gamma=A+\mathbb{R}B\subset L$ *is controllable if and only if the following conditions are satisfied:*

1. $B\notin L^{(1)}$,
2. $A(\lambda)\neq 0$ *and* $A(\mu)\neq 0$.

Theorem 16.11. *Let* $L=L_{6,II}(\lambda,\mu,k)$, $\lambda,\mu\in\mathbb{C}\setminus\mathbb{R}$, $\operatorname{Re}\lambda=\operatorname{Re}\mu$, $\lambda\neq\mu,\bar{\mu}$, $k\in\mathbb{R}\setminus\{0\}$, *and let* $A,B\in L$. *The system* $\Gamma=A+\mathbb{R}B\subset L$ *is controllable if and only if the following conditions are satisfied:*

1. $B\notin L^{(1)}$,
2. $A(\lambda)\neq 0$ *and* $A(\mu)\neq 0$.

Theorem 16.12. *Let* $L=L_{6,III}(\lambda,k,l)$, $\lambda\in\mathbb{C}\setminus(\mathbb{R}\cup i\mathbb{R})$, $k,l\in\mathbb{R}$, $k^2+l^2\neq 0$, *and let* $A,B\in L$. *The system* $\Gamma=A+\mathbb{R}B\subset L$ *is controllable if and only if the following conditions are satisfied:*

1. $B\notin L^{(1)}$,
2. $A(\lambda)\neq 0$.

Theorem 16.13. *Let* $L=L_{6,IV}(\lambda,k,l)$, $\lambda\in\mathbb{C}\setminus(\mathbb{R}\cup i\mathbb{R})$, $k,l\in\mathbb{R}$, $k^2+l^2\neq 0$, *and let* $A,B\in L$. *The system* $\Gamma=A+\mathbb{R}B\subset L$ *is controllable if and only if the following conditions are satisfied:*

1. $B\notin L^{(1)}$,
2. $A(\lambda)\neq 0$ *and* $A(-\lambda)\neq 0$.

Theorem 16.14. *Let* $L=L_{6,V}(\lambda,k,l)$, $\lambda\in\mathbb{C}\setminus(\mathbb{R}\cup i\mathbb{R})$, $k,l\in\mathbb{R}$, $k^2+l^2\neq 0$, *and let* $A,B\in L$. *The system* $\Gamma=A+\mathbb{R}B\subset L$ *is controllable if and only if the following conditions are satisfied:*

1. $B\notin L^{(1)}$,

2. $\operatorname{top}(A, \lambda) \neq 0$.

Remark. The class $L_{6,VI}(bi)$, $b \in \mathbb{R} \setminus \{0\}$, contains both controllable and noncontrollable Lie algebras.

Theorem 16.15. *Let a six-dimensional solvable Lie algebra L not belong to the class $L_{6,VI}(bi)$, $b \in \mathbb{R} \setminus \{0\}$. Then L is controllable if and only if it is isomorphic to one of the following Lie algebras:*

1. $L_{6,I}(\lambda, \mu)$, $\lambda, \mu \in \mathbb{C} \setminus \mathbb{R}$, $\lambda \neq \mu, \bar{\mu}$;
2. $L_{6,II}(\lambda, \mu, k)$, $\lambda, \mu \in \mathbb{C} \setminus \mathbb{R}$, $\operatorname{Re} \lambda = \operatorname{Re} \mu$, $\lambda \neq \mu, \bar{\mu}$, $k \in \mathbb{R} \setminus \{0\}$;
3. $L_{6,III}(\lambda, k, l)$, $\lambda \in \mathbb{C} \setminus (\mathbb{R} \cup i\,\mathbb{R})$, $k, l \in \mathbb{R}$, $k^2 + l^2 \neq 0$;
4. $L_{6,IV}(\lambda, k, l)$, $\lambda \in \mathbb{C} \setminus (\mathbb{R} \cup i\,\mathbb{R})$, $k, l \in \mathbb{R}$, $k^2 + l^2 \neq 0$;
5. $L_{6,V}(\lambda, k, l)$, $\lambda \in \mathbb{C} \setminus (\mathbb{R} \cup i\,\mathbb{R})$, $k, l \in \mathbb{R}$, $k^2 + l^2 \neq 0$.

Controllable solvable Lie algebras up to dimension 6:

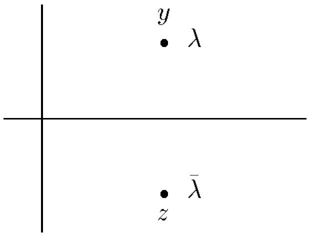

Fig. 1. $L_3(\lambda)$.

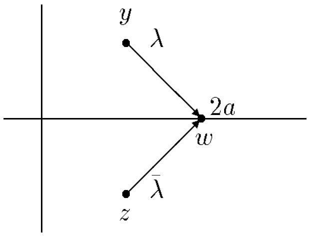

Fig. 2. $L_4(\lambda)$, $\operatorname{Re} \lambda = a$.

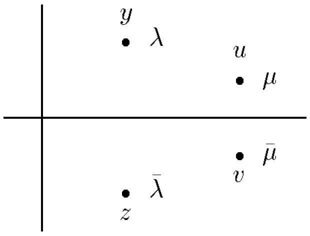

Fig. 3. $L_{5,I}(\lambda, \mu)$.

