The left-invariant sub-Riemannian problem on the group of motions of a plane

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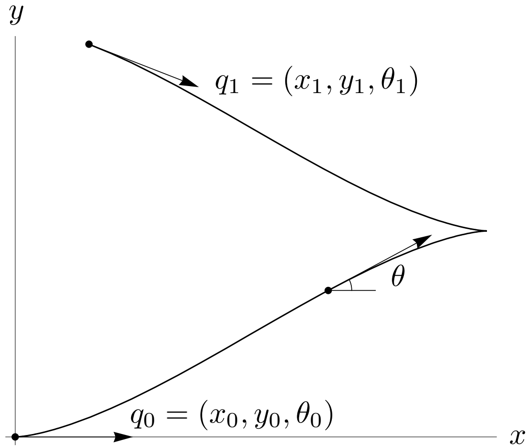
Problem statement: Optimal motion of a mobile robot in the plane



$$q(0) = q_0, \;\; q(t_1) = q_1, \quad l = \int_0^{t_1} \sqrt{\dot{x}^2 + \dot{y}^2 + \alpha^2 \; \dot{\theta}^2} \, dt \to \min, \quad \alpha = 1$$

Optimal control problem

$$\dot{x} = u\cos\theta, \qquad \dot{y} = u\sin\theta, \qquad \dot{\theta} = v,$$

$$(x, y) \in \mathbb{R}^2, \qquad \theta \in S^1 = \mathbb{R}/(2\pi\,\mathbb{Z}),$$

$$q = (x, y, \theta) \in M = \mathbb{R}^2 \times S^1,$$

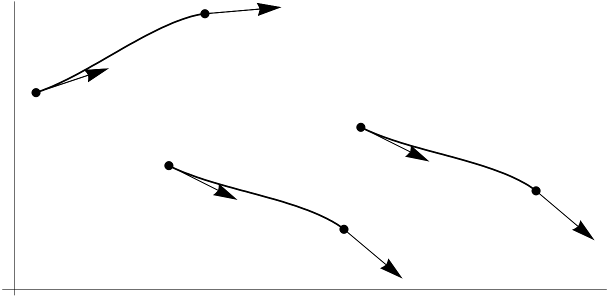
$$(u, v) \in \mathbb{R}^2,$$

$$q(0) = q_0, \qquad q(t_1) = q_1,$$

$$l = \int_0^{t_1} \sqrt{u^2 + v^2}\, dt \to \min .$$

Continuous symmetries of the problem

- rotations
- translations



Group of motions (rototranslations) of a plane

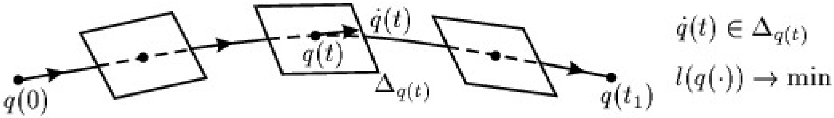
$$\mathrm{SE}(2) = \mathbb{R}^2 \ltimes \mathrm{SO}(2) = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{pmatrix} \mid (x, y) \in \mathbb{R}^2,\ \theta \in S^1 \right\}$$

Left-invariant frame on SE(2):

$$X_1(q) = qE_{13},\ X_2(q) = q(E_{21} - E_{12}),\ X_3(q) = [X_1, X_2](q) = -qE_{23}.$$

Left-invariant sub-Riemannian problem on SE(2)

$$
\begin{aligned}
&\dot q = uX_1(q) + vX_2(q), \quad q \in \mathrm{SE}(2),\ (u,v) \in \mathbb{R}^2,\\
&q(0) = q_0, \qquad q(t_1) = q_1,\\
&l = \int_0^{t_1} \langle \dot q, \dot q\rangle^{1/2}\, dt \to \min,\\
&\langle X_i, X_j\rangle = \delta_{ij}, \quad i,\ j = 1, 2.
\end{aligned}
$$



Known results on 3-dimensional sub-Riemannian problems

- Left-invariant problem on the Heisenberg group: global solution (A.Vershik, V.Gershkovich, 1987),
- Contact problems in $\mathbb{R}^3$: local study (A.Agrachev, 1996; J.-P.Gauthier, 1996),
- Martinet case: global solution (A.Agrachev, B.Bonnard, M.Chyba, I.Kupka, 1997),
- Left-invariant problems on SO(3), SU(2), SL(2): global solution (U.Boscain, F.Rossi, 2008).

SR problem on SE(2)
Existence of solutions

- $\dot{q} = uX_1(q) + vX_2(q)$,

  $\mathrm{span}(X_1(q), X_2(q), [X_1, X_2](q)) = T_qM \quad \forall\ q \in M$

  $\Rightarrow$ complete controllability (Rashevskii-Chow theorem)

- Filippov's theorem

  $\Rightarrow$ existence of optimal trajectories $q(t)$.

