Euler's elastic problem

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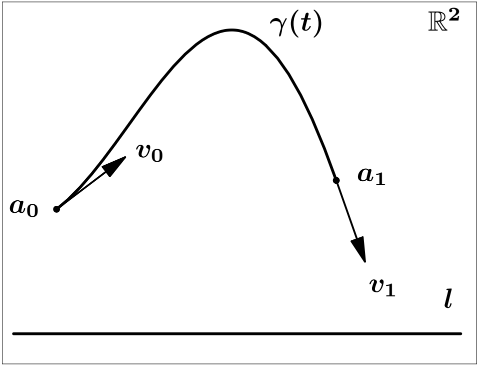
Summary of the talk

Problem statement: Stationary configurations of elastic rod



Given: $l > 0$, $a_0, a_1 \in \mathbb{R}^2$, $v_0 \in T_{a_0}\mathbb{R}^2$, $v_1 \in T_{a_1}\mathbb{R}^2$, $|v_0| = |v_1| = 1$.
Find: $\gamma(t)$, $t \in [0, t_1]$:
$\gamma(0) = a_0$, $\gamma(t_1) = a_1$, $\dot\gamma(0) = v_0$, $\dot\gamma(t_1) = v_1$. $|\dot\gamma(t)| \equiv 1 \quad \Rightarrow \quad t_1 = l$

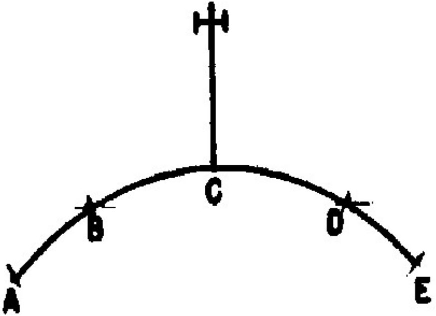
Elastic energy $J = \dfrac{1}{2}\displaystyle\int_0^{t_1} k^2\, dt \to \min$, $k(t)$ — curvature of $\gamma(t)$.

XIII century: Jordanus de Nemore

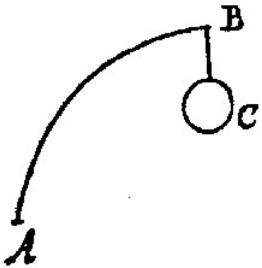
De Ratione Ponderis,
Book 4, Proposition 13:

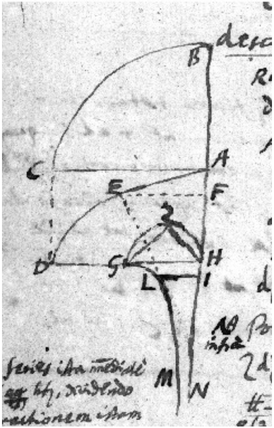
> All elastic curves are circles.

(False solution).



1691: James (Jacob) Bernoulli, rectangular elastica





$$dy = \frac{x^2\,dx}{\sqrt{1-x^4}}, \qquad ds = \frac{dx}{\sqrt{1-x^4}}, \qquad x \in [0,1]$$

1742: Daniel Bernoulli

- Elastic energy

$$E = \text{const} \cdot \int \frac{ds}{R^2},$$

  R — radius of curvature,

- Attempts to solve the variational problem

$$E \to \min .$$

- Letter to Leonhard Euler: proposal to solve this problem.

1744: Leonhard Euler

- "Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes, sive Solutio problematis isoperimitrici latissimo sensu accepti", Lausanne, Geneva, 1744,
- Appendix "De curvis elasticis",
- *"That among all curves of the same length which not only pass through the points A and B, but are also tangent to given straight lines at these points, that curve be determined in which the value of* $\int_A^B \frac{ds}{R^2}$ *be a minimum."*

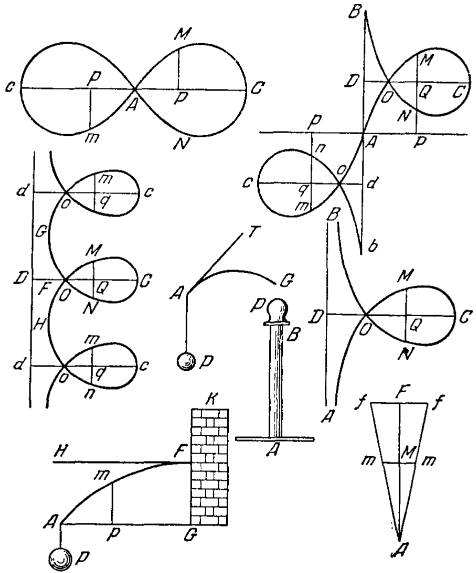
1744: Leonhard Euler

- Problem of calculus of variations,
- Euler-Lagrange equation,
- Reduction to quadratures

$$dy = \frac{(\alpha + \beta x + \gamma x^2)\, dx}{\sqrt{a^4 - (\alpha + \beta x + \gamma x^2)^2}}, \qquad ds = \frac{a^2\, dx}{\sqrt{a^4 - (\alpha + \beta x + \gamma x^2)^2}},$$

- Qualitative analysis of the integrals
- Types of solutions (elasticae)

Euler's sketches



1807: Pierre Simon Laplace

Profile of capillary surface between vertical planes:

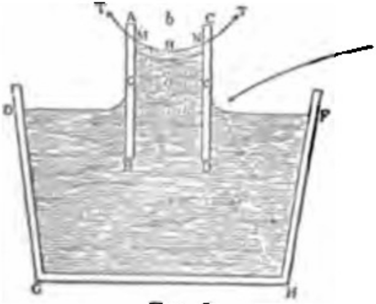


FIG. 6.

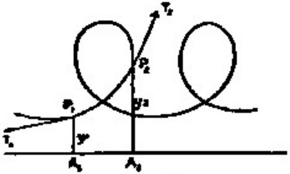
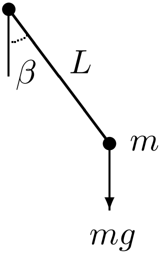


FIG. 8.

