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Maxwell strata and symmetries in the problem of optimal rolling of a sphere over a plane

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Abstract. The problem of rolling of a sphere over a plane without slipping or twisting is considered. It is required to roll the sphere from one contact configuration into another one so that the curve traced by the contact point be of minimum length. Extremal trajectories in this problem were described by Arthur, Walsh and Jurdjevic.

In this work, discrete and continuous symmetries of the problem are constructed and fixed points of the action of these symmetries in the inverse image and image of the exponential map are studied. This analysis is used to derive necessary conditions for optimality; namely, upper bounds on the cut time along the extremal trajectories.

Bibliography: 21 titles.

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§ 1. Introduction

We study a mechanical system that consists of two horizontal planes and a sphere contacting both planes. The lower plane is stationary and the sphere is rolling without slipping or twisting due to horizontal motion of the upper plane. The state of such a system is described by the point of contact of the sphere with the lower plane and by the orientation of the sphere in the three-dimensional space. It is required to roll the sphere from the prescribed initial state into some terminal state so that the curve traced by the point of contact on the plane be of minimum length. Here, the control is the velocity of the upper plane or, equivalently, the velocity of the centre of the sphere.

Since the object of investigation is the kinematics of this system, we can ignore the presence of the upper plane and study the rolling of the sphere over the (lower) plane without slipping or twisting. The no-slip condition means that the instantaneous velocity of the point of contact of the sphere with the plane is equal to zero, and the no-twist condition means that the angular velocity vector is directed horizontally. The rolling of one surface over another surface without slipping or twisting is used to simulate the operation of a robotic arm. Problems related to

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this kind of motion are of considerable interest in mechanics, robotics, and control theory (see, for example, [1]–[4]).

The problem of optimal rolling of a sphere over a plane was posed by Hammersley [5]. Arthur and Walsh (see [6]) showed that equations for extremal trajectories in this problem can be integrated analytically, and Jurdjevic (see [7], [8]) performed a qualitative study of these trajectories and discovered that, in the case of optimal rolling, the point of contact of the sphere and the plane traces Euler elasticae (stationary configurations of an elastic rod on a plane, see [9], [10]).

The problem of the optimality of extremal trajectories remained open. Small arcs of extremal trajectories are optimal, but large arcs are, in general, not. A point at which an extremal trajectory loses optimality is referred to as a *cut point* (using the terminology of Riemannian geometry). In this work, we initiate investigation of the cut points and the corresponding cut time in the problem of optimal rolling of a sphere over a plane. We derive upper estimates for the cut time along extremal trajectories, which are thus necessary conditions for optimality. These estimates are obtained through the study of discrete symmetries of the problem and the corresponding Maxwell points (points of intersection of different extremal trajectories with the same values of the functional and time). This approach was successfully used for the investigation of a number of problems of optimal control (see [11]–[13]); the current experience suggests that the aforementioned upper estimates for the cut time may actually coincide with the cut time.

We proceed to the mathematical statement of the problem. Let e_1, e_2, e_3 be a fixed right-handed frame in the space $\mathbb{R}^3$ such that the vectors e_1 and e_2 span a plane $\mathbb{R}^2 \subset \mathbb{R}^3$ and the vector e_3 points into the half-space containing a unit sphere S^2 rolling over the plane. The frame e_1, e_2, e_3 is attached to the point $O \in \mathbb{R}^2$. Let f_1, f_2, f_3 be a moving right-handed frame attached to the rolling sphere S^2. We denote the coordinates of a point in $\mathbb{R}^3$ relative to the basis e_1, e_2, e_3 by (x, y, z) and the coordinates of the same point relative to the basis f_1, f_2, f_3 carried over to the point O by (X, Y, Z). Thus,

$$xe_1 + ye_2 + ze_3 = Xf_1 + Yf_2 + Zf_3.$$

Let the rotation of the three-dimensional space that carries the moving frame f_1, f_2, f_3 into the fixed frame e_1, e_2, e_3 be determined by the matrix $R \in \mathrm{SO}(3)$ in the moving frame:

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = R \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

The state of the system 'sphere S^2 and plane $\mathbb{R}^2$' is described by the coordinates (x, y) of the point of contact between S^2 and $\mathbb{R}^2$ and by the rotation matrix R. For the control we use the vector (u_1, u_2) of velocity of the centre of the sphere. As is shown in [7], the problem of optimal rolling of a sphere over a plane can be stated as the following optimal control problem:

$$\dot{x} = u_1, \tag{1.1}$$

$$\dot{y} = u_2, \tag{1.2}$$

$$\dot{R} = R(u_2 A_1 - u_1 A_2), \tag{1.3}$$

$$Q = (x, y, R) \in M = \mathbb{R}^2 \times \mathrm{SO}(3), \tag{1.4}$$

$$u = (u_1, u_2) \in \mathbb{R}^2, \tag{1.5}$$

$$Q(0) = Q_0 = (0, 0, \mathrm{Id}), \qquad Q(t_1) = Q_1, \tag{1.6}$$

$$l = \int_0^{t_1} \sqrt{u_1^2 + u_2^2}\, dt \to \min. \tag{1.7}$$

Here and below we use the basis matrices in the Lie algebra so(3)

$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \qquad A_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{1.8}$$

The admissible controls are measurable and essentially bounded, and the admissible trajectories are Lipschitz continuous.

The problem (1.1)–(1.7) is a left-invariant sub-Riemannian problem on the Lie group $M = \mathbb{R}^2 \times \mathrm{SO}(3)$. We introduce the following left-invariant frame on this Lie group:

$$e_1 = \frac{\partial}{\partial x}, \qquad e_2 = \frac{\partial}{\partial y}, \qquad V_i(R) = RA_i, \quad i = 1, 2, 3.$$

In terms of the left-invariant fields

$$X_1 = e_1 - V_2, \qquad X_2 = e_2 + V_1$$

the controlled system (1.1)–(1.4) takes the form

$$\dot{Q} = u_1 X_1(Q) + u_2 X_2(Q), \qquad Q \in M = \mathbb{R}^2 \times \mathrm{SO}(3), \quad (u_1, u_2) \in \mathbb{R}^2. \tag{1.9}$$

The functional (1.7) is the sub-Riemannian length functional for the left-invariant sub-Riemannian structure defined by the fields X_1, X_2 as an orthonormal basis:

$$l = \int_0^{t_1} \langle \dot{Q}, \dot{Q} \rangle^{1/2}\, dt \to \min, \tag{1.10}$$

$$\langle X_i, X_j \rangle = \delta_{ij}, \qquad i, j = 1, 2.$$

The commutators $[A_i, A_j] = A_i A_j - A_j A_i$ satisfy the relations

$$[A_1, A_2] = A_3, \qquad [A_2, A_3] = A_1, \qquad [A_3, A_1] = A_2.$$

The multiplication table for the Lie algebra $L = \mathrm{span}(e_1, e_2, V_1, V_2, V_3)$ of the Lie group M has the form

$$\operatorname{ad} e_i = 0, \qquad [V_1, V_2] = V_3, \qquad [V_2, V_3] = V_1, \qquad [V_3, V_1] = V_2.$$

By virtue of the relations

$$[X_1, X_2] = V_3, \qquad [X_1, V_3] = -V_1, \qquad [X_2, V_3] = -V_2$$

the vector fields X_1, X_2 on the right-hand side of the system (1.9) generate the Lie algebra L. By the Rashevskiĭ-Chow theorem (see [4]) the system (1.9) is completely controllable, which means that any two points $Q_0, Q_1 \in M$ can be joined by a trajectory of the system. It follows from Filippov's theorem (see [4]) that there exist optimal controls in the problem (1.1)–(1.7) for any $Q_0, Q_1 \in M$ in the class of essentially bounded measurable controls.

This work continues the study of invariant sub-Riemannian problems on Lie groups, which has been actively carried out in the framework of the geometric control theory (see [14]–[16]). It is organized as follows. In § 2 we approach the problem of optimal rolling of a sphere over a plane using Pontryagin's maximum principle and derive differential equations for extremal trajectories (some of the results obtained by Jurdjevic in [7] and [8] are reformulated in convenient terms). In particular, we demonstrate that a subsystem of the Hamiltonian system of the maximum principle for costate variables in appropriate coordinates reduces to the equation of motion of the mathematical pendulum, and the projections of the extremal trajectories onto the plane (x, y) are Euler elasticae. We introduce an exponential map which is used to parametrize extremal trajectories. In § 3 symmetries of a family of elasticae on the plane (rotations and reflections) are first extended to symmetries of a family of extremals and then to symmetries of the exponential map (in other words, to transformations of the inverse image and image of the exponential map which commute with the latter). The corresponding group G of symmetries of the exponential map is described. § 4 contains the main results of the paper, namely, necessary conditions for the optimality of extremal trajectories in the form of upper estimates on the cut time. In the analytic problem an extremal trajectory is known to lose optimality after it intersects another extremal trajectory with the same values of time and the functional (such points are referred to as Maxwell points). We study fixed points of the action of the group G in the inverse image and image of the exponential map and employ the results obtained to describe Maxwell points corresponding to the group of symmetries G, that is, the common terminal points of the extremal trajectories interchanged by elements of this group. As a consequence, we derive upper bounds for the cut time. These results are obtained in terms of unit quaternions which parametrize the rotations of the three-dimensional space; in § 5, some basic facts related to this parametrization are recalled.

§ 2. Extremal trajectories

By the Cauchy-Bunyakovsky-Schwarz inequality, the problem of minimization of the sub-Riemannian length (1.7) is equivalent to the problem of minimization of the action

$$J = \frac{1}{2} \int_0^{t_1} (u_1^2 + u_2^2)\, dt \to \min \tag{2.1}$$

for a given terminal time instant t_1.