Pontryagin maximum principle

- Abnormal extremal trajectories constant.
- Normal extremals:

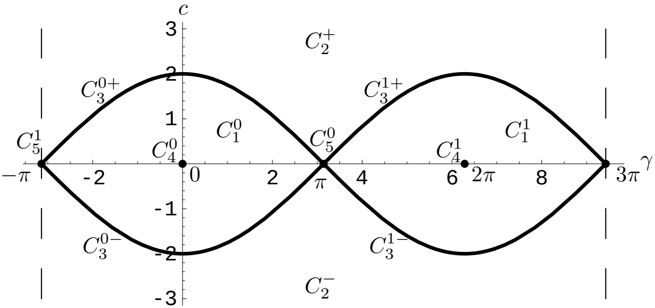
$$\dot{\gamma} = c, \quad \dot{c} = -\sin\gamma, \qquad (\gamma, c) \in C \cong (2S^1_\gamma) \times \mathbb{R}_c,$$
$$\dot{x} = \sin\frac{\gamma}{2}\cos\theta, \quad \dot{y} = \sin\frac{\gamma}{2}\sin\theta, \quad \dot{\theta} = -\cos\frac{\gamma}{2}.$$

- Arc length parametrization:

$$\dot{x}^2 + \dot{y}^2 + \dot{\theta}^2 \equiv 1 \quad \Rightarrow \quad l = t_1 \to \min$$

Stratification of phase cylinder of pendulum $C = \cup_{i=1}^{5} C_i$

- Energy integral $E = c^2/2 - \cos\gamma \in [-1, +\infty)$
- $C_1 = \{\lambda \in C \mid E \in (-1, 1)\}$,
- $C_2 = \{\lambda \in C \mid E \in (1, +\infty)\}$,
- $C_3 = \{\lambda \in C \mid E = 1, \ c \neq 0\}$,
- $C_4 = \{\lambda \in C \mid E = -1\}$,
- $C_5 = \{\lambda \in C \mid E = 1, \ c = 0\}$.



Parameterisation of extremal trajectories

- $\lambda = (\gamma, c) \in C_1 \quad \Rightarrow$

$$\begin{aligned}
\theta_t &= s_1(\operatorname{am}\varphi - \operatorname{am}\varphi_t) \pmod{2\pi},\\
x_t &= (s_1/k)[\operatorname{cn}\varphi(\operatorname{dn}\varphi - \operatorname{dn}\varphi_t) + \operatorname{sn}\varphi(t + \operatorname{E}(\varphi) - \operatorname{E}(\varphi_t))],\\
y_t &= (1/k)[\operatorname{sn}\varphi(\operatorname{dn}\varphi - \operatorname{dn}\varphi_t) - \operatorname{cn}\varphi(t + \operatorname{E}(\varphi) - \operatorname{E}(\varphi_t))].
\end{aligned}$$

- $\lambda = (\gamma, c) \in C_2 \quad \Rightarrow$

$$\begin{aligned}
\cos\theta_t &= k^2 \operatorname{sn}\psi \operatorname{sn}\psi_t + \operatorname{dn}\psi \operatorname{dn}\psi_t,\\
\sin\theta_t &= k(\operatorname{sn}\psi \operatorname{dn}\psi_t - \operatorname{dn}\psi \operatorname{sn}\psi_t),\\
x_t &= s_2 k[\operatorname{dn}\psi(\operatorname{cn}\psi - \operatorname{cn}\psi_t) + \operatorname{sn}\psi(t/k + \operatorname{E}(\psi) - \operatorname{E}(\psi_t))],\\
y_t &= s_2[k^2 \operatorname{sn}\psi(\operatorname{cn}\psi - \operatorname{cn}\psi_t) - \operatorname{dn}\psi(t/k + \operatorname{E}(\psi) - \operatorname{E}(\psi_t))].
\end{aligned}$$

- $\lambda = (\gamma, c) \in C_3 \cup C_4 \cup C_5 \quad \Rightarrow$ hyperbolic and linear functions.

Extremal trajectories: generic cases

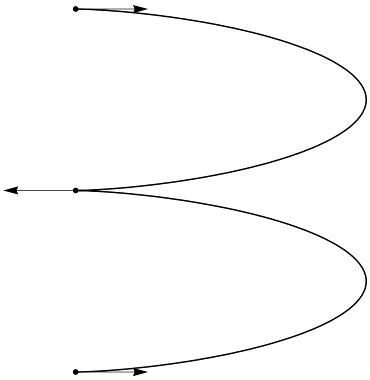


Figure: $\lambda \in C_1$

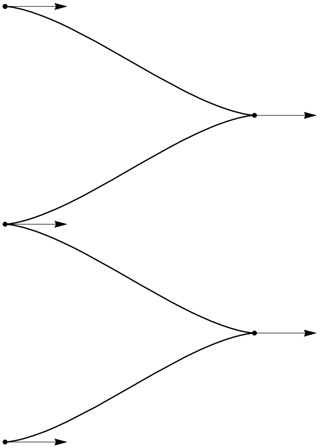


Figure: $\lambda \in C_2$

Extremal trajectories: special cases

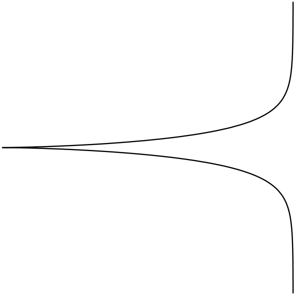


Figure: $\lambda \in C_3$

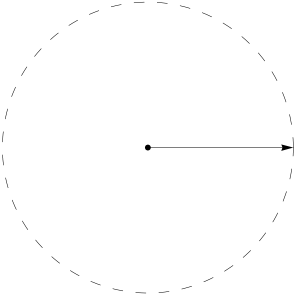


Figure: $\lambda \in C_4$

Figure: $\lambda \in C_5$

Optimality of extremal trajectories

$q(t)$ is locally optimal:

$$\exists \varepsilon > 0 \quad \forall \text{ trajectory } \widetilde{q} : \quad \|\widetilde{q} - q\|_C < \varepsilon, \\ q(0) = \widetilde{q}(0),\ q(t_1) = \widetilde{q}(\tilde{t}_1) \quad \Rightarrow \quad t_1 \leq \tilde{t}_1$$

$q(t)$ is globally optimal:

$$\forall \text{ trajectory } \widetilde{q} : \quad q(0) = \widetilde{q}(0),\ q(t_1) = \widetilde{q}(\tilde{t}_1) \quad \Rightarrow \quad t_1 \leq \tilde{t}_1$$

Loss of optimality

- Strong Legendre condition:
  $\dfrac{\partial^2 h_u^{-1}}{\partial u^2} < 0 \quad \Rightarrow \quad$ short arcs $q(t)$ are optimal.

- Cut time:

$$t_{\text{cut}}(q) = \sup\{t > 0 \mid q(s) \text{ is optimal for } s \in [0, t]\}.$$

Reasons for loss of optimality:
(1) Maxwell point (global)

Maxwell point q_t:

$$\exists \text{ extremal trajectory } \widetilde{q}_s \neq q_s : \quad q_0 = \widetilde{q}_0, \; q_t = \widetilde{q}_t$$

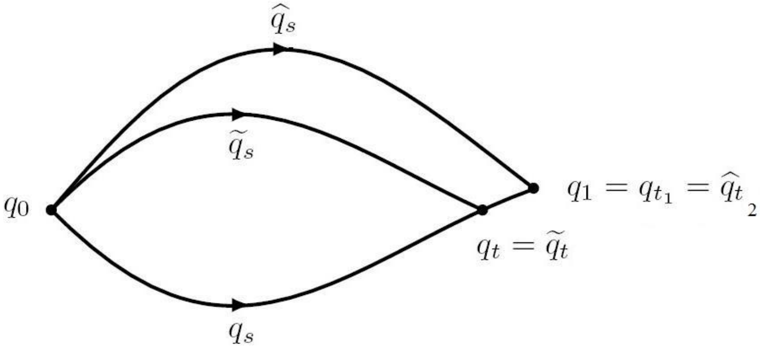
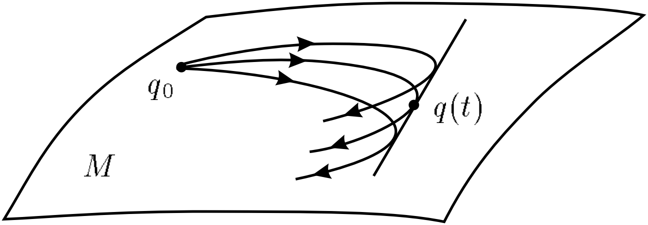


Figure: $t_2 < t_1$

Reasons for loss of optimality: (2) Conjugate point (local)

$q_t \in$ envelope of the family of extremal trajectories



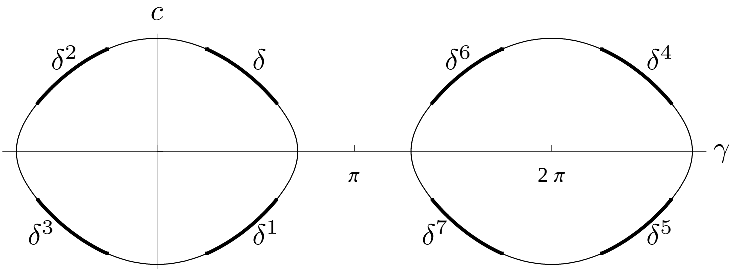
$t_{\text{cut}} \leq \min(t_{\text{Max}}, t_{\text{conj}})$

Reflections ε^i in the phase cylinder of pendulum $\ddot{\gamma} = -\sin\gamma$

- Group of symmetries of parallelepiped

$$G = \{\mathrm{Id}, \varepsilon^1, \dots, \varepsilon^7\} = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2.$$

- Action of reflections $\varepsilon^i \; : \; \delta \mapsto \delta^i$ on trajectories of pendulum:



Action of reflections ε^i on curves (x_t, y_t) modulo rotations

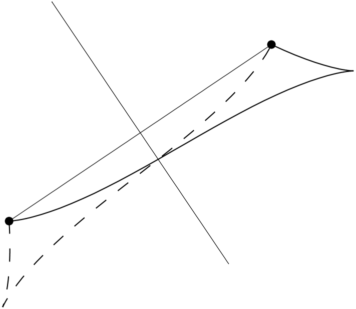


Figure: ε^1, ε^2

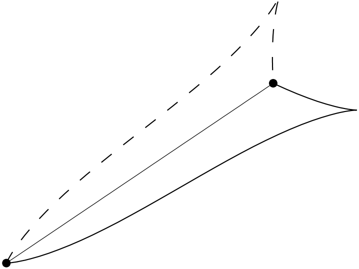


Figure: ε^4, ε^7

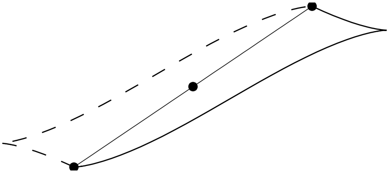


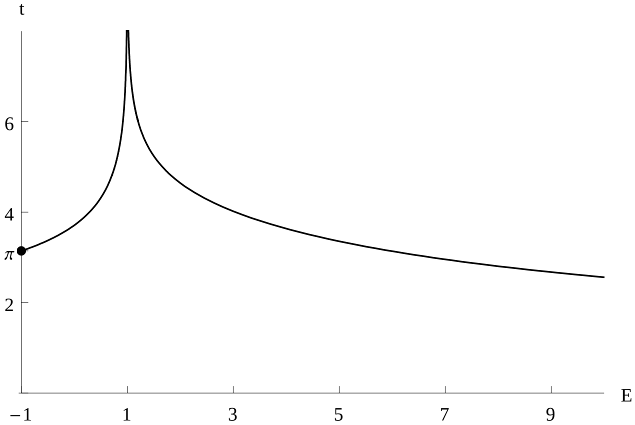
Figure: ε^5, ε^6

Maxwell points corresponding to reflections

- Fixed points of reflections ε^i:

$$t = t^n_{\varepsilon^i}, \qquad i = 1, 2, \dots, 7, \quad n = 1, 2, \dots$$

- Upper bound of cut time: $t_{\text{cut}} \le \mathbf{t} := \min(t^1_{\varepsilon^i})$.
- Plot of function $\mathbf{t} = \mathbf{t}(E)$:



Exponential mapping and conjugate points

- Exponential mapping

$$\begin{aligned} \mathrm{Exp} &: (\lambda, t) = (\gamma, c, t) \mapsto q(t), \\ \mathrm{Exp} &: N = C \times \mathbb{R}_+ \to M \end{aligned}$$

- q — conjugate point $\iff$ q — critical value of Exp
- $\mathrm{Exp}(\gamma, c, t) = (x, y, \theta)$
- $\dfrac{\partial(x, y, \theta)}{\partial(\gamma, c, t)} = 0$

Bounds of conjugate time

- Trajectories without inflexion points:

$$\lambda \in C_1 \cup C_3 \cup C_4 \cup C_5 \quad \Rightarrow \quad t^1_{\text{conj}}(\lambda) = +\infty.$$

- Trajectories with inflexion points:

$$\lambda \in C_2 \quad \Rightarrow \quad t^1_{\varepsilon^6}(\lambda) \geq t^1_{\text{conj}}(\lambda) \geq t^1_{\varepsilon^2}(\lambda) = \mathbf{t}(\lambda).$$

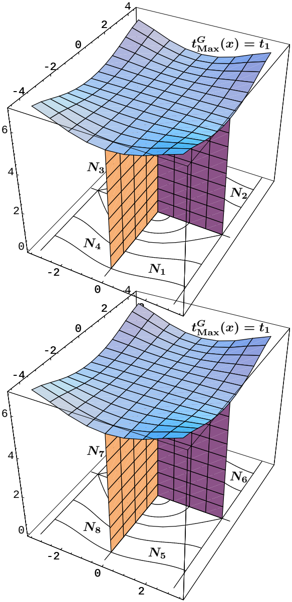
Stratifications in preimage and image of exponential mapping

- $\widehat{N} = \{(\lambda, t) \in C \times \mathbb{R}_+ \mid t \leq \mathbf{t}(\lambda)\}, \qquad \widehat{M} = M \setminus \{q_0\}$
- $\mathrm{Exp} \; : \; \widehat{N} \to \widehat{M}$ surjective
- $\widehat{N} = \cup_{i \in I} N_i, \qquad N_i \cap N_j = \emptyset$ for $i \neq j \in I$
- $\widehat{M} = \cup_{i \in I} M_i, \qquad I = J \cup K, \quad J \cap K = \emptyset$
- $\forall i \neq j \in J \quad M_i \cap M_j = \emptyset$
- $\forall i \in K \; \exists! \, j \in K, \; j \neq i \; : \; M_i = M_j$
- N_i, M_i smooth manifolds of dim $\in \{0, \dots, 3\}$
- $\# I = 66, \quad \# J = 32, \quad \# K = 34.$