Figures by J. Maxwell (Encyclopaedia Britannica, 1890.)

Kinetic analogue: mathematical pendulum



$$\ddot{\beta} = -r \sin \beta, \qquad r = \frac{g}{L}$$

1880: L.Saalchütz

The first explicit parametrization of Euler elasticae by Jacobi's functions:

L.Saalschütz, *Der belastete Stab unter Einwirkung einer seitlichen Kraft*, 1880.

1906: Max Born

- Ph.D. thesis "Stability of elastic lines in the plane and the space"
- Euler-Lagrange equation $\Rightarrow$

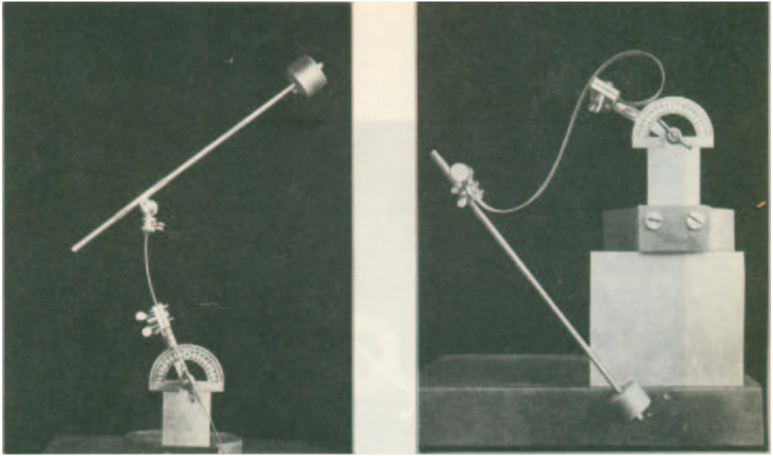
$$\dot{x} = \cos\theta, \qquad \dot{y} = \sin\theta,$$
$$A\ddot{\theta} + B\sin(\theta - \gamma) = 0, \qquad A, \ B, \ \gamma = \text{const},$$

equation of pendulum,

- elastic arc without inflection points $\Rightarrow$ stable,
- elastic arc with inflection points $\Rightarrow$ numerical investigation,
- numeric plots of elasticae.

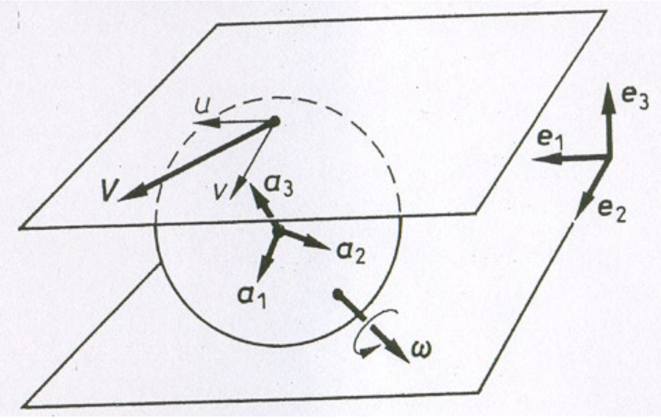
1906: Max Born

Experiments on elastic rods:



1993: Velimir Jurdjevic

Euler elasticae in the ball-plate problem:
Rolling of sphere on a plane without slipping or twisting



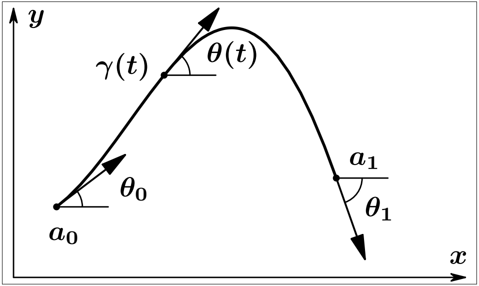
Euler elasticae in the nilpotent sub-Riemannian problem with the growth vector (2,3,5):

$$\dot{q} = u_1 X_1 + u_2 X_2, \qquad q \in \mathbb{R}^5, \quad u = (u_1, u_2) \in \mathbb{R}^2,$$
$$q(0) = q_0, \qquad q(t_1) = q_1,$$
$$l = \int_0^{t_1} \sqrt{u_1^2 + u_2^2}\, dt \to \min,$$
$$[X_1, X_2] = X_3, \qquad [X_1, X_3] = X_4, \qquad [X_2, X_3] = X_5.$$

Applications of planar and space elasticae

- elasticity theory and strength of materials (modeling of columns, beams, elastic rods),
- size and shape in biology (maximal height of a tree, curvature of palms under the action of wind, curvature of the spine, mechanics of insect wings),
- analogues of splines in approximation theory (G. Birkhoff, K.R. de Boor, 1964),
- recovery of images in computer vision (D. Mumford, 1994),
- modeling of optical fibers and flexible circuit boards in microelectronics (V. Jairazbhoy et al., 2008)
- dynamics of vortices and cubic Schroedinger equation (H. Hasimoto, 1971),
- modeling of DNA molecules (R.S. Manning, 1996), ...