We shall look for optimal trajectories in the problem (1.1)–(1.6), (2.1) using Pontryagin's maximum principle [17] in the invariant statement. To do this, we first recall some facts of Hamiltonian formalism on the cotangent bundle (see [4]). Consider the cotangent bundle T^*M of a smooth manifold M with the canonical projection $\pi\colon T^*M \to M$, $\pi(\lambda) = q$ for the covector $\lambda \in T_q^*M$. The tautological one-form $s \in \Lambda^1(T^*M)$ on the cotangent bundle is defined as follows. Let $\lambda \in T^*M$ and $v \in T_\lambda(T^*M)$; then $\langle s_\lambda, v\rangle = \langle \lambda, \pi_* v\rangle$ (in the coordinates $s = p\,dq$). The canonical symplectic structure $\sigma \in \Lambda^2(T^*M)$ on the cotangent bundle is defined as $\sigma = ds$ (in the coordinates $\sigma = dp \wedge dq$). With any Hamiltonian $h \in C^\infty(T^*M)$ we can associate a Hamiltonian vector field on the cotangent bundle $\vec{h} \in \operatorname{Vec}(T^*M)$ by the rule $\sigma_\lambda(\,\cdot\,, \vec{h}) = d_\lambda h$.

Pontryagin's maximum principle for the problem (1.1)–(1.6), (2.1) has the following invariant statement (see [4]). Let us introduce the Hamiltonians which are linear on the fibres of the cotangent bundle T^*M and correspond to the basis fields:

$$h_i(\lambda) = \langle \lambda, e_i\rangle, \quad i = 1, 2, \qquad H_i(\lambda) = \langle \lambda, V_i\rangle, \quad i = 1, 2, 3,$$

as well as a family of Hamiltonians

$$\begin{gathered} h_{u,\nu}(\lambda) = \langle \lambda, u_1X_1 + u_2X_2\rangle + \frac{\nu}{2}(u_1^2 + u_2^2) \\ = u_1(h_1 - H_2) + u_2(h_2 + H_1) + \frac{\nu}{2}(u_1^2 + u_2^2), \\ \lambda \in T^*M, \qquad u = (u_1, u_2) \in \mathbb{R}^2, \qquad \nu \in \mathbb{R}. \end{gathered}$$

Theorem 1 (Pontryagin's maximum principle). *Let Q_t and $u(t)$, $t \in [0, t_1]$, be an optimal trajectory and the corresponding optimal control in the problem* (1.1)–(1.6), (2.1). *Then there exists a nontrivial Lipschitz continuous curve*

$$(\nu, \lambda_t) \neq 0, \qquad \nu \in \mathbb{R}, \quad \lambda_t \in T^*_{Q_t}M,$$

which satisfies the conditions

$$\dot\lambda_t = \vec{h}_{u(t),\nu}(\lambda_t) = u_1(t)\bigl(\vec{h}_1(\lambda_t) - \vec{H}_2(\lambda_t)\bigr) + u_2(t)\bigl(\vec{h}_2(\lambda_t) + \vec{H}_1(\lambda_t)\bigr), \tag{2.2}$$

$$h_{u(t),\nu}(\lambda_t) = \max_{u\in\mathbb{R}^2} h_{u,\nu}(\lambda_t), \qquad t \in [0, t_1], \tag{2.3}$$

$$\nu \leqslant 0.$$

A curve $\lambda_t \in T^*M$ that satisfies the conditions of Theorem 1 is referred to as an *extremal*, and its projection $Q_t = \pi(\lambda_t) \in M$ is an *extremal trajectory* (or a *sub-Riemannian geodesic*).

Using the coordinates $(h_1, h_2, H_1, H_2, H_3)$ in the fibres of the cotangent bundle T^*M, we rewrite the Hamiltonian system (2.2) in the form

$$\dot h_1 = \dot h_2 = 0, \tag{2.4}$$

$$\dot H_1 = u_1 H_3, \tag{2.5}$$

$$\dot H_2 = u_2 H_3, \tag{2.6}$$

$$\dot H_3 = -u_1H_1 - u_2H_2, \tag{2.7}$$

$$\dot Q = u_1X_1 + u_2X_2. \tag{2.8}$$

2.1. Abnormal case. First, let us consider the case $\nu = 0$. The maximum condition (2.3) implies that along the abnormal extremals the following relations hold:

$$h_1 - H_2 \equiv 0, \qquad h_2 + H_1 \equiv 0.$$

Differentiating these relations, by virtue of the Hamiltonian system (2.4)–(2.8) we obtain expressions for the vertical coordinates

$$H_1 = \text{const}, \qquad H_2 = \text{const}, \qquad H_3 = 0, \qquad h_1 = H_2, \qquad h_2 = -H_1,$$

for the extremal controls

$$u_1 = kH_2 = \text{const}, \qquad u_2 = -kH_1 = \text{const}, \qquad k = \text{const} \neq 0$$

and for the extremal trajectories

$$x_t = u_1 t, \qquad y_t = u_2 t, \qquad R_t = \exp(t(u_2 A_1 - u_1 A_2)).$$

In the abnormal case the sphere is rolling steadily along a straight line. Any abnormal trajectory is optimal, that is, contains no cut points.

2.2. Normal case. Now let us consider the case $\nu = -1$. From the maximum condition (2.3) we get

$$u_1 = h_1 - H_2, \qquad u_2 = h_2 + H_1.$$

The maximized Hamiltonian has the form

$$H = \frac{1}{2}\big((h_1 - H_2)^2 + (h_2 + H_1)^2\big),$$

and the corresponding Hamiltonian system is

$$\dot{h}_1 = \dot{h}_2 = 0, \tag{2.9}$$

$$\dot{H}_1 = (h_1 - H_2)H_3, \tag{2.10}$$

$$\dot{H}_2 = (h_2 + H_1)H_3, \tag{2.11}$$

$$\dot{H}_3 = -h_1 H_1 - h_2 H_2, \tag{2.12}$$

$$\dot{Q} = (h_1 - H_2)X_1 + (h_2 + H_1)X_2. \tag{2.13}$$

As is usual in sub-Riemannian problems, we can restrict our consideration to unit speed geodesics, that is, extremal trajectories along which

$$H = \frac{1}{2}((h_1 - H_2)^2 + (h_2 + H_1)^2) \equiv \frac{1}{2}.$$

Under such restriction it is convenient to pass in the dual space from the coordinates $(h_1, h_2, H_1, H_2, H_3)$ to the new coordinates (r, α, θ, c):

$$h_1 = r\cos\alpha, \qquad h_2 = r\sin\alpha, \tag{2.14}$$

$$h_1 - H_2 = \cos(\theta + \alpha), \qquad h_2 + H_1 = \sin(\theta + \alpha), \tag{2.15}$$

$$c = H_3.$$

In the new coordinates the Hamiltonian system for the normal extremals (2.9)–(2.13) takes the form that we shall use throughout this paper:

$$\dot{\theta} = c, \tag{2.16}$$

$$\dot{c} = -r \sin\theta, \tag{2.17}$$

$$\dot{\alpha} = \dot{r} = 0, \tag{2.18}$$

$$\dot{x} = \cos(\theta + \alpha), \tag{2.19}$$

$$\dot{y} = \sin(\theta + \alpha), \tag{2.20}$$

$$\dot{R} = R\Omega, \qquad \Omega = \sin(\theta + \alpha)A_1 - \cos(\theta + \alpha)A_2. \tag{2.21}$$

The variables α and θ are uniquely determined by the variables h_1, h_2, H_1, H_2 from relations (2.14), (2.15) only if $r^2 = h_1^2 + h_2^2 \neq 0$. In this case the trajectories of the systems (2.9)–(2.13) and (2.16)–(2.21) are in one-to-one correspondence.

If $r^2 = h_1^2 + h_2^2 = 0$, then the value of the parameter α cannot be found from (2.14). In this case we assume that the angle α takes an arbitrary value on the circle $S^1 = \mathbb{R}/(2\pi\mathbb{Z})$. Then each trajectory of the system (2.9)–(2.13) corresponds to an infinite family of trajectories of the system (2.16)–(2.21) with the same function $\theta_t + \alpha$. The multivalued correspondence between the trajectories of the systems (2.9)–(2.13) and (2.16)–(2.21) reduces to changes of the kind $\widetilde{\theta}_t = \theta_t + \gamma$, $\widetilde{\alpha} = \alpha - \gamma$.

The family of normal extremals λ_t is parametrized by the cylinder C consisting of the initial points $\lambda = \lambda_t\big|_{t=0}$:

$$\begin{aligned} C &= \left\{\lambda \in T^*_{Q_0}M \mid H(\lambda) = \frac{1}{2}\right\} \\ &\cong \left\{(h_1, h_2, H_1, H_2, H_3) \in \mathbb{R}^5 \mid (h_1 - H_2)^2 + (h_2 + H_1)^2 = 1\right\} \\ &\cong \left\{(\theta, c, \alpha, r) \mid \theta \in S^1,\ c \in \mathbb{R},\ \alpha \in S^1,\ r \geqslant 0\right\}. \end{aligned}$$

Each initial point $\lambda \in C$ corresponds to the extremal $\lambda_t = e^{t\vec{H}}(\lambda)$. The main object of our further study is the exponential map Exp that takes a pair consisting of an initial point $\lambda \in C$ and a time instant $t > 0$ to the end-point of the corresponding extremal trajectory:

$$\operatorname{Exp}(\lambda, t) = \pi \circ e^{t\vec{H}}(\lambda) = Q_t,$$
$$\operatorname{Exp}\colon N \to M, \qquad N = C \times \mathbb{R}_+ = \{(\lambda, t) \mid \lambda \in C,\ t > 0\}.$$

In the case when $r = 0$ the elastica (x_t, y_t) is a straight line (if $H_3 = c = 0$) or a circle (if $H_3 = c \neq 0$); such elasticae are called *degenerate*.