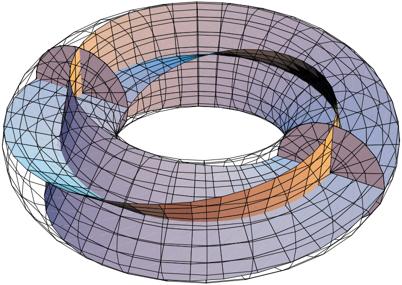
Global structure of exponential mapping

- $\mathrm{Exp} \ : \ N_i \longrightarrow M_i$ diffeomorphism $\quad \forall \ i \in I.$
- $\widehat{M} = \mathrm{Max} \cup \widetilde{M}, \qquad \mathrm{Max} = \cup_{i \in K} M_i, \qquad \widetilde{M} = \cup_{i \in J} M_i$
- $\widehat{N} = N_{\mathrm{Max}} \cup \widetilde{N}, \qquad N_{\mathrm{Max}} = \cup_{i \in K} N_i, \qquad \widetilde{N} = \cup_{i \in J} N_i$
- $\mathrm{Exp} \ : \ \widetilde{N} \longrightarrow \widetilde{M}$ bijection
- $\mathrm{Exp} \ : \ N_{\mathrm{Max}} \longrightarrow \mathrm{Max}$ double mapping

Global structure of exponential mapping



$\xrightarrow{\text{Exp}}$



Cut time and cut points

$$t_{\mathrm{cut}}(\lambda) = \mathbf{t}(\lambda) = \begin{cases} t^1_{\varepsilon^5} = 2K(k) = T/2, & \lambda \in C_1, \\ t^1_{\varepsilon^2} = 2kp^1_1(k) \in (T, 2T), & \lambda \in C_2, \\ +\infty, & \lambda \in C_3 \cup C_5, \\ t^1_{\varepsilon^5} = \pi = T/2, & \lambda \in C_4 \end{cases}$$

$$p = p^1_1(k) \quad : \quad \mathrm{cn}(p,k)(\mathrm{E}(p,k) - p) - \mathrm{dn}(p,k)\,\mathrm{sn}(p,k) = 0$$

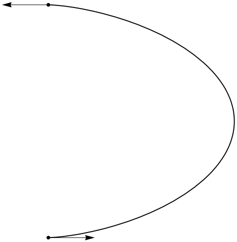


Figure: $\lambda \in C_1$

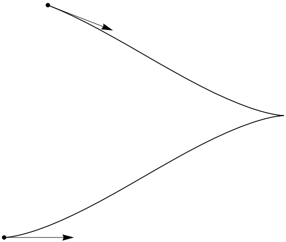


Figure: $\lambda \in C_2$

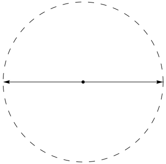


Figure: $\lambda \in C_4$

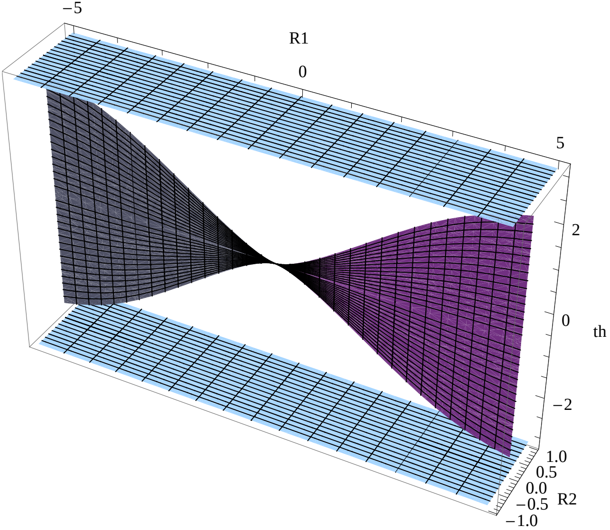
Maxwell strata

- $\mathrm{Max} = \mathrm{Max}_{\mathrm{loc}} \cup \mathrm{Max}_{\mathrm{glob}}$
- $\mathrm{Max}_{\mathrm{glob}} = \{q \in M \mid \theta = \pi\}$
- $\mathrm{Max}_{\mathrm{loc}} = \{q \in M \mid \theta \in (-\pi, \pi),\ R_2 = 0,\ |R_1| > R_1^1(|\theta|)\},$
  $R_1 = y\cos\frac{\theta}{2} - x\sin\frac{\theta}{2}, \qquad R_2 = x\cos\frac{\theta}{2} + x\sin\frac{\theta}{2},$
  $R_1^1(\theta) = 2(p_1^1(k) - \mathrm{E}(p_1^1(k), k)),$
  $k = k_1^1(\theta)$ inverse of $\theta = k\,\mathrm{sn}(p_1^1(k), k).$
- $q_1 \in \mathrm{Max} \quad \Rightarrow \quad$ 2 optimal trajectories,
- $q_1 \in M \setminus \mathrm{Max} \quad \Rightarrow \quad$ 1 optimal trajectory.