Euler's problem: Coordinates in $\mathbb{R}^2 \times S^1$



- $(x, y) \in \mathbb{R}^2, \quad \theta \in S^1,$
- $\gamma(t) = (x(t), y(t)), \quad t \in [0, t_1],$
- $a_0 = (x_0, y_0), \quad a_1 = (x_1, y_1),$
- $v_0 = (\cos\theta_0, \sin\theta_0), \quad v_1 = (\cos\theta_1, \sin\theta_1).$

Optimal control problem

$$\begin{aligned}
&\dot{x} = \cos\theta, \\
&\dot{y} = \sin\theta, \\
&\dot{\theta} = u, \\
&q = (x, y, \theta) \in \mathbb{R}^2_{x,y} \times S^1_\theta, \qquad u \in \mathbb{R}, \\
&q(0) = q_0 = (x_0, y_0, \theta_0), \;\; q(t_1) = q_1 = (x_1, y_1, \theta_1), \;\; t_1 \text{ fixed.}
\end{aligned}$$

$$k^2 = \dot{\theta}^2 = u^2 \quad \Rightarrow \quad J = \frac{1}{2}\int_0^{t_1} u^2\, dt \to \min .$$

Admissible controls $u(t) \in L_2[0, t_1]$,
trajectories $q(t) \in AC[0, t_1]$

Left-invariant problem on the group of motions of a plane

$$\mathrm{SE}(2) = \mathbb{R}^2 \ltimes \mathrm{SO}(2) = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{pmatrix} \mid (x, y) \in \mathbb{R}^2,\ \theta \in S^1 \right\}$$

$$\begin{aligned}
&\dot{q} = X_1(q) + uX_2(q), \qquad q \in \mathrm{SE}(2), \quad u \in \mathbb{R}. \\
&q(0) = q_0, \qquad q(t_1) = q_1, \qquad t_1 \text{ fixed}, \\
&J = \frac{1}{2}\int_0^{t_1} u^2 dt \to \min,
\end{aligned}$$

Left-invariant frame on $\mathrm{SE}(2)$:

$$X_1(q) = qE_{13}, \quad X_2(q) = q(E_{21} - E_{12}), \quad X_3(q) = -qE_{23}$$

Continuous symmetries and normalization of conditions of the problem

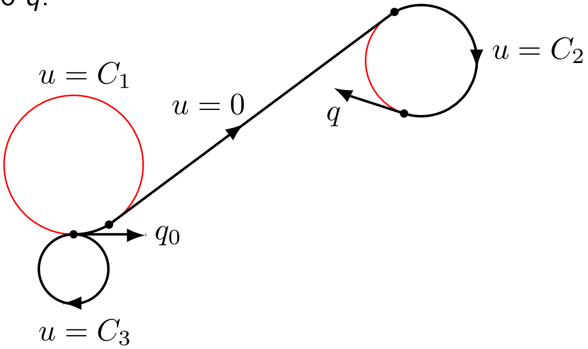
- Left translations on $\mathrm{SE}(2) \quad \Rightarrow \quad q_0 = \mathrm{Id} \in \mathrm{SE}(2)$:
  - Parallel translations in $\mathbb{R}^2 \quad \Rightarrow \quad (x_0, y_0) = (0, 0)$
  - Rotations in $\mathbb{R}^2 \quad \Rightarrow \quad \theta_0 = 0$
- Dilations in $\mathbb{R}^2 \quad \Rightarrow \quad t_1 = 1$

Attainable set

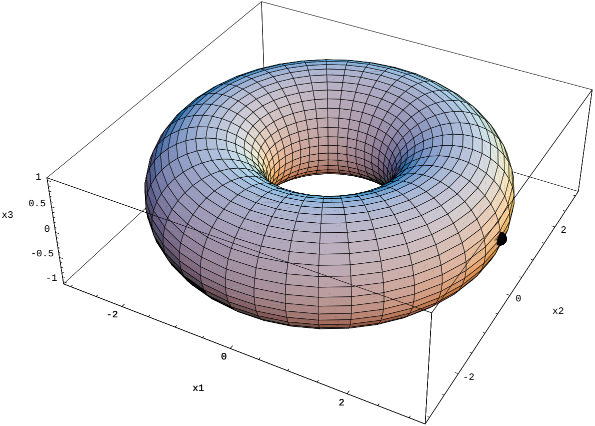
$q_0 = \mathrm{Id} = (0, 0, 0), \qquad t_1 = 1$

$$\mathcal{A}_{q_0}(1) = \{(x, y, \theta) \mid x^2 + y^2 < 1 \ \forall \theta \in S^1 \text{ or } (x, y, \theta) = (1, 0, 0)\}.$$

Steering q_0 to q:



Attainable set



In the sequel: $q_1 \in \mathcal{A}_{q_0}(t_1)$

Existence and regularity of optimal solutions

$$\dot{q} = X_1(q) + uX_2(q), \qquad q \in \mathbb{R}^2 \times S^1, \quad u \in \mathbb{R} \text{ unbounded}$$
$$q(0) = q_0, \quad q(t_1) = q_1,$$
$$J = \frac{1}{2}\int_0^{t_1} u^2\, dt \to \min,$$

- General existence theorem $\Rightarrow$ $\exists$ optimal $u(t) \in L_2$
- Compactification of the space of control parameters
  $\Rightarrow$ $\exists$ optimal $u(t) \in L_\infty$

$\Rightarrow$ Pontryagin Maximum Principle applicable

Pontryagin Maximum Principle in invariant form

$$\dot{q} = X_1(q) + uX_2(q),\ q \in M = \mathbb{R}^2 \times S^1,\ u \in \mathbb{R}, \quad J = \frac{1}{2}\int_0^{t_1} u^2\, dt \to \min$$

- $T_q M = \mathrm{span}(X_1(q), X_2(q), X_3(q)), \quad X_3 = [X_1, X_2]$
- $T_q^* M = \{(h_1, h_2, h_3)\}, \quad h_i(\lambda) = \langle \lambda, X_i \rangle,\ \lambda \in T^* M$
- Hamiltonian vector fields $\vec{h}_i \in \mathrm{Vec}(T^* M)$
- $h_u^\nu = \langle \lambda, X_1 + uX_2 \rangle + \frac{\nu}{2} u^2 = h_1(\lambda) + uh_2(\lambda) + \frac{\nu}{2} u^2$

Theorem (Pontryagin Maximum Principle)

$u(t)$ and $q(t)$ optimal $\Rightarrow$ *$\exists$ lipshchitzian $\lambda_t \in T_{q(t)}^* M$ and $\nu \leq 0$:*