If $r \neq 0$, then the elastica (x_t, y_t) belongs to one of four classes depending on the total energy $E = c^2/2 - r\cos\theta$ of the pendulum (2.16), (2.17):

1) inflectional if $E \in (-r, r)$;
2) non-inflectional if $E \in (r, +\infty)$;
3) critical if $E = r$ and $c \neq 0$;
4) a straight line if $E = -r$ or $E = r$ and $c = 0$.

Elasticae which belong to the classes 1)–3) will be called *nondegenerate*.

§ 3. Continuous and discrete symmetries

Consider an invertible transformation g that acts in both the inverse image and image of the exponential map:

$$g\colon N \to N, \qquad g\colon M \to M.$$

Formally, we are talking about two different maps, but it is convenient to denote them by the same symbol; the domain of the map (either N or M) will be clear from the context. The map g is called a *symmetry of the exponential map* if $g \circ \operatorname{Exp} = \operatorname{Exp} \circ g$, which means that the following diagram is commutative:

$$\begin{tikzcd} N \arrow[r, "\operatorname{Exp}"] \arrow[d, "g"] & M \arrow[d, "g"] \\ N \arrow[r, "\operatorname{Exp}"] & M \end{tikzcd}$$

In this section we construct the group of symmetries of the exponential map which correspond to symmetries of the elasticae (2.16)–(2.20), that is, rotations and reflections. In § 4 the analysis of the action of this group is used to derive estimates for the cut time in the problem of rolling a sphere over a plane.

3.1. Transformation of normal extremals. In this subsection we consider transformations of normal extremals $\lambda_s = (\theta_s, c_s, \alpha, r, Q_s)$, $s \in [0, t]$, that is, trajectories of the Hamiltonian system

$$\dot{\theta}_s = c_s, \tag{3.1}$$

$$\dot{c}_s = -r \sin \theta_s, \tag{3.2}$$

$$\dot{\alpha} = \dot{r} = 0, \tag{3.3}$$

$$\dot{x}_s = \cos(\theta_s + \alpha), \tag{3.4}$$

$$\dot{y}_s = \sin(\theta_s + \alpha), \tag{3.5}$$

$$\dot{R}_s = R_s \Omega_s, \qquad \Omega_s = \sin(\theta_s + \alpha) A_1 - \cos(\theta_s + \alpha) A_2, \tag{3.6}$$

$$(x_0, y_0, R_0) = (0, 0, \mathrm{Id}). \tag{3.7}$$

In the calculations we use the standard isomorphism between the Lie algebras $\operatorname{so}(3)$ and $\mathbb{R}^3$ (with the algebraic operations defined by the matrix commutator and the cross product, respectively); see, for instance, [4]. A matrix $A \in \operatorname{so}(3)$ is associated with a vector $\widetilde{A} \in \mathbb{R}^3$ by the following rule:

$$A = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix}, \qquad \widetilde{A} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}.$$

Basis matrices A_i correspond to the vectors $\widetilde{A}_i = e_i$ of the standard basis in $\mathbb{R}^3$.

The following relations are useful in calculations (see [7]):

$$\widetilde{e^{tB} A e^{-tB}} = e^{tB} \widetilde{A}, \qquad A, B \in \operatorname{so}(3), \tag{3.8}$$

$$\widetilde{RAR^{-1}} = R\widetilde{A}, \qquad R \in \operatorname{SO}(3), \quad A \in \operatorname{so}(3).$$

3.1.1. *Rotations.* Rotations of elasticae (x_s, y_s) about the origin in the plane (x, y) generate a one-parameter group of symmetries of the trajectories of the Hamiltonian system (3.1)–(3.7)

$$\{\Phi^\beta \mid \beta \in S^1\};$$

here the rotation Φ^β is defined as follows:

$$\Phi^\beta \colon \{\lambda_s \mid s \in [0,t]\} \to \{\lambda_s^\beta \mid s \in [0,t]\}, \tag{3.9}$$

$$\lambda_s = (\theta_s, c_s, \alpha, r, Q_s), \qquad Q_s = (x_s, y_s, R_s), \tag{3.10}$$

$$\lambda_s^\beta = (\theta_s^\beta, c_s^\beta, \alpha^\beta, r, Q_s^\beta), \qquad Q_s^\beta = (x_s^\beta, y_s^\beta, R_s^\beta), \tag{3.11}$$

$$\theta_s^\beta = \theta_s, \qquad c_s^\beta = c_s, \qquad \alpha^\beta = \alpha + \beta, \tag{3.12}$$

$$\begin{pmatrix} x_s^\beta \\ y_s^\beta \end{pmatrix} = \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} x_s \\ y_s \end{pmatrix}, \tag{3.13}$$

$$R_s^\beta = e^{\beta A_3} R_s e^{-\beta A_3}, \qquad \Omega_s^\beta = e^{\beta A_3} \Omega_s e^{-\beta A_3}. \tag{3.14}$$

Proposition 1. *If $\{\lambda_s \mid s \in [0,t]\}$ is a trajectory of the system (3.1)–(3.7), then for any $\beta \in S^1$ the curve $\{\lambda_s^\beta \mid s \in [0,t]\}$ is also a trajectory of this system.*

Proof. This reduces to differentiation. For instance, let us verify the relations

$$\dot R_s^\beta = R_s^\beta \Omega_s^\beta, \qquad \Omega_s^\beta = \sin(\theta_s^\beta + \alpha^\beta) A_1 - \cos(\theta_s^\beta + \alpha^\beta) A_2.$$

We have

$$\begin{aligned} \Omega_s^\beta &= e^{\beta A_3}(\sin(\theta_s + \alpha) A_1 - \cos(\theta_s + \alpha) A_2) e^{-\beta A_3} \\ &= \sin(\theta_s + \alpha) e^{\beta A_3} A_1 e^{-\beta A_3} - \cos(\theta_s + \alpha) e^{\beta A_3} A_2 e^{-\beta A_3}. \end{aligned}$$

Using equality (3.8) we obtain

$$e^{\beta A_3}\widetilde{A_1 e^{-\beta A_3}} = e^{\beta A_3} e_1 = \cos\beta e_1 + \sin\beta e_2,$$

and therefore $e^{\beta A_3} A_1 e^{-\beta A_3} = \cos\beta A_1 + \sin\beta A_2$. A similar calculation gives

$$e^{\beta A_3} A_2 e^{-\beta A_3} = -\sin\beta A_1 + \cos\beta A_2.$$

Hence,

$$\begin{aligned} \Omega_s^\beta &= \sin(\theta_s + \alpha)(\cos\beta A_1 + \sin\beta A_2) - \cos(\theta_s + \alpha)(-\sin\beta A_1 + \cos\beta A_2) \\ &= \sin(\theta_s + \alpha + \beta) A_1 - \cos(\theta_s + \alpha + \beta) A_2 = \sin(\theta_s^\beta + \alpha^\beta) A_1 - \cos(\theta_s^\beta + \alpha^\beta) A_2. \end{aligned}$$

Further, we find

$$\dot R_s^\beta = e^{\beta A_3} \dot R_s e^{-\beta A_3} = e^{\beta A_3} R_s \Omega_s e^{-\beta A_3} = (e^{\beta A_3} R_s e^{-\beta A_3})(e^{\beta A_3} \Omega_s e^{-\beta A_3}) = R_s^\beta \Omega_s^\beta.$$

3.1.2. *Reflections.* In this paragraph, the reflections of the trajectories (θ_s, c_s) of the pendulum (3.1), (3.2) through the coordinate axes θ, c and through the origin are extended to discrete symmetries ε^1, ε^2, ε^3 of the family of trajectories of the Hamiltonian system (3.1)–(3.7):

$$\begin{aligned} \varepsilon^i \colon \{\lambda_s \mid s \in [0,t]\} &\to \{\lambda_s^i \mid s \in [0,t]\}, \qquad i = 1,2,3, \\ \lambda_s = (\theta_s, c_s, \alpha, r, Q_s), &\qquad Q_s = (x_s, y_s, R_s), \\ \lambda_s^i = (\theta_s^i, c_s^i, \alpha^i, r, Q_s^i), &\qquad Q_s^i = (x_s^i, y_s^i, R_s^i). \end{aligned}$$

A similar extension for the generalized Dido problem (the nilpotent sub-Riemannian problem with the growth vector $(2,3,5)$) was obtained in [15].