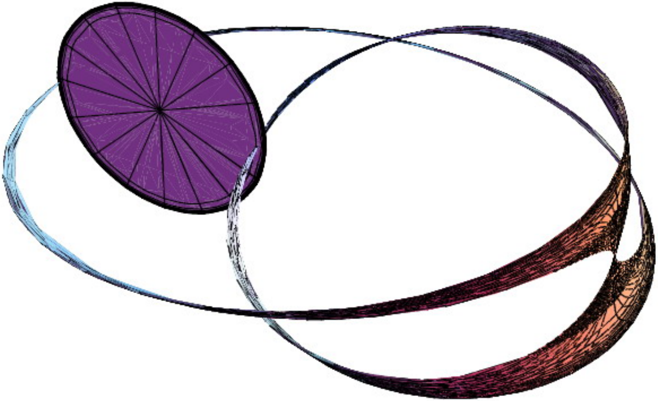
Cut locus

- $\mathrm{Cut} = \{\mathrm{Exp}(\lambda, t) \mid \lambda \in N,\ \ t = t_{\mathrm{cut}}(\lambda)\}$
- $\mathrm{Cut} = \mathrm{cl}(\mathrm{Max}) \setminus \{q_0\} = \mathrm{Cut}_{\mathrm{loc}} \cup \mathrm{Cut}_{\mathrm{glob}}$
- $\mathrm{Cut}_{\mathrm{loc}} = \mathrm{cl}(\mathrm{Max}_{\mathrm{loc}}) \setminus \{q_0\}$
- $\mathrm{Cut}_{\mathrm{glob}} = \mathrm{Max}_{\mathrm{glob}}$

Cut locus in rectifying coordinates (R_1, R_2, θ)



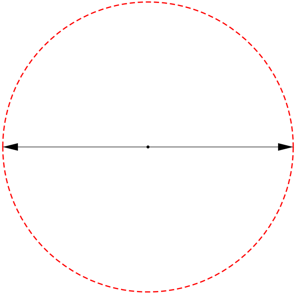
Cut locus: global view



Optimal solutions

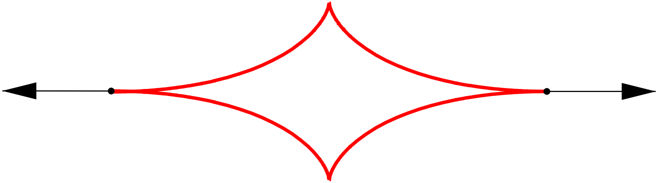
$$x_1 \neq 0, \qquad y_1 = 0, \qquad \theta_1 = 0$$

$$x_1 = 0, \qquad y_1 = 0, \qquad \theta_1 \neq 0$$



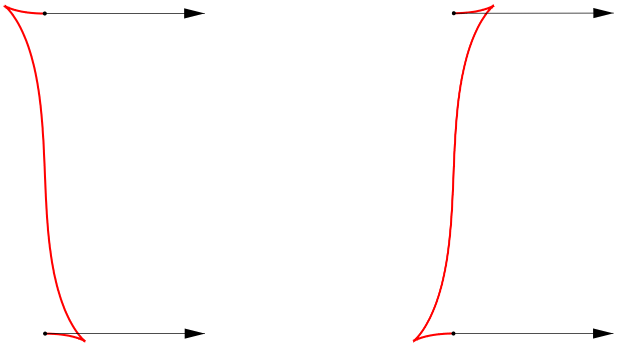
Optimal solutions

$$x_1 \neq 0, \qquad y_1 = 0, \qquad \theta_1 = \pi$$



Optimal solutions

$$x_1 = 0, \qquad y_1 \neq 0, \qquad \theta_1 = 0$$



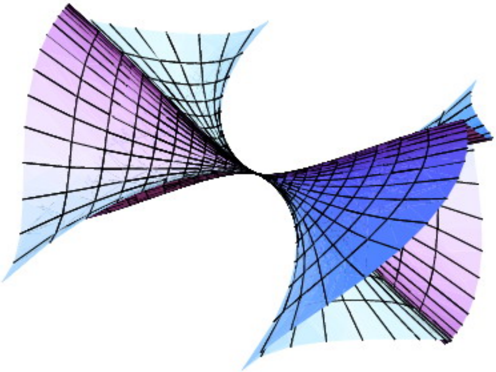
Optimal solutions

Generic boundary conditions:

systems of equations in Jacobi's functions $\Rightarrow$

$\Rightarrow$ software (MATHEMATICA).

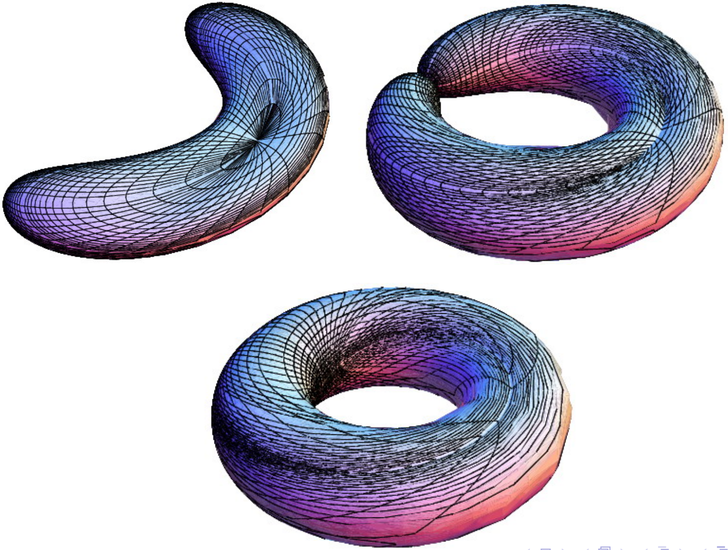
Sub-Riemannian caustic $\{\mathrm{Exp}(\lambda, t) \mid \lambda \in N,\ t = t^1_{\mathrm{conj}}(\lambda)\}$