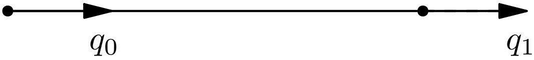
$$\dot{\lambda}_t = \vec{h}_{u(t)}^\nu(\lambda_t) = \vec{h}_1(\lambda_t) + u(t)\vec{h}_2(\lambda_t),$$

$$h_{u(t)}^\nu(\lambda_t) = \max_{u \in \mathbb{R}} h_u^\nu(\lambda_t),$$

$$(\nu, \lambda_t) \neq 0, \qquad t \in [0, t_1].$$

Abnormal extremal trajectories

$\nu = 0 \quad \Rightarrow \quad u(t) \equiv 0 \quad \Rightarrow \quad \theta \equiv 0, \qquad x = t, \qquad y \equiv 0$



$J = 0 = \min \quad \Rightarrow$

$\Rightarrow$ abnormal extremal trajectories optimal for $t \in [0, t_1]$

Unique trajectory from $q_0 = (0, 0, 0)$ to $(t_1, 0, 0) \in \partial \mathcal{A}_{q_0}(t_1)$.

Normal Hamiltonian system

$\nu = -1 \quad \Rightarrow \quad$ nonuniqueness of extremal trajectories

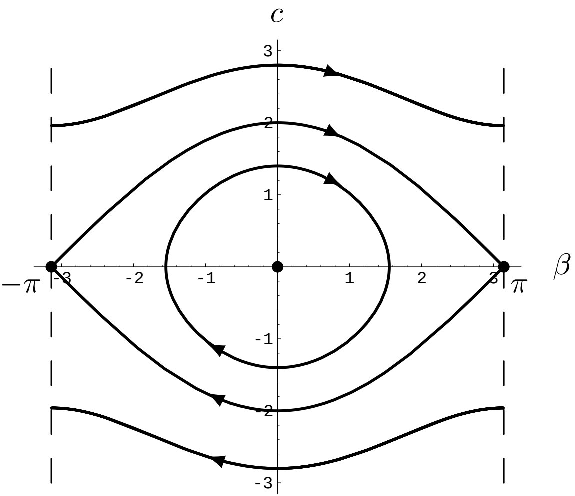
Hamiltonian system:

$$\dot h_1 = -h_2 h_3, \qquad \dot x = \cos\theta$$
$$\dot h_2 = h_3, \qquad \dot y = \sin\theta$$
$$\dot h_3 = h_1 h_2, \qquad \dot\theta = h_2$$

$$r^2 = h_1^2 + h_3^2 \equiv \text{const} \quad \Rightarrow \quad h_1 = -r\cos\beta, \;\; h_3 = -r\sin\beta$$

Equation of pendulum

$$\ddot{\beta} = -r \sin \beta \quad \Leftrightarrow \quad \begin{cases} \dot{\beta} = c, \\ \dot{c} = -r \sin \beta \end{cases}$$



Normal extremal trajectories

$$
\begin{aligned}
\ddot{\theta} &= -r \sin(\theta - \gamma), \qquad r,\ \gamma = \text{const}, \\
\dot{x} &= \cos\theta, \\
\dot{y} &= \sin\theta.
\end{aligned}
$$

Integrable in Jacobi's functions.

$\theta(t), x(t), y(t)$ parametrized by Jacobi's functions

$$
\operatorname{cn}(u,k), \quad \operatorname{sn}(u,k), \quad \operatorname{dn}(u,k), \quad \mathrm{E}(u,k).
$$

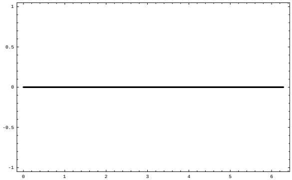
Euler elasticae

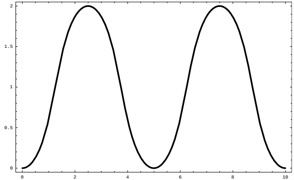
Energy of pendulum

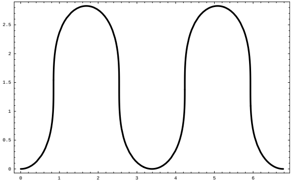
$$E = \frac{\dot{\theta}^2}{2} - r\cos(\theta - \gamma) \equiv \text{const} \in [-r, +\infty)$$

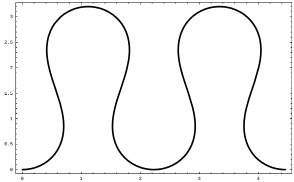
- $E = -r \neq 0 \quad \Rightarrow \quad$ straight lines
- $E \in (-r, r),\ r \neq 0 \quad \Rightarrow \quad$ inflectional elasticae
- $E = r \neq 0,\ \theta - \gamma = \pi \quad \Rightarrow \quad$ straight lines
- $E = r \neq 0,\ \theta - \gamma \neq \pi \quad \Rightarrow \quad$ critical elasticae
- $E > r \neq 0 \quad \Rightarrow \quad$ non-inflectional elasticae
- $r = 0 \quad \Rightarrow \quad$ straight lines and circles

Euler elasticae

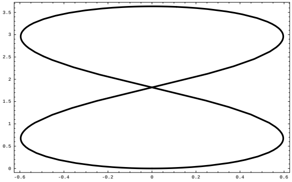


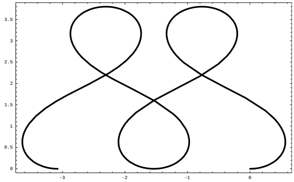


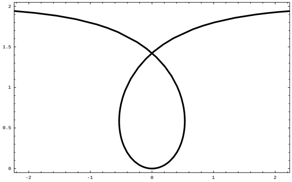


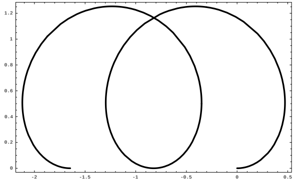


Euler elasticae

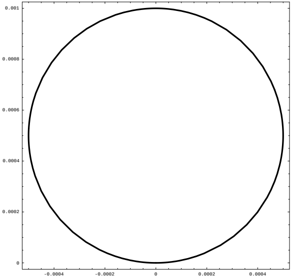








Euler elasticae



Optimality of normal extremal trajectories

$q(t)$ locally optimal:

$$\exists \varepsilon > 0 \quad \forall \widetilde{q} : \quad \|\widetilde{q} - q\|_C < \varepsilon, \ q(0) = \widetilde{q}(0), \ q(t_1) = \widetilde{q}(t_1) \quad \Rightarrow \quad J(q) \leq J(\widetilde{q})$$