Reflection ε^1*.* The reflection of the trajectories (θ_s, c_s) of the pendulum (3.1), (3.2) through the coordinate axis θ corresponds to the following discrete symmetry of the family of extremal trajectories:

$$\theta_s^1 = \theta_{t-s}, \qquad c_s^1 = -c_{t-s}, \qquad \alpha^1 = \alpha + \pi, \tag{3.15}$$

$$x_s^1 = x_{t-s} - x_t, \qquad y_s^1 = y_{t-s} - y_t, \tag{3.16}$$

$$R_s^1 = (R_t)^{-1} R_{t-s}, \qquad \Omega_s^1 = -\Omega_{t-s}. \tag{3.17}$$

Proposition 2. *If* $\{\lambda_s \mid s \in [0,t]\}$ *is a trajectory of the system* (3.1)–(3.7)*, then the curve* $\{\lambda_s^1 \mid s \in [0,t]\}$ *also is a trajectory of this system.*

Proof. Let us check the differential equations for the reflected curve λ_s^1:

$$\begin{aligned} \dot\theta_s^1 &= -\dot\theta_{t-s} = -c_{t-s} = c_s^1, \\ \dot c_s^1 &= c_{t-s} = -r \sin\theta_{t-s} = -r\sin\theta_s^1, \\ \dot x_s^1 &= -\dot x_{t-s} = -\cos(\theta_{t-s} + \alpha) = \cos(\theta_{t-s} + \alpha + \pi) = \cos(\theta_s^1 + \alpha^1), \\ \dot y_s^1 &= -\dot y_{t-s} = -\sin(\theta_{t-s} + \alpha) = \sin(\theta_{t-s} + \alpha + \pi) = \sin(\theta_s^1 + \alpha^1), \\ \Omega_s^1 &= \sin(\theta_s^1 + \alpha^1) A_1 - \cos(\theta_s^1 + \alpha^1) A_2 \\ &= -\sin(\theta_{t-s} + \alpha) A_1 + \cos(\theta_{t-s} + \alpha) A_2 = -\Omega_{t-s}, \\ \dot R_s^1 &= -(R_t)^{-1} \dot R_{t-s} = -(R_t)^{-1} R_{t-s} \Omega_{t-s} = R_s^1 \Omega_s^1. \end{aligned}$$

The initial conditions for Q_s^1 are also satisfied:

$$x_s^1\big|_{s=0} = x_t - x_t = 0, \qquad y_s^1\big|_{s=0} = y_t - y_t = 0, \qquad R_s^1\big|_{s=0} = (R_t)^{-1} \dot R_t = \mathrm{Id}.$$

To clarify the geometric meaning of the action of the symmetry ε^1 on the elasticae, we represent this action as follows:

$$\varepsilon^1 \colon \begin{pmatrix} x_s \\ y_s \end{pmatrix} \overset{(1)}{\mapsto} \begin{pmatrix} x_{t-s} \\ y_{t-s} \end{pmatrix} \overset{(2)}{\mapsto} \begin{pmatrix} x_t - x_{t-s} \\ y_t - y_{t-s} \end{pmatrix} \overset{(3)}{\mapsto} \begin{pmatrix} x_{t-s} - x_t \\ y_{t-s} - y_t \end{pmatrix} = \begin{pmatrix} x_s^1 \\ y_s^1 \end{pmatrix}.$$

Here, (1) corresponds to time reversal on the elastica; (2) is the reflection of the plane (x, y) through the midpoint $p_c = (x_t/2, y_t/2)$ of the chord of the elastica, that is, the line segment that joins its initial point $(x_0, y_0) = (0, 0)$ and terminal point (x_t, y_t); (3) is the rotation of the elastica about the origin $(x_0, y_0) = (0, 0)$ through angle π.

On the other hand, the transformation $(3) \circ (2)$ is the translation that shifts the initial point of the elastica to the origin.

Reflection ε^2. The reflection of the trajectories of the pendulum (θ_s, c_s) through the axis c induces the following symmetry of extremal trajectories:

$$\theta_s^2 = -\theta_{t-s}, \qquad c_s^2 = c_{t-s}, \qquad \alpha^2 = \pi - \alpha, \tag{3.18}$$

$$x_s^2 = x_{t-s} - x_t, \qquad y_s^2 = y_t - y_{t-s}, \tag{3.19}$$

$$R_s^2 = I_2(R_t)^{-1}R_{t-s}I_2, \qquad \Omega_s^2 = -I_2\Omega_{t-s}I_2, \tag{3.20}$$

$$I_2 = I_2^{-1} = e^{\pi A_2} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Proposition 3. *If* $\{\lambda_s \mid s \in [0,t]\}$ *is a trajectory of the system* (3.1)–(3.7), *then the curve* $\{\lambda_s^2 \mid s \in [0,t]\}$ *also is a trajectory of this system.*

Proof. As in the proof of Proposition 2, we verify, for example, the validity of the relation for the matrix R:

$$\Omega_s^2 = \sin(\theta_s^2 + \alpha^2)A_1 - \cos(\theta_s^2 + \alpha^2)A_2 = \sin(\theta_{t-s} + \alpha)A_1 + \cos(\theta_{t-s} + \alpha)A_2,$$

$$\begin{aligned} -I_2\Omega_{t-s}I_2 &= -e^{\pi A_2}\big[\sin(\theta_{t-s} + \alpha)A_1 - \cos(\theta_{t-s} + \alpha)A_2\big]e^{-\pi A_2} \\ &= -\sin(\theta_{t-s} + \alpha)e^{\pi A_2}A_1e^{-\pi A_2} + \cos(\theta_{t-s} + \alpha)e^{\pi A_2}A_2e^{-\pi A_2} \\ &= \sin(\theta_{t-s} + \alpha)A_1 + \cos(\theta_{t-s} + \alpha)A_2 = \Omega_s^2, \end{aligned}$$

$$\begin{aligned} \dot R_s^2 &= -I_2(R_t)^{-1}\dot R_{t-s}I_2 = -I_2(R_t)^{-1}R_{t-s}\Omega_{t-s}I_2 \\ &= \big(I_2(R_t)^{-1}R_{t-s}I_2\big)(-I_2\Omega_{t-s}I_2) = R_s^2\Omega_s^2. \end{aligned}$$

The action of the reflection ε^2 on the elasticae may be represented as follows:

$$\begin{aligned} \varepsilon^2\colon \begin{pmatrix} x_s \\ y_s \end{pmatrix} &\overset{(1)}{\mapsto} \begin{pmatrix} x_{t-s} \\ y_{t-s} \end{pmatrix} \overset{(2)}{\mapsto} \begin{pmatrix} x_t \\ y_t \end{pmatrix} + \begin{pmatrix} -\cos 2\chi & -\sin 2\chi \\ -\sin 2\chi & \cos 2\chi \end{pmatrix} \begin{pmatrix} x_{t-s} \\ y_{t-s} \end{pmatrix} \\ &\overset{(3)}{\mapsto} \begin{pmatrix} -\cos 2\chi & -\sin 2\chi \\ \sin 2\chi & -\cos 2\chi \end{pmatrix} \left[\begin{pmatrix} x_t \\ y_t \end{pmatrix} + \begin{pmatrix} -\cos 2\chi & -\sin 2\chi \\ -\sin 2\chi & \cos 2\chi \end{pmatrix} \begin{pmatrix} x_{t-s} \\ y_{t-s} \end{pmatrix}\right] \\ &= \begin{pmatrix} x_{t-s} - x_t \\ y_t - y_{t-s} \end{pmatrix} = \begin{pmatrix} x_s^2 \\ y_s^2 \end{pmatrix}; \end{aligned}$$

here, χ is the polar angle of the point (x_t, y_t):

$$\cos\chi = \frac{x_t}{\sqrt{x_t^2 + y_t^2}}, \qquad \sin\chi = \frac{y_t}{\sqrt{x_t^2 + y_t^2}}.$$

In other words, the reflection ε^2 acts on the elasticae as a composition of three transformations: (1) the time reversal on the elastica, (2) the reflection of the plane (x, y) across the perpendicular bisector $l^\perp$ of the chord of the elastica, and (3) the rotation through angle $(\pi - 2\chi)$.

Reflection ε^3. The reflection of the trajectories of the pendulum (θ_s, c_s) through the origin $(\theta, c) = (0, 0)$ is extended to the following symmetry of extremal trajectories:

$$\theta_s^3 = -\theta_s, \qquad c_s^3 = -c_s, \qquad \alpha^3 = -\alpha, \tag{3.21}$$

$$x_s^3 = x_s, \qquad y_s^3 = -y_s, \tag{3.22}$$

$$R_s^3 = I_2R_sI_2, \qquad \Omega_s^3 = I_2\Omega_sI_2. \tag{3.23}$$

Proposition 4. *If $\{\lambda_s \mid s \in [0,t]\}$ is a trajectory of the system (3.1)–(3.7), then the curve $\{\lambda_s^3 \mid s \in [0,t]\}$ also is a trajectory of this system.*

Proof. As in the proof of Proposition 3, we shall verify the relation for the rotation matrix R:

$$\Omega_s^3 = \sin(\theta_s^3 + \alpha^3)A_1 - \cos(\theta_s^3 + \alpha^3)A_2 = -\sin(\theta_s + \alpha)A_1 - \cos(\theta_s + \alpha)A_2,$$

$$\begin{aligned} I_2\Omega_s I_2 &= I_2[\sin(\theta_s + \alpha)A_1 - \cos(\theta_s + \alpha)A_2]I_2 \\ &= -\sin(\theta_s + \alpha)A_1 - \cos(\theta_s + \alpha)A_2 = \Omega_s^3, \end{aligned}$$

$$\dot{R}_s^3 = I_2\dot{R}_s I_2 = I_2 R_s \Omega_s I_2 = (I_2 R_s I_2)(I_2\Omega_s I_2) = R_s^3\Omega_s^3.$$

Unlike the reflections ε^1 and ε^2, the reflection ε^3 not only takes arcs of extremal trajectories to arcs of extremal trajectories, but also represents a symmetry of the Hamiltonian system $\dot{\lambda} = \vec{H}(\lambda)$. Formulae (3.21)–(3.23) specify the map $\varepsilon^3 \colon T^*M \to T^*M$ such that for any trajectory λ_s of the Hamiltonian system, its image $\varepsilon^3(\lambda_s)$ is a trajectory of this system as well. Moreover, in the case of the reflection ε^3, the restriction to the time segment $s \in [0,t]$ is not required, since the map ε^3 (3.21)–(3.23) does not depend on the terminal instant by contrast with the map ε^1 (3.15)–(3.17) and ε^2 (3.18)–(3.20).