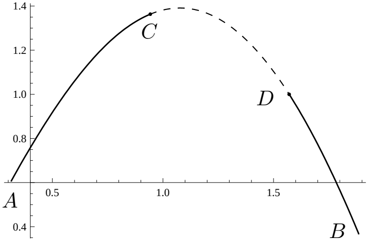
Sub-Riemannian spheres

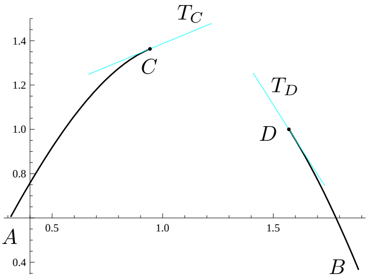
- $d(q_0, q_1) = \inf\{l(q(\cdot)) \mid q(0) = q_0,\ q(t_1) = q_1\}$,
- $S_R = \{q \in M \mid d(q_0, q) = R\}$,
- $R = 0 \quad \Rightarrow \quad S_R = \{q_0\}$,
- $R \in (0, \pi) \quad \Rightarrow \quad S_R \cong S^2$,
- $R = \pi \quad \Rightarrow \quad S_R \cong S^2/\{N = S\}$,
- $R > \pi \quad \Rightarrow \quad S_R \cong T^2$.

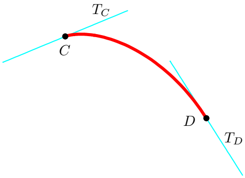
Global structure of sub-Riemannian spheres: $R < \pi, \quad R = \pi, \quad R > \pi$

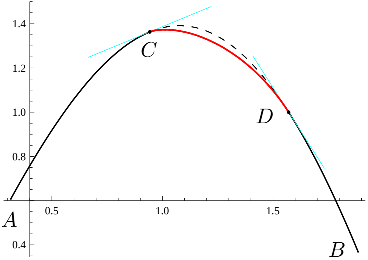


Application: Antropomorphic restoration of isophotes







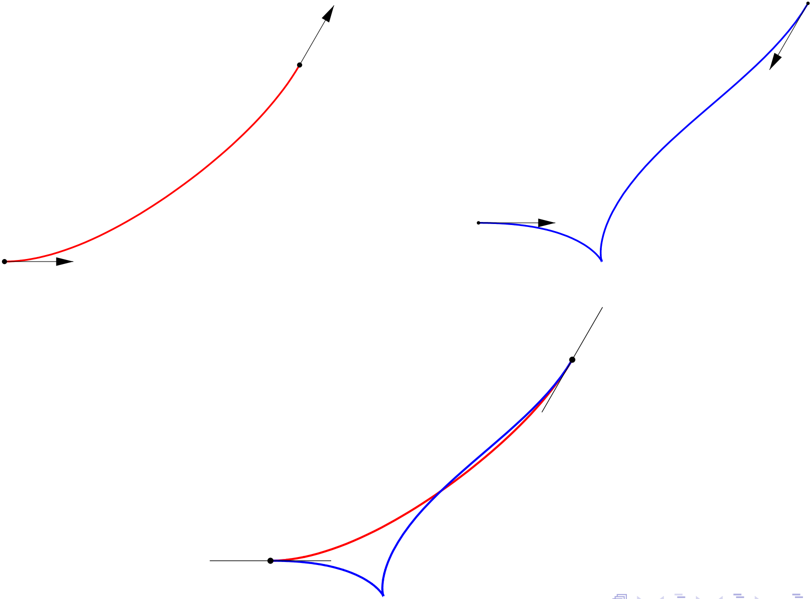


Neurogeometry and sub-Riemannian problem on $\mathbb{R}^2 \times \mathbb{R}P^1$

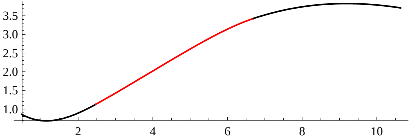
- J.Petitot, The neurogeometry of pinwheels as a sub-Riemannian contact structure, *J. Physiology - Paris* 97 (2003), 265–309.
- J.Petitot, *Neurogeometrie de la vision — Modeles mathematiques et physiques des architectures fonctionnelles*, 2008, Editions de l'Ecole Polytechnique.

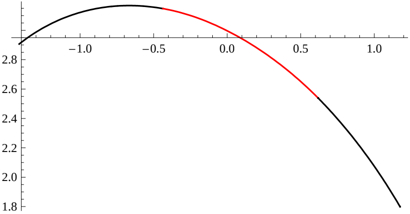
$$
\begin{aligned}
&\dot{x} = u\cos\theta, \qquad \dot{y} = u\sin\theta, \qquad \dot{\theta} = v,\\
&q = (x, y, \theta), \qquad (x, y) \in \mathbb{R}^2, \quad \theta \in \mathbb{R}P^1 = \mathbb{R}/(\pi\,\mathbb{Z}),\\
&(u, v) \in \mathbb{R}^2,\\
&q(0) = q_0, \qquad q(t_1) = q_1,\\
&l = \int_0^{t_1} \sqrt{u^2 + v^2}\, dt \to \min .
\end{aligned}
$$