Stable elastica $(x(t), y(t))$

$q(t)$ globally optimal:

$$\forall \widetilde{q} : \quad q(0) = \widetilde{q}(0), \ q(t_1) = \widetilde{q}(t_1) \quad \Rightarrow \quad J(q) \leq J(\widetilde{q})$$

Elastica $(x(t), y(t))$ of minimal energy.

Loss of optimality

Theorem (Strong Legendre condition)

$$\left.\frac{\partial^2}{\partial u^2}\right|_{u(s)} h_u^{-1}(\lambda_s) < -\delta < 0 \quad \Rightarrow$$

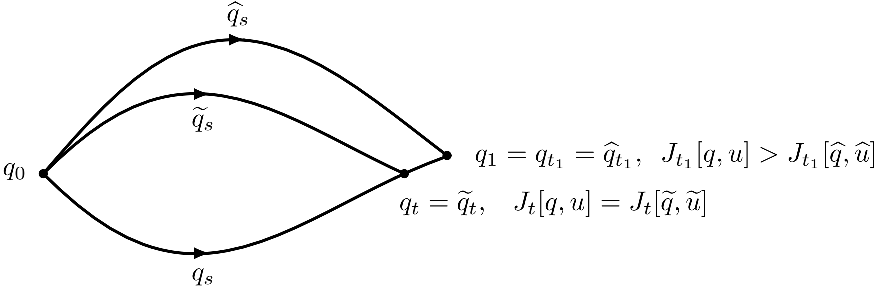
$\Rightarrow$ *small arcs of normal extremal trajectories $q(s)$ are optimal.*

Cut time along $q(s)$:

$$t_{\mathrm{cut}}(q) = \sup\{t > 0 \mid q(s),\ s \in [0, t],\ \text{optimal}\ \}.$$

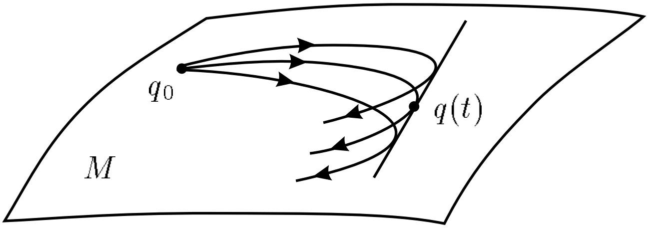
Reasons for loss of optimality: (1) Maxwell point

Maxwell point q_t: $\quad \exists\ \widetilde{q}_s \not\equiv q_s : \quad q_t = \widetilde{q}_t,\ J_t[q,u] = J_t[\widetilde{q},\widetilde{u}]$



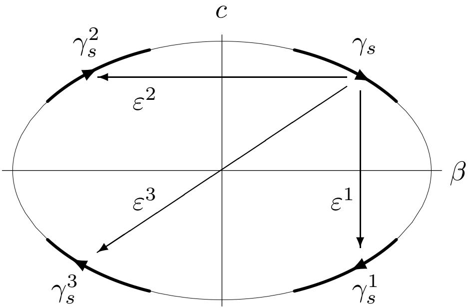
Reasons for loss of optimality: (2) Conjugate point

Conjugate point: $q_t \in$ envelope of the family of extremal trajectories



$$t_{\text{cut}} \leq \min(t_{\text{Max}}, t_{\text{conj}})$$

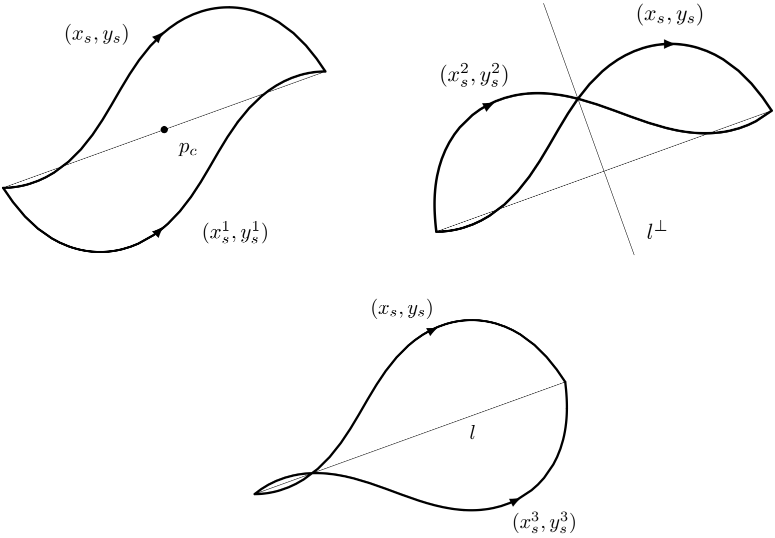
Reflections in the phase cylinder of pendulum $\ddot{\beta} = -r \sin \beta$