The reflection ε^3 has the following geometric meaning for the elasticae (x_s, y_s):

$$\begin{aligned} \varepsilon^3 \colon \begin{pmatrix} x_s \\ y_s \end{pmatrix} &\overset{(1)}{\mapsto} \begin{pmatrix} \cos 2\chi & \sin 2\chi \\ \sin 2\chi & -\cos 2\chi \end{pmatrix} \begin{pmatrix} x_s \\ y_s \end{pmatrix} \\ &\overset{(2)}{\mapsto} \begin{pmatrix} \cos 2\chi & \sin 2\chi \\ -\sin 2\chi & \cos 2\chi \end{pmatrix} \left[\begin{pmatrix} \cos 2\chi & \sin 2\chi \\ \sin 2\chi & -\cos 2\chi \end{pmatrix} \begin{pmatrix} x_s \\ y_s \end{pmatrix} \right] = \begin{pmatrix} x_s \\ -y_s \end{pmatrix} = \begin{pmatrix} x_s^3 \\ y_s^3 \end{pmatrix}. \end{aligned}$$

In other words, it is a composition of (1) the reflection of the plane (x, y) across the chord of the elastica and (2) the rotation through angle (-2χ). On the other hand,

$$\varepsilon^3 \colon \begin{pmatrix} x_s \\ y_s \end{pmatrix} \mapsto \begin{pmatrix} x_s \\ -y_s \end{pmatrix} = \begin{pmatrix} x_s^3 \\ y_s^3 \end{pmatrix}$$

is the reflection of the elastica through the x-axis.

3.2. Symmetries of the exponential mapping. In this paragraph the action of the rotations Φ^β and reflections ε^i in the inverse image and image of the exponential map is defined so as to commute with the action of the exponential map.

The rotations $\Phi^\beta \colon \lambda \mapsto \lambda^\beta$ (3.9)–(3.14) are symmetries of the Hamiltonian system; therefore, the action of these rotations in T^*M is naturally represented as a direct sum of the actions in $N = T^*_{Q_0}M \times \mathbb{R}_+$ (on (λ, t), where λ is the initial point of the extremal) and in M (on Q_t, which is the end of the corresponding extremal trajectory):

$$\Phi^\beta \colon N \to N, \qquad (\lambda, t) \mapsto (\lambda^\beta, t),$$

$$\lambda = (\theta, c, \alpha, r), \qquad \lambda^\beta = (\theta, c, \alpha^\beta, r), \qquad \alpha^\beta = \alpha + \beta,$$

and

$$\Phi^\beta\colon M \to M, \qquad Q \mapsto Q^\beta,$$
$$Q = (x,y,R), \qquad Q^\beta = (x^\beta, y^\beta, R^\beta),$$
$$\begin{pmatrix} x^\beta \\ y^\beta \end{pmatrix} = \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} x \\ y, \end{pmatrix}, \qquad R^\beta = e^{\beta A_3} R e^{-\beta A_3}.$$

The action of the reflections ε^i in N is defined as the restriction of their action on the vertical components of the extremal trajectories at the initial time instant $s=0$:

$$\varepsilon^i\colon N \to N, \qquad (\lambda, t) \mapsto (\lambda^i, t), \qquad i = 1,2,3,$$
$$\lambda = (\theta, c, \alpha, r), \qquad \lambda^i = (\theta^i, c^i, \alpha^i, r),$$

where $\lambda = \lambda_s\big|_{s=0}$ and $\lambda^i = \lambda^i_s\big|_{s=0}$. Taking into account the explicit expressions for the action of the reflections on the arcs of extremal trajectories derived in § 3.1, we obtain explicit expressions for the action of ε^i in N:

$$\varepsilon^1\colon (\theta, c, \alpha, r, t) \mapsto (\theta^1, c^1, \alpha^1, r, t) = (\theta_t, -c_t, \alpha + \pi, r, t),$$
$$\varepsilon^2\colon (\theta, c, \alpha, r, t) \mapsto (\theta^2, c^2, \alpha^2, r, t) = (-\theta_t, c_t, \pi - \alpha, r, t),$$
$$\varepsilon^3\colon (\theta, c, \alpha, r, t) \mapsto (\theta^3, c^3, \alpha^3, r, t) = (-\theta, -c, -\alpha, r, t).$$

The action of the reflections in M is defined as their action on the extremal trajectories at the terminal time instant $s=t$:

$$\varepsilon^i\colon M \to M, \qquad Q \mapsto Q^i, \qquad i = 1,2,3,$$
$$Q = (x,y,R), \qquad Q^i = (x^i, y^i, R^i),$$

where $Q = Q_s\big|_{s=t}$, $Q^i = Q^i_s\big|_{s=t}$. Explicit formulae (according to the results of § 3.1) have the form

$$\varepsilon^1\colon (x,y,R) \mapsto (x^1, y^1, R^1) = (-x, -y, (R)^{-1}),$$
$$\varepsilon^2\colon (x,y,R) \mapsto (x^2, y^2, R^2) = (-x, y, I_2 (R)^{-1} I_2),$$
$$\varepsilon^3\colon (x,y,R) \mapsto (x^3, y^3, R^3) = (x, -y, I_2 R I_2).$$

Thus, we have defined the action of the rotations and reflections in the inverse image and image of the exponential map:

$$\Phi^\beta, \varepsilon^i\colon N \to N, \qquad (\lambda, t) \mapsto (\lambda^\beta, t), (\lambda^i, t), \tag{3.24}$$
$$\Phi^\beta, \varepsilon^i\colon M \to M, \qquad Q \mapsto Q^\beta, Q^i. \tag{3.25}$$

It is important that the image $Q^i = \varepsilon^i(Q)$ depends only on the inverse image Q and not on the time instant t.

Proposition 5. *The maps Φ^β and ε^i are symmetries of the exponential map.*

Proof. This follows from the fact that Φ^β and ε^i take arcs of extremal trajectories into arcs of extremal trajectories and from the definition of the action of Φ^β and ε^i in N and M.

Consider the group of symmetries of the exponential map generated by the rotations and reflections:

$$G = \langle \Phi^\beta, \varepsilon^1, \varepsilon^2, \varepsilon^3 \rangle.$$

The multiplication table in this group for the actions $g\colon N \to N$ and $g\colon M \to M$, $g \in G$, coincides with the multiplication table for the action on the trajectories of the pendulum $g\colon \{(\theta_s, c_s)\} \to \{(\theta_s^i, c_s^i)\}$ and has the form

$\cdot \circ \cdot$	ε^1	ε^2	ε^3	Φ^β
ε^1	Id	ε^3	ε^2	$\Phi^\beta \circ \varepsilon^1$
ε^2	ε^3	Id	ε^1	$\Phi^{-\beta} \circ \varepsilon^2$
ε^3	ε^2	ε^1	Id	$\Phi^{-\beta} \circ \varepsilon^3$
Φ^γ	$\varepsilon^1 \circ \Phi^\gamma$	$\varepsilon^2 \circ \Phi^{-\gamma}$	$\varepsilon^3 \circ \Phi^{-\gamma}$	$\Phi^{\beta+\gamma}$

Hence we obtain an explicit description of the symmetry group:

$$G = \{\Phi^\beta,\ \Phi^\beta \circ \varepsilon^i \mid \beta \in S^1,\ i = 1, 2, 3\}.$$

§ 4. The Maxwell strata corresponding to symmetries

A point Q_t of an extremal trajectory $Q_s = \operatorname{Exp}(\lambda, s)$ is said to be a *Maxwell point* if there exists another extremal trajectory $\widetilde{Q}_s \not\equiv Q_s$ such that $\widetilde{Q}_t = Q_t$. In the analytic sub-Riemannian problem an extremal trajectory is known to lose optimality after the Maxwell point (see, for instance, [18]).

Let us define the *Maxwell set* in the inverse image of the exponential map:

$$\mathrm{MAX} = \bigl\{(\lambda, t) \in N \mid \exists\, (\widetilde{\lambda}, t) \in N : \operatorname{Exp}(\lambda, s) \not\equiv \operatorname{Exp}(\widetilde{\lambda}, s),\ \operatorname{Exp}(\lambda, t) = \operatorname{Exp}(\widetilde{\lambda}, t)\bigr\}.$$

If $(\lambda, t) \in \mathrm{MAX}$, then the extremal trajectory $\operatorname{Exp}(\lambda, s)$ is not optimal for $s > t$.