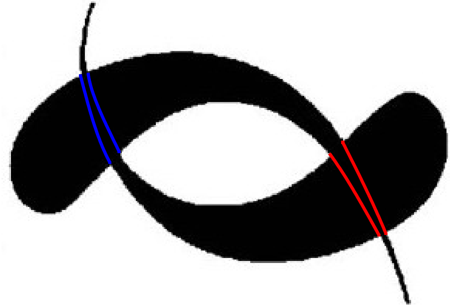
Optimal solution for the problem on $\mathbb{R}^2 \times \mathbb{R}P^1$



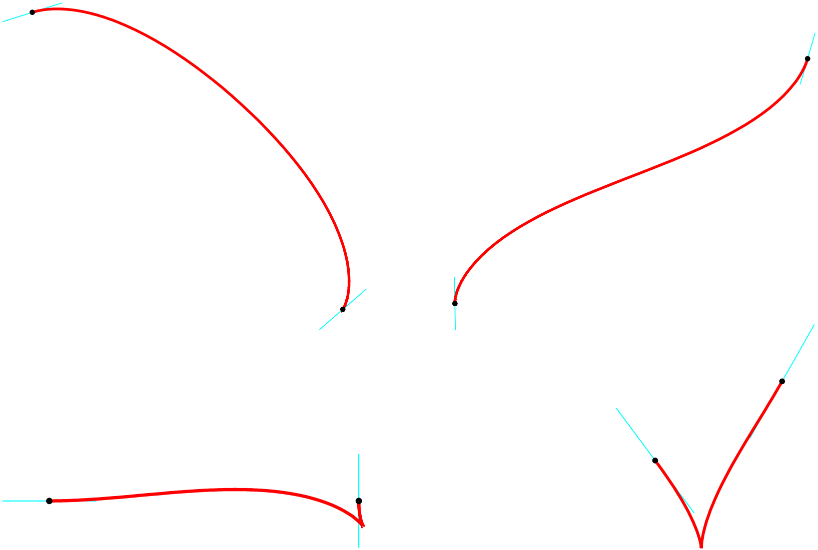
Restored curves



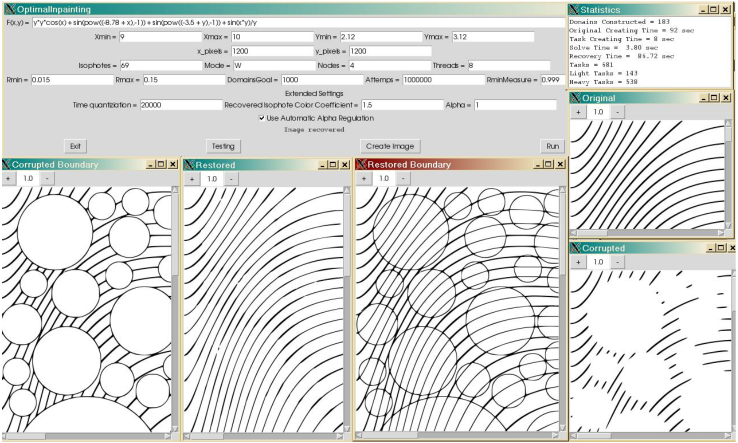




Smooth and non-smooth arcs



Parallel software for restoration of images



F(x,y) = y*y*cos(x) + sin(pow((-8.78 + x),-1)) + sin(pow((-3.5 + y),-1)) + sin(x*y)/y

Xmin = 9 Xmax = 10 Ymin = 2.12 Ymax = 3.12

x_pixels = 1200 y_pixels = 1200

Isophotes = 69 Mode = W Nodes = 4 Threads = 8

Rmin = 0.015 Rmax = 0.15 DomainsGoal = 1000 Attemps = 1000000 RminMeasure = 0.999

Extended Settings

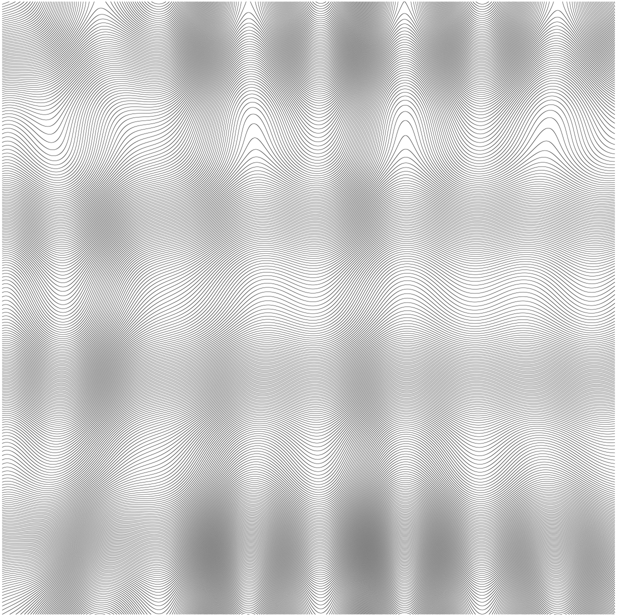
Time quantiziation = 20000 Recovered Isophote Color Coefficient = 1.5 Alpha = 1

☑ Use Automatic Alpha Regulation