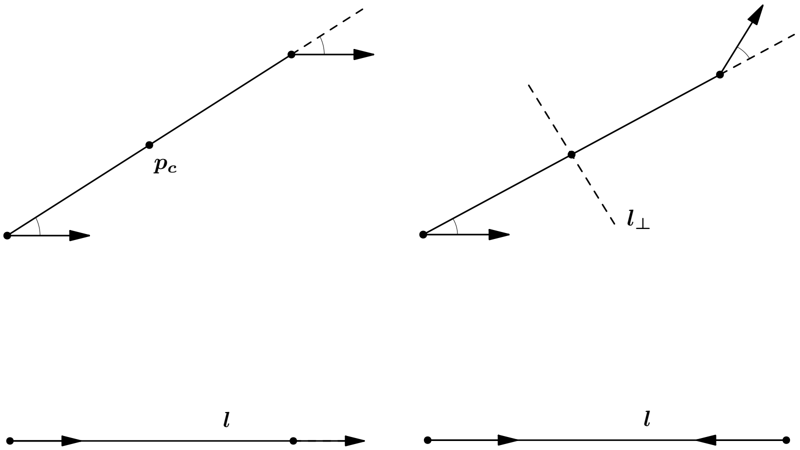
Dihedral group

$$D_2 = \{\mathrm{Id}, \varepsilon^1, \varepsilon^2, \varepsilon^3\} = \mathbb{Z}_2 \times \mathbb{Z}_2.$$

Action of reflections ε^1, ε^2, ε^3 on elasticae



Fixed points of reflections ε^1, ε^2, ε^3



Maxwell points corresponding to reflections

Fixed points of reflections ε^i $\quad\Rightarrow\quad$ Maxwell times:

$$t = t^n_{\varepsilon^i}, \qquad i = 1, 2, \quad n = 1, 2, \ldots$$

T = period of pendulum $\quad\Rightarrow$

$$t^n_{\varepsilon^1} = nT, \qquad\qquad \left(n - \frac{1}{2}\right) T < t^n_{\varepsilon^2} < \left(n + \frac{1}{2}\right) T.$$

Upper bound of cut time:

$$t_{\mathrm{cut}} \leq \min(t^1_{\varepsilon^1}, t^1_{\varepsilon^2}) \leq T.$$

Conjugate points

Exponential mapping

$$\mathrm{Exp}_t \; : \; T^*_{q_0} M \to M, \qquad \lambda_0 \mapsto q = q(t) = \pi \circ e^{t\vec{h}}(\lambda_0)$$

q — conjugate point $\quad \Leftrightarrow \quad$ q — critical value of Exp_t

$\mathrm{Exp}_t(h_1, h_2, h_3) = (x, y, \theta)$

$$\frac{\partial(x, y, \theta)}{\partial(h_1, h_2, h_3)} = 0$$

Local optimality of normal extremal trajectories

Theorem (Jacobi condition)

Normal extremal trajectories lose their local optimality at the first conjugate point.

The first conjugate point $t^1_{\text{conj}} \in (0, +\infty]$.

- No inflection points $\Rightarrow$ no conjugate points
- Inflectional case $\Rightarrow$ $t^1_{\text{conj}} \in [t^1_{\varepsilon^1}, t^1_{\varepsilon^2}] \subset [\frac{1}{2}T, \frac{3}{2}T]$

Stability of inflectional elasticae

- $t_1 \leq \frac{1}{2}T \quad \Rightarrow \quad$ stability,
- $t_1 \geq \frac{3}{2}T \quad \Rightarrow \quad$ instability

In particular:

- no inflection points $\quad \Rightarrow \quad$ stability (M. Born),
- 1 or 2 inflection points $\quad \Rightarrow \quad$ stability or instability,
- 3 inflection points $\quad \Rightarrow \quad$ instability.

Global optimality of elasticae

$$q_1 \in \mathcal{A}_{q_0}(t_1), \qquad \text{optimal } q(t) = ?$$

$$q(t) = \mathrm{Exp}_t(\lambda) \text{ optimal for } t \in [0, t_1] \quad \Rightarrow \quad t_1 \leq \min(t^1_{\varepsilon_1}(\lambda), t^1_{\varepsilon_2}(\lambda))$$

$$N' = \{\lambda \in T^*_{q_0}M \mid t_1 \leq \min(t^1_{\varepsilon_1}(\lambda), t^1_{\varepsilon_2}(\lambda))\}$$

$\mathrm{Exp}_{t_1} \; : \; N' \to \mathcal{A}_{q_0}(t_1)$ surjective, with singularities and multiple points

$\exists$ open dense $\widetilde{N} \subset N'$, $\widetilde{M} \subset \mathcal{A}_{q_0}(t_1)$ such that

$\mathrm{Exp}_{t_1} \; : \; \widetilde{N} \to \widetilde{M}$ is a direct sum of diffeomorphisms

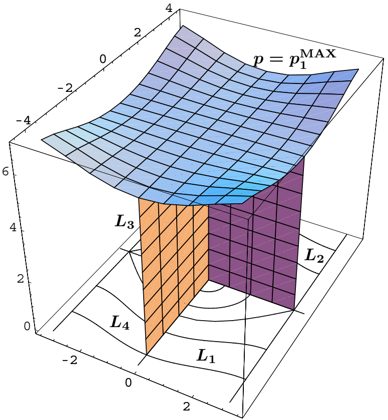


Figure: $\widetilde{N} = \cup_{i=1}^{4} L_i$

$\xrightarrow{\mathrm{Exp}_{t_1}}$

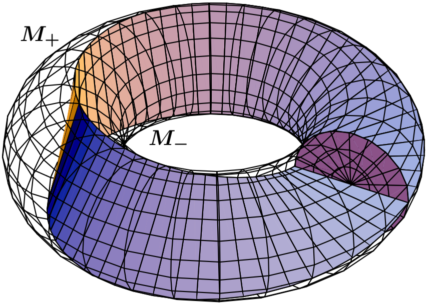
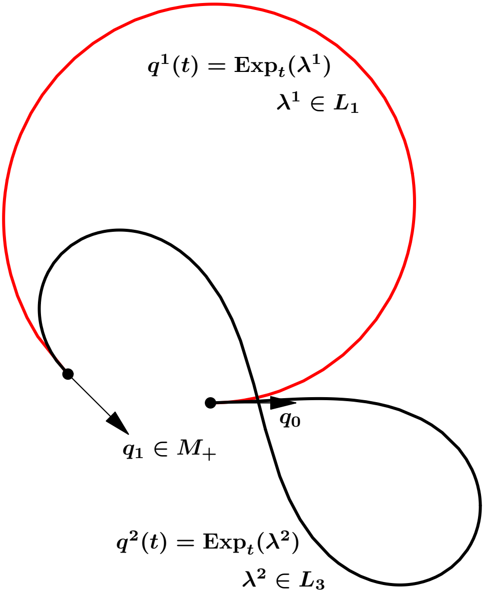