Also, let us define the *Maxwell set corresponding to the group* G:

$$\begin{aligned}\mathrm{MAX}^G = \bigl\{(\lambda, t) \in N \mid \exists\, g \in G :\\ (\widetilde{\lambda}, t) = g(\lambda, t),\ \operatorname{Exp}(\lambda, s) \not\equiv \operatorname{Exp}(\widetilde{\lambda}, s),\ \operatorname{Exp}(\lambda, t) \not\equiv \operatorname{Exp}(\widetilde{\lambda}, t)\bigr\},\end{aligned}$$

and the *Maxwell set corresponding to the group* $\langle \varepsilon^i, \Phi^\beta \rangle$, $i = 1, 2, 3$:

$$\begin{aligned}\mathrm{MAX}^i = \bigl\{(\lambda, t) \in N \mid \exists\, \beta \in S^1 :\\ (\widetilde{\lambda}, t) = \varepsilon^i \circ \Phi^\beta(\lambda, t),\ \operatorname{Exp}(\lambda, s) \not\equiv \operatorname{Exp}(\widetilde{\lambda}, s),\ \operatorname{Exp}(\lambda, t) = \operatorname{Exp}(\widetilde{\lambda}, t)\bigr\}.\end{aligned}$$

There are obvious inclusions

$$\mathrm{MAX}^i \subset \mathrm{MAX}^G \subset \mathrm{MAX}, \qquad i = 1, 2, 3.$$

Below, we shall use the following notation:

$$\varepsilon^i \circ \Phi^\beta(\lambda, t) = (\lambda^{\beta,i}, t), \qquad \operatorname{Exp}(\lambda, s) = Q_s, \qquad \operatorname{Exp}(\lambda^{\beta,i}, s) = Q_s^{\beta,i}.$$

With this notation, we have

$$\mathrm{MAX}^i = \bigl\{(\lambda, t) \in N \mid \exists\, \beta \in S^1 : Q_s \not\equiv Q_s^{\beta,i},\ Q_t = Q_t^{\beta,i}\bigr\}.$$

The main aim of this section is to derive equations that specify the Maxwell strata MAX^i. In § 4.1 we study the relations $Q_s \equiv Q_s^{\beta,i}$, and in § 4.2, the relation $Q_t = Q_t^{\beta,i}$. In the final § 4.3 we present the final results related to the strata MAX^i and, as a corollary, give necessary conditions for the optimality in the problem of a sphere rolling along a plane.

The proposition below shows that distinct initial convectors λ and $\widetilde{\lambda}$ may correspond to the same extremal trajectory $Q_s \equiv \widetilde{Q}_s$ only when the elastica $(x_s, y_s) \equiv (\widetilde{x}_x, \widetilde{y}_s)$ is a straight line.

Proposition 6. *Let* $\lambda = (\theta, c, r, \alpha, Q_0)$, $\widetilde{\lambda} = (\widetilde{\theta}, \widetilde{c}, \widetilde{r}, \widetilde{\alpha}, Q_0) \in C$, $\lambda_s = e^{s\vec{H}}(\lambda) = (\theta_s, c_s, \alpha, r, Q_s)$, $\widetilde{\lambda}_s = e^{s\vec{H}}(\widetilde{\lambda}) = (\widetilde{\theta}_s, \widetilde{c}_s, \widetilde{\alpha}, \widetilde{r}, \widetilde{Q}_s)$. *Then*

$$
\begin{aligned}
Q_s \equiv \widetilde{Q}_s \quad &\Longleftrightarrow \quad \theta_s + \alpha \equiv \widetilde{\theta}_s + \widetilde{\alpha} \\
&\Longleftrightarrow \quad \left[\begin{array}{ll} (1) & \lambda = \widetilde{\lambda} \quad \textit{or} \\ (2) & c = \widetilde{c} = 0, \quad \theta + \alpha = \widetilde{\theta} + \widetilde{\alpha}, \quad r \sin\theta = \widetilde{r}\sin\widetilde{\theta} = 0; \end{array} \right.
\end{aligned}
$$

moreover, in case (2) *the elastica* $(x_s, y_s) \equiv (\widetilde{x}_x, \widetilde{y}_s)$ *is a straight line.*

Proof. We have the chain of equivalent relations

$$
\begin{aligned}
Q_s \equiv \widetilde{Q}_s \quad &\Longleftrightarrow \quad x_s \equiv \widetilde{x}_s, \quad y_s \equiv \widetilde{y}_s, \quad R_s \equiv \widetilde{R}_s \quad \Longleftrightarrow \quad \theta_s + \alpha \equiv \widetilde{\theta}_s + \widetilde{\alpha} \\
&\Longleftrightarrow \quad \theta_s + \alpha \equiv \widetilde{\theta}_s + \widetilde{\alpha}, \qquad \widetilde{c}_s \equiv c_s, \qquad r\sin\theta_s \equiv \widetilde{r}\sin\widetilde{\theta}_s. \qquad (4.1)
\end{aligned}
$$

A. If $\widetilde{c}_s \equiv c_s \equiv 0$, then θ_s, $\widetilde{\theta}_s \equiv \mathrm{const}$, and relations (4.1) are equivalent to equalities (2) in the statement of this proposition.

B. Let $\widetilde{c}_s \equiv c_s \not\equiv 0$. Differentiating the third relation in (4.1), by virtue of the Hamiltonian system of the maximum principle we obtain $r\cos\theta_s \equiv \widetilde{r}\cos\widetilde{\theta}_s$, and hence $r = \widetilde{r}$.
   1) If $r = \widetilde{r} = 0$, then $\lambda_s \equiv \widetilde{\lambda}_s$ and $\lambda = \widetilde{\lambda}$.
   2) If $r = \widetilde{r} \neq 0$, then $\sin\theta_s \equiv \sin\widetilde{\theta}_s$, whence follows the identity

$$
\sin\theta_s - \sin\widetilde{\theta}_s = 2\cos\frac{\theta_s + \widetilde{\theta}_s}{2}\sin\frac{\widetilde{\alpha} - \alpha}{2} \equiv 0.
$$

   2.1) If $\widetilde{\alpha} - \alpha = 0$, then $\lambda = \widetilde{\lambda}$.
   2.2) If $\theta_s + \widetilde{\theta}_s = \pi + 2\pi k$, then $\theta_s \equiv \pi n$, $\widetilde{\theta}_s \equiv \pi m$, and this gives equalities (2) in the statement of this proposition.

4.1. Fixed points of symmetries in the inverse image of the exponential map. In this subsection we derive necessary conditions for the validity of the relation $Q_s \equiv Q_s^{\beta,i}$ in the case of a nondegenerate elastica (x_s, y_s). As follows from Proposition 6, in this case the relation $Q_s \equiv Q_s^{\beta,i}$ is equivalent to the equality $\lambda = \lambda^{\beta,i}$.

Proposition 7. *Let* $(\lambda, t) \in N$, $(\lambda^{\beta,i}, t) = \varepsilon^i \circ \Phi^\beta(\lambda, t)$, $i \in \{1, 2, 3\}$, *and let elastica* $\gamma = \{(x_s, y_s) \mid s \in [0, t]\}$ *be nondegenerate. If* $Q_s \equiv Q_s^{\beta,i}$, *then:*

(i) *in the case* $i = 1$ *the elastica* γ *is centred at the vertex;*
(ii) *in the case* $i = 2$ *the elastica* γ *is centred at the inflection point;*
(iii) *in the case* $i = 3$ *the relation* $Q_s \equiv Q_s^{\beta,i}$ *cannot hold.*

We say that an elastica $\gamma = \{(x_s, y_s) \mid s \in [0,t]\}$ is *centred at a vertex* (at an inflection point) if its midpoint $(x_{t/2}, y_{t/2})$ is a vertex, that is, a local extremum of the curvature (respectively, an inflection point, that is, a point where the curvature changes sign).

Proof. If $Q_s \equiv Q_s^{\beta,i}$, then by Proposition 6 we have the equality $\lambda = \lambda^{\beta,i}$. Therefore, for the trajectories $\lambda_s = (\theta_s, c_s, \alpha, r, Q_s)$ and $\lambda_s^{\beta,i} = (\theta_s^{\beta,i}, c_s^{\beta,i}, \alpha^{\beta,i}, r, Q_s^{\beta,i})$ of the Hamiltonian system of the maximum principle we obtain $c_s \equiv c_s^{\beta,i}$.

(i) If $i = 1$, then $c_s \equiv c_s^{\beta,1} \equiv c_s^1 \equiv -c_{t-s}$. The curvature of the elastica c_s is odd with respect to the midpoint $s = t/2$; therefore, $c_{t/2} = 0$. In the case of a nondegenerate elastica, any zero of the curvature is regular ($c_s = 0 \Longrightarrow \dot{c}_s \neq 0$); hence the midpoint $(x_{t/2}, y_{t/2})$ of the elastica γ is an inflection point.

(ii) If $i = 2$, then $c_s \equiv c_s^{\beta,2} \equiv c_s^2 \equiv c_{t-s}$. The curvature of the elastica c_s is even with respect to the midpoint $s = t/2$. Therefore, the curvature has a local extremum at the point $s = t/2$, and the elastica γ has a vertex at the midpoint $(x_{t/2}, y_{t/2})$.

(iii) If $i = 3$, then $\theta_s \equiv \theta_s^{3,\beta} \equiv -\theta_s$, which means that $\theta_s \equiv 0$. The elastica γ degenerates into a straight line, which contradicts the hypothesis of this proposition.

4.2. Fixed points of symmetries in the image of the exponential map. In this subsection we investigate points Q_t of extremal trajectories which satisfy the equation $Q_t = Q_t^{\beta,i}$ mentioned in the definition of the Maxwell strata MAX^i. We derive equations of manifolds in the state space M which contain these points. In these equations we employ the representation of the rotation matrix $Q \in \mathrm{SO}(3)$ with the use of quaternions $q = q_0 + q_1 i + q_2 j + q_3 k \in S^3 \subset \mathbb{H}$ (see § 5). The correspondence $Q \mapsto q$ determines the quaternion q up to its sign; this agrees with the fact that the equations derived below for the quaternion are invariant under the inversion $q \mapsto -q$ (see, for example, (4.3)).

4.2.1. *Equation* $Q_t = Q_t^{\beta,1}$. Proposition 8 below describes fixed points of the action of the group of symmetries $\langle \varepsilon^1, \Phi^\beta \rangle$ in the image of the exponential map; namely, points $Q \in M$ such that

$$\exists\, \beta \in S^1 \ \text{ satisfying the condition } \varepsilon^1 \circ \Phi^\beta(Q) = Q. \tag{4.2}$$

Proposition 8. *Let $Q = (x, y, R) \in M$ and let $q = q_0 + q_1 i + q_2 j + q_3 k \in S^3$ be the quaternion that corresponds to the rotation $R \in \mathrm{SO}(3)$.*

(i) *If $(x, y) \neq (0,0)$, then condition (4.2) is equivalent to the disjunction*

$$q_3 = 0 \quad \text{or } q = \pm k, \tag{4.3}$$

moreover, in this case $\beta = \pi$.