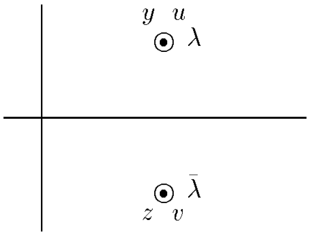

Fig. 4. $L_{5,II}(\lambda)$.

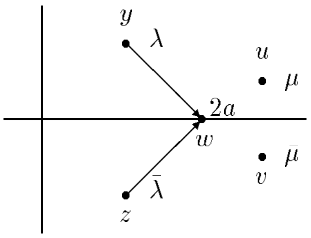

Fig. 5. $L_{6,I}(\lambda, \mu)$, $\operatorname{Re}\lambda = a$.

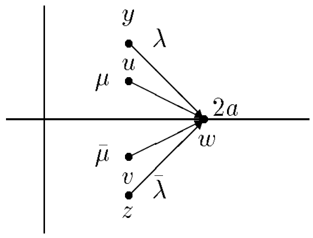

Fig. 6. $L_{6,II}(\lambda, \mu, k)$, $\operatorname{Re}\lambda = \operatorname{Re}\mu = a$.

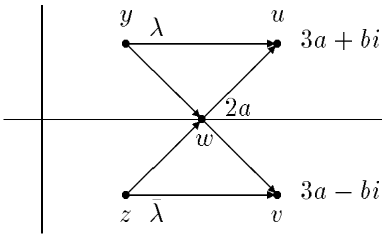

Fig. 7. $L_{6,III}(\lambda, k, l)$, $\lambda = a + bi$.

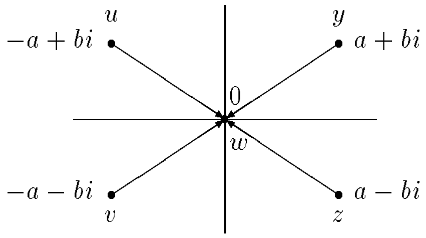

Fig. 8. $L_{6,IV}(\lambda, k, l)$, $\lambda = a + bi$.

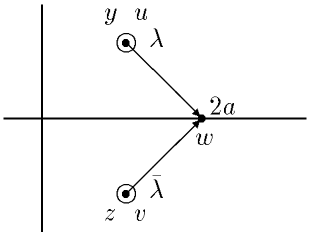

Fig. 9. $L_{6,V}(\lambda, \mu)$, $\operatorname{Re}\lambda = a$.

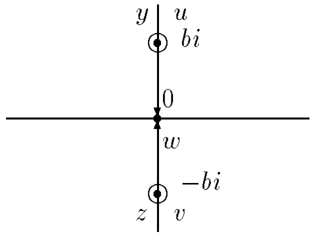

Fig. 10. $L_{6,VI}(bi)$.

16.7. **Notes.** The classification of small-dimensional controllable solvable Lie algebras is due to Sachkov [115, 116].

A natural next step would be a complete and visual classification of controllable systems on general small-dimensional Lie algebras by synthesizing the “semisimple” and “solvable” theory via Levi decomposition (Kupka [85]).

17. FINAL REMARKS

In this section we collect some references related to the subject of this survey:

Surveys and reference works on controllability of right-invariant systems on Lie groups: Chong and Lawson [37], Kupka [85], Sachkov [115], Sallet [118, 119, 120].

Textbooks and surveys on geometric control theory: Agrachev, Vakhrameev, and Gamkrelidze [4], Brockett [33], Jurdjevic [72], Sussmann [134].

Textbooks and surveys on Lie groups and Lie algebras: Bourbaki [30, 31], Varadarajan [137], Vinberg and Onishchik [138], Vinberg, Gorbatcevich, and Onishchik [139].

Textbooks on Lie semigroup theory: Hofmann and Lawson [60], Hilgert and Neeb [53], Hilgert, Hofmann and Lawson [52].

Controllability of nonlinear systems: Agrachev [3], Bacciotti and Stefani [19], Basto Gonçalves [14, 15, 16, 17], Bianchini and Stefani [20], Hermann [44], Hermes [45, 46, 47], Hermes and Kawski [48], Kawski [77], Krener [79], Levitt and Sussmann [88], Lobry [90, 91], Stefani [130], Sussmann [135], Sussmann and Jurdjevic [132], Tretyak [136].

Controllability of bilinear and affine systems: Adda [1], Adda and Sallet [2], Bacciotti [18], Bonnard [21, 22], Boothby [26], Brockett [34], Bruni, Di Pillo and Koch [35], Koditschek and Narendra [78], Elliott and Tarn [40], Imbert, Clique and Fossard [63], Joo and Tuan [65], Jurdjevic and Sallet [75], Kučera [82, 83, 84], Lepe [89], Lobry [92], Piechottka [104], Piechottka and Frank [105], Rink and Möhler [106], Sachkov [107, 108, 109, 110, 112].

Linear systems on Lie groups: Ayala and Tirao [13], Markus [93].

Motion planning on Lie groups: Krishnaprasad and Tsakiris [80, 81], Leonard [99], Leonard and Krishnaprasad [100, 101], Sarti, Walsh, and Sastry [124], Walsh, Montgomery, and Sastry [140].

Control problems on Lie groups: Bonnard [23], Cheng, Dayawansa, and Martin [36], Enos [41], Hirschorn [55], Jurdjevic [68, 69, 70, 71], Lovric [102], Mittenhuber [94, 96], Monroy-Pérez [97], Sussmann [131], Yakovenko [141, 142], Yatcenko [143].

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INDEX

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