Figure: $\widetilde{M} = M_+ \cup M_-$

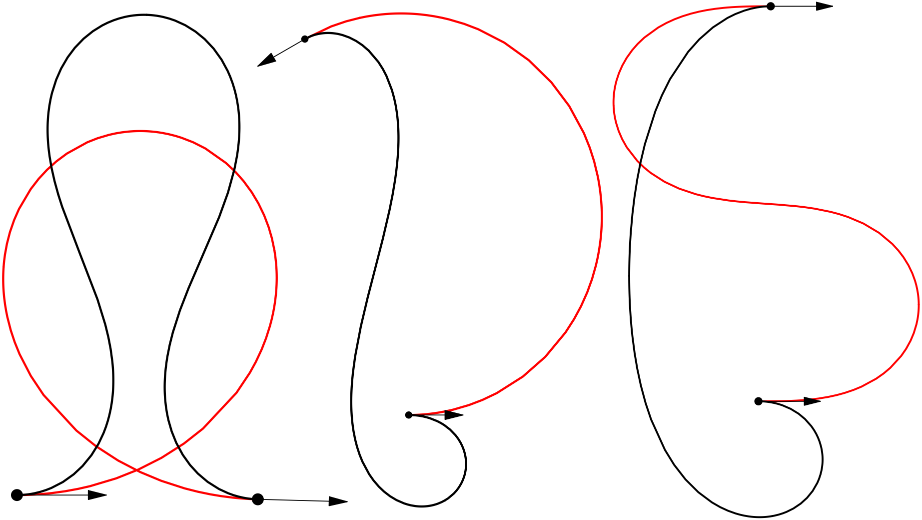
$\mathrm{Exp}_{t_1} \ : \ L_1, \ L_3 \to M_+$ diffeo, $\quad \mathrm{Exp}_{t_1} \ : \ L_2, \ L_4 \to M_-$ diffeo

Competing elasticae



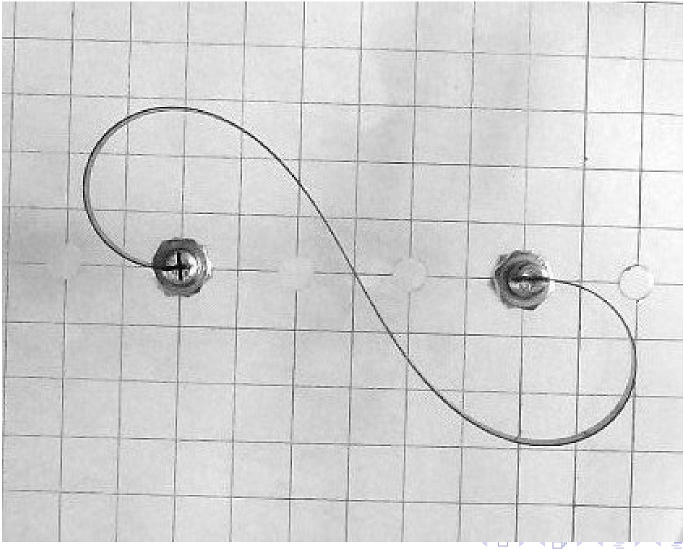
$$? \;:\; J[q^1] \lesseqgtr J[q^2]$$

Competing elasticae

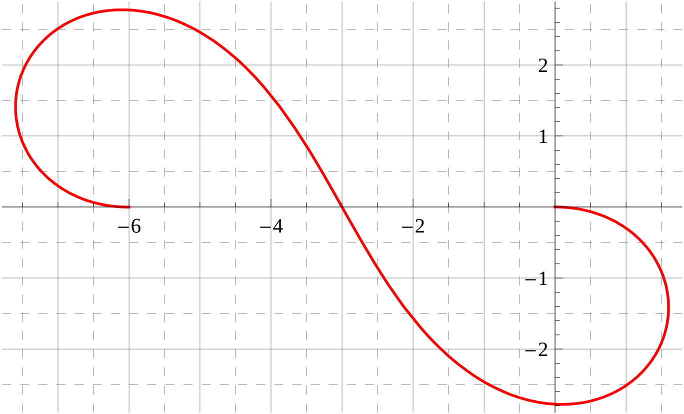


Animations . . .

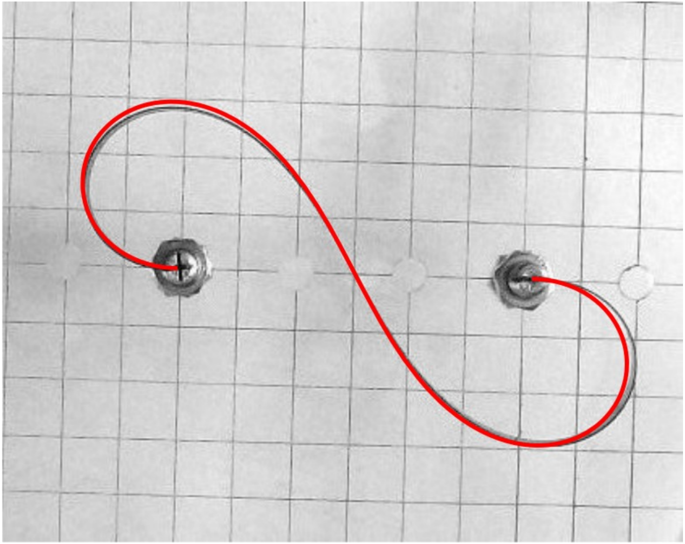
Work in progress:
Experimental elastica (with S.V.Levyakov)