(ii) *If $(x, y) = (0,0)$, then condition (4.3) implies (4.2), and it may be assumed that $\beta = \pi$.*

Proof. We have

$$\varepsilon^1 \circ \Phi^\beta \begin{pmatrix} x \\ y \\ R \end{pmatrix} = \begin{pmatrix} -x\cos\beta + y\sin\beta \\ -x\sin\beta - y\cos\beta \\ e^{\beta A_3} R^{-1} e^{-\beta A_3} \end{pmatrix}.$$

Further, $e^{\beta A_3}R^{-1}e^{-\beta A_3} = R \iff Re^{\beta A_3}Re^{-\beta A_3} = \mathrm{Id}$.

(i) Suppose that $(x,y) \neq (0,0)$. Then

$$\begin{pmatrix} -x\cos\beta + y\sin\beta \\ -x\sin\beta - y\cos\beta \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \quad \iff \quad \beta = \pi.$$

Further,

$$\begin{aligned} \varepsilon^1 \circ \Phi^\beta(Q) = Q, \quad \beta = \pi \quad &\iff \quad RI_3RI_3 = \mathrm{Id} \quad \iff \quad (RI_3)^2 = \mathrm{Id} \\ \iff \quad (qk)^2 = \pm 1 \quad &\iff \quad q_3 = 0 \quad \text{or } q = \pm k. \end{aligned}$$

(ii) If $(x,y) = (0,0)$, then we assume that $\beta = \pi$ and, similarly to item (i), conclude that condition (4.3) implies (4.2).

4.2.2. *Equation* $Q_t = Q_t^{\beta,2}$. Proposition 9 below describes the fixed points of the action of the group of symmetries $\langle \varepsilon^2, \Phi^\beta \rangle$ in the image of the exponential map, namely, the points $Q \in M$ such that

$$\exists\, \beta \in S^1 \ \text{ satisfying the condition } \varepsilon^2 \circ \Phi^\beta(Q) = Q. \tag{4.4}$$

Proposition 9. *Let $Q = (x,y,R) \in M$ and let $q = q_0 + q_1 i + q_2 j + q_3 k \in S^3$ be the quaternion that corresponds to the rotation $R \in \mathrm{SO}(3)$.*

(i) *If $(x,y) = (\rho\cos\chi, \rho\sin\chi) \neq (0,0)$, then condition (4.4) is equivalent to the disjunction*

$$xq_1 + yq_2 = 0 \quad \text{or } q = \pm(\cos\chi\, i + \sin\chi\, j);$$

moreover, in this case $\beta = \pi - 2\chi$.

(ii) *If $(x,y) = (0,0)$, then condition (4.4) is satisfied and β can be found from the equation $q_1\sin(\beta/2) + q_2\cos(\beta/2) = 0$.*

Proof. We have

$$\varepsilon^2 \circ \Phi^\beta \begin{pmatrix} x \\ y \\ R \end{pmatrix} = \begin{pmatrix} -x\cos\beta + y\sin\beta \\ x\sin\beta + y\cos\beta \\ I_2 e^{\beta A_3} R^{-1} e^{-\beta A_3} I_2 \end{pmatrix} = \begin{pmatrix} -x\cos\beta + y\sin\beta \\ x\sin\beta + y\cos\beta \\ e^{-\beta A_3} I_2 R^{-1} e^{-\beta A_3} I_2 \end{pmatrix}.$$

Further,

$$\begin{aligned} &e^{-\beta A_3} I_2 R^{-1} e^{-\beta A_3} I_2 = R \quad \iff \quad RI_2 e^{\beta A_3} RI_2 e^{\beta A_3} = \mathrm{Id} \\ &\iff \quad (RI_2 e^{\beta A_3})^2 = \mathrm{Id} \quad \iff \quad \left[qj\left(\cos\left(\frac{\beta}{2}\right) + \sin\left(\frac{\beta}{2}\right) k \right] ^2 = \pm 1 \\ &\iff \quad q\left[\sin\left(\frac{\beta}{2}\right) i + \cos\left(\frac{\beta}{2}\right) j \right]^2 = \pm 1 \\ &\iff \quad \left[\begin{array}{l} \mathrm{Re}\left[q\left(\sin\left(\frac{\beta}{2}\right) i + \cos\left(\frac{\beta}{2}\right) j \right) \right] = 0, \\ q\left(\sin\left(\frac{\beta}{2}\right) i + \cos\left(\frac{\beta}{2}\right) j \right) = \pm 1 \end{array} \right. \\ &\iff \quad \left[\begin{array}{l} q_1 \sin\left(\frac{\beta}{2}\right) + q_2 \cos\left(\frac{\beta}{2}\right) = 0, \\ q = \pm\left(\sin\left(\frac{\beta}{2}\right) i + \cos\left(\frac{\beta}{2}\right) j \right). \end{array} \right. \end{aligned} \tag{4.5}$$

(i) Suppose that $(x, y) = (\rho \cos \chi, \rho \sin \chi) \neq (0, 0)$. Then

$$\begin{pmatrix} -\cos \beta x + \sin \beta y \\ \sin \beta x + \cos \beta y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \quad \Longleftrightarrow \quad \beta = \pi - 2\chi$$

and we have

$$\varepsilon^2 \circ \Phi^\beta(Q) = Q, \quad \beta = \pi - 2\chi \quad \Longleftrightarrow \quad \left[\begin{array}{l} q_1 \cos \chi + q_2 \sin \chi = 0, \\ q = \pm(\cos \chi i + \sin \chi j), \end{array} \right.$$

which proves item (i) of this proposition.

(ii) If $(x, y) = (0, 0)$, then the equality

$$\varepsilon^2 \circ \Phi^\beta(Q) = Q$$

is equivalent to the disjunction (4.5). The first equation

$$q_1 \sin \frac{\beta}{2} + q_2 \cos \frac{\beta}{2} = 0$$

may be satisfied by choosing an appropriate value of the angle β.

4.2.3. *Equation* $Q_t = Q_t^{\beta,3}$. Proposition 10 below describes the points $Q \in M$ such that

$$\exists\, \beta \in S^1 \ \text{ satisfying the condition } \varepsilon^3 \circ \Phi^\beta(Q) = Q. \tag{4.6}$$

Proposition 10. *Let* $Q = (x, y, R) \in M$ *and let*

$$q = q_0 + q_1 i + q_2 j + q_3 k \in S^3$$

be the quaternion that corresponds to the rotation $R \in \mathrm{SO}(3)$.

(i) *If* $(x, y) \neq (0, 0)$, *then condition* (4.6) *is equivalent to the disjunction*

$$\begin{cases} x q_1 + y q_2 = 0, \\ q_3 = 0 \end{cases} \quad or \quad \begin{cases} y q_1 - x q_2 = 0, \\ q_0 = 0; \end{cases} \tag{4.7}$$

moreover, in this case $\beta = -2\chi$.

(ii) *If* $(x, y) = (0, 0)$, *then condition* (4.6) *is equivalent to the disjunction*

$$q_3 = 0 \quad or \quad q_0 = 0, \tag{4.8}$$

and β *may be determined from the equations*

$$\cos\left(\frac{\beta}{2}\right) q_1 - \sin\left(\frac{\beta}{2}\right) q_2 = 0 \quad or \quad \sin\left(\frac{\beta}{2}\right) q_1 + \cos\left(\frac{\beta}{2}\right) q_2 = 0,$$

respectively.

Proof. We have

$$\varepsilon^3 \circ \Phi^\beta(Q) = \begin{pmatrix} x \cos \beta - y \sin \beta \\ -x \sin \beta - y \cos \beta \\ I_2 e^{\beta A_3} R e^{-\beta A_3} I_2 \end{pmatrix}.$$

Then, for any $\beta \in S^1$ and $R \in \mathrm{SO}(3)$, there is a chain of equivalent relations

$$
\begin{aligned}
&I_2 e^{\beta A_3} R e^{-\beta A_3} I_2 \iff I_2 e^{\beta A_3} R = R I_2 e^{\beta A_3} \\
&\iff j\left(\cos\left(\frac{\beta}{2}\right) + \sin\left(\frac{\beta}{2}\right)k\right)(q_0 + q_1 i + q_2 j + q_3 k) \\
&\qquad = \pm(q_0 + q_1 i + q_2 j + q_3 k) j \left(\cos\left(\frac{\beta}{2}\right) + \sin\left(\frac{\beta}{2}\right)k\right) \\
&\iff \left[\begin{array}{l} \cos\left(\frac{\beta}{2}\right) q_3 i - \sin\left(\frac{\beta}{2}\right) q_3 j + \left(\sin\left(\frac{\beta}{2}\right) q_2 - \cos\left(\frac{\beta}{2}\right) q_1\right) k = 0, \\ \sin\left(\frac{\beta}{2}\right) q_1 + \cos\left(\frac{\beta}{2}\right) q_2 - \sin\left(\frac{\beta}{2}\right) q_0 i - \cos\left(\frac{\beta}{2}\right) q_0 j = 0 \end{array}\right. \\
&\iff \begin{cases} \cos\left(\frac{\beta}{2}\right) q_1 - \sin\left(\frac{\beta}{2}\right) q_2 = 0, \\ q_3 = 0 \end{cases} \quad \text{or} \quad \begin{cases} \sin\left(\frac{\beta}{2}\right) q_1 + \cos\left(\frac{\beta}{2}\right) q_2 = 0, \\ q_0 = 0. \end{cases}
\end{aligned}
\tag{4.9}
$$

(i) Suppose that $(x, y) = (\rho\cos\chi, \rho\sin\chi) \neq (0,0)$. Then

$$
\begin{cases} x\cos\beta - y\sin\beta = x, \\ -x\sin\beta - y\cos\beta = y \end{cases} \iff \beta = -2\chi.
$$

Therefore, $\varepsilon^3 \circ \Phi^\beta(Q) = Q$, $\beta = -2\chi$ is equivalent to the disjunction of the conditions

$$
\begin{cases} q_1\cos\chi + q_2\sin\chi = 0, \\ q_3 = 0 \end{cases} \quad \text{or} \quad \begin{cases} q_1\sin\chi - q_2\cos\chi = 0, \\ q_0 = 0, \end{cases}
$$

which is equivalent to conditions (4.7) when $(x, y) \neq (0, 0)$.