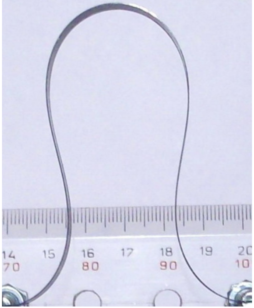
Plot of elastica

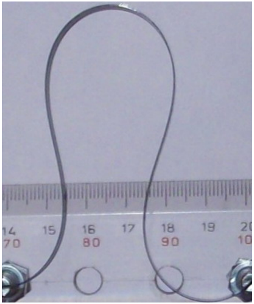


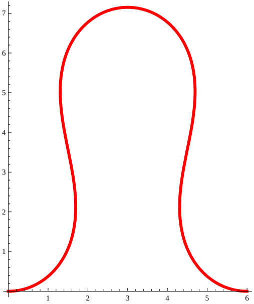
Comparison of experimental and mathematical elastica

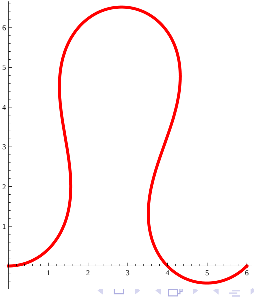


One-parameter family of elasticae losing stability: Series A

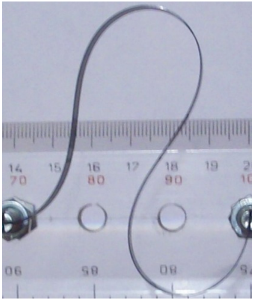


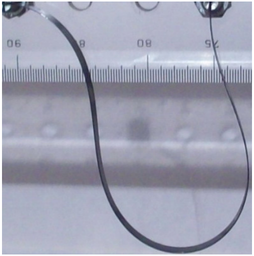


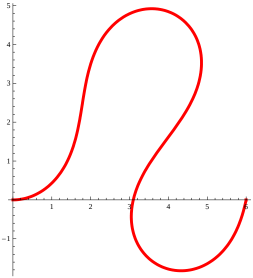


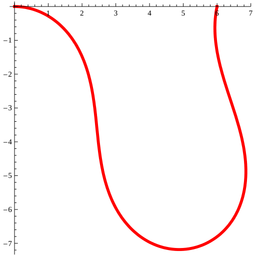


One-parameter family of elasticae losing stability: Series A

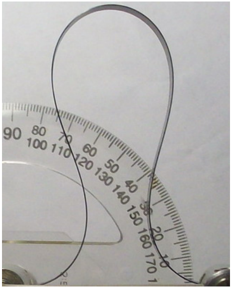


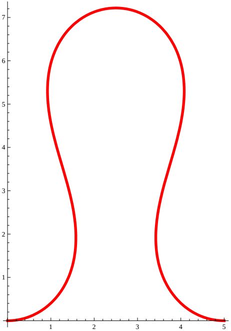


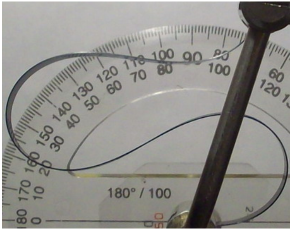


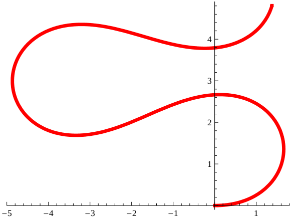


One-parameter family of elasticae losing stability: Series B

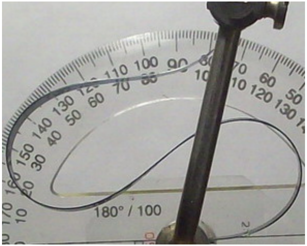


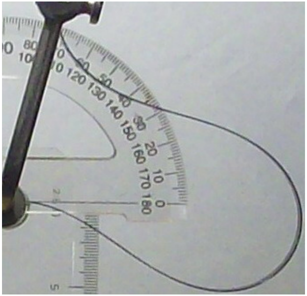


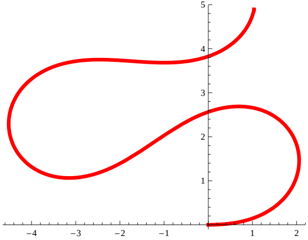


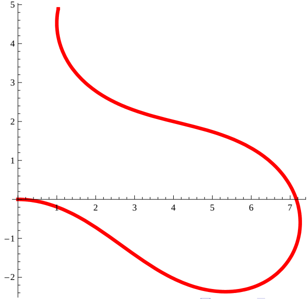


One-parameter family of elasticae losing stability: Series B









Questions and related problems

- Cut time $t_{\text{cut}} = ?$
- Optimal synthesis in Euler's elastic problem
- Nilpotent $(2, 3, 5)$ sub-Riemannian problem
- The ball-plate problem
- Sub-Riemannian problem on $\mathrm{SE}(2)$

Conclusion: Euler's elastic problem

- Optimal control problem,
- Extremal trajectories,
- Local and global optimality of extremal trajectories,
- Stability of Euler elasticae,
- Software for finding globally optimal elastica,
- Publications:

    - Yu. L. Sachkov, Maxwell strata in Euler's elastic problem, *Journal of Dynamical and Control Systems*, Vol. 14 (2008), No. 2, 169–234, `arXiv:0705.0614 [math.OC]`.
    - Yu. L. Sachkov, Conjugate points in Euler's elastic problem, *Journal of Dynamical and Control Systems*, Vol. 14 (2008), No. 3, 409–439, `arXiv:0705.1003 [math.OC]`.
    - Yu. L. Sachkov, Optimality of Euler elasticae, *Doklady Mathematics*, Vol. 417, No. 1, November 2007, 23–25.