(ii) Suppose that $(x, y) = (0, 0)$. Then one can choose an appropriate value of the angle β in the first equalities of (4.9), and condition (4.6) is equivalent to (4.8).

4.3. Optimality conditions for extremal trajectories. In this subsection we obtain the main results of this work, namely, conditions under which the end-points of extremal trajectories belong to the Maxwell strata and, as a consequence, necessary conditions for the optimality of extremal trajectories of general position.

Theorem 2. *Suppose that $t > 0$ and let $Q_s = (x_s, y_s, R_s) = \mathrm{Exp}(\lambda, s)$ be an extremal trajectory such that*:

(i) $q_3(t) = 0$;

(ii) *the elastica $\{(x_s, y_s) \mid s \in [0, t]\}$ is nondegenerate and is not centred at an inflection point.*

Then $(\lambda, t) \in \mathrm{MAX}^1$, and hence for any $t_1 > t$ the trajectory Q_s, $s \in [0, t_1]$, is not optimal.

Proof. Let us set $\beta = \pi$. Then by Proposition 8 we have $Q_t^{\beta,1} = Q_t$, and by Proposition 7 we have $Q_s^{\beta,1} \not\equiv Q_s$. Therefore, $(\lambda, t) \in \mathrm{MAX}^1$.

Theorem 3. *Suppose that $t > 0$ and let $Q_s = (x_s, y_s, R_s) = \mathrm{Exp}(\lambda, s)$ be an extremal trajectory such that*:

(i) $(xq_1 + yq_2)(t) = 0$;

(ii) *the elastica $\{(x_s, y_s) \mid s \in [0, t]\}$ is nondegenerate and is not centered at a vertex.*

Then $(\lambda, t) \in \mathrm{MAX}^2$, and hence for any $t_1 > t$ the trajectory Q_s, $s \in [0, t_1]$, is not optimal.

Proof. We shall choose the value of angle $\beta \in S^1$ as follows. We take $\beta = \pi - 2\chi$ if $(x, y)(t) = (\rho\cos\chi, \rho\sin\chi) \neq (0, 0)$, and in the case when $(x, y)(t) = (0, 0)$ we find the value of β from the equation $q_1(t)\sin(\beta/2) + q_2(t)\cos(\beta/2) = 0$.

Then by Proposition 9 we have $Q_t^{\beta,2} = Q_t$, and Proposition 7 yields $Q_s^{\beta,2} \not\equiv Q_s$. Therefore, $(\lambda, t) \in \mathrm{MAX}^2$.

Theorem 4. *Suppose that $t > 0$ and let $Q_s = (x_s, y_s, R_s) = \mathrm{Exp}(\lambda, s)$ be an extremal trajectory such that*:

(i) $(xq_1 + yq_2)(t) = q_3(t) = 0$ *or* $(yq_1 - xq_2)(t) = q_0(t) = 0$;

(ii) *the elastica $\{(x_s, y_s) \mid s \in [0, t]\}$ is nondegenerate.*

Then $(\lambda, t) \in \mathrm{MAX}^3$, and hence for any $t_1 > t$ the trajectory Q_s, $s \in [0, t_1]$, is not optimal.

Proof. It follows from Proposition 10 that the value of the angle $\beta \in S^1$ can be chosen in such a way that $Q_t^{\beta,3} = Q_t$. By item (iii) of Proposition 7, $Q_s^{\beta,3} \not\equiv Q_s$. Therefore, $(\lambda, t) \in \mathrm{MAX}^3$.

Remark. Taking into account that for any quaternion $q = q_0 + iq_1 + jq_2 + kq_3 \in S^3$ the corresponding motion $R_q \colon \mathbb{R}^3 \to \mathbb{R}^3$ is a rotation about the vector $(q_1, q_2, q_3) \in \mathbb{R}^3$, we can suggest the following interpretation of conditions (i) in Theorems 2–4:

- condition (i) in Theorem 2 means that the rotation of the sphere R_t is a rotation about some horizontal axis;
- condition (i) in Theorem 3 means that the rotation R_t is a rotation about an axis orthogonal to the displacement vector of the point of contact between the sphere and the plane $(x_t, y_t, 0)$;
- condition (i) in Theorem 4 means that the rotation R_t is a rotation about the horizontal axis which is orthogonal to the vector $(x_t, y_t, 0)$, or that R_t is the rotation through angle π about the axis which lies in the vertical plane and contains the vector $(x_t, y_t, 0)$.

For a fuller investigation of the cut points, it is desirable to possess efficient estimates for the first roots of the equations $q_3(t) = 0$ and $(xq_1 + yq_2)(t) = 0$. The asymptotic behaviour of these estimates near the stable equilibrium of the pendulum (that is, as $(\theta, c) \to (0, 0)$) was found by Mashtakov. In fact, even this asymptotic is much more complicated than similar global estimates for the first Maxwell time in the Euler elastica problem (see [12]), the generalized Dido problem (see [18]), and the sub-Riemannian problem on the group of motions of a plane (see [19]). Therefore, the problem of optimal rolling of a sphere over a plane is much more complicated than these related optimal control problems.

§ 5. Appendix: representation of rotations of three-dimensional space by quaternions

To represent the orientation of the rolling sphere, apart from the matrix $R \in \mathrm{SO}(3)$ it is convenient to use quaternions. Let us recall some facts related to this representation (see, for example, [20] and [21]). Consider the quaternion division ring

$$\mathbb{H} = \{q = q_0 + iq_1 + jq_2 + kq_3 \mid q_0, \dots, q_3 \in \mathbb{R}\},$$

the unit sphere

$$S^3 = \{q \in \mathbb{H} \mid |q|^2 = q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1\}$$

in this ring, and the subspace of purely imaginary quaternions

$$I = \{q \in \mathbb{H} \mid \operatorname{Re} q = q_0 = 0\} \simeq \mathbb{R}^3.$$

Any quaternion $q \in S^3$ acts on the Euclidean space I:

$$q \in S^3 \quad \Longrightarrow \quad R_q(a) = qaq^{-1}, \quad a \in I,$$

and the map $R_q\colon I \to I$ is a rotation, that is, $R_q \in \mathrm{SO}_3(I)$. The map

$$p\colon q \to R_q, \qquad S^3 \to \mathrm{SO}_3(I) \simeq \mathrm{SO}(3)$$

is a double cover:

$$R_q = R_{\widetilde{q}} \quad \Longleftrightarrow \quad q = \pm\widetilde{q}.$$

For any quaternion $q = q_0 + iq_1 + jq_2 + kq_3 \in S^3$ the motion $R_q\colon I \to I$ is a rotation about the vector $a = \operatorname{Im} q = iq_1 + jq_2 + kq_3$ since

$$R_q(a) = qaq^{-1} = (q_0 + a)a(q_0 + a)^{-1} = (q_0 + a)a(q_0 - a) = (q_0 + a)(q_0 - a)a = a.$$

In other words, $\operatorname{Im} q$ is an eigenvector of the map R_q.

Any quaternion $q \in S^3$ can be written in the form

$$q = \cos\alpha + \sin\alpha\, u, \qquad u \in S^3 \cap I, \quad \alpha \in S^1.$$

Then R_q is the rotation of the space I through angle 2α about the vector u: in the right-handed orthonormal basis (u, v, w) of the space I this rotation is described by the matrix

$$R_q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(2\alpha) & -\sin(2\alpha) \\ 0 & \sin(2\alpha) & \cos(2\alpha) \end{pmatrix}.$$

Therefore,

$$p\biggl(\cos\Bigl(\frac{t}{2}\Bigr) + \sin\Bigl(\frac{t}{2}\Bigr)i\biggr) = e^{tA_1}, \qquad p\biggl(\cos\Bigl(\frac{t}{2}\Bigr) + \sin\Bigl(\frac{t}{2}\Bigr)j\biggr) = e^{tA_2},$$
$$p\biggl(\cos\Bigl(\frac{t}{2}\Bigr) + \sin\Bigl(\frac{t}{2}\Bigr)k\biggr) = e^{tA_3}.$$

In this paper, we used the following simple assertion.

Lemma. *Let $q \in S^3 \subset \mathbb{H}$. Then:*
(i) $q^2 = 1 \iff q = \pm 1 \iff \operatorname{Im} q = 0$;
(ii) $q^2 = -1 \iff \operatorname{Re} q = 0$.

The proof follows from the equality

$$(q_0 + q_1 i + q_2 j + q_3 k)^2 = q_0^2 - q_1^2 - q_2^2 - q_3^2 + 2q_0(q_1 i + q_2 j + q_3 k).$$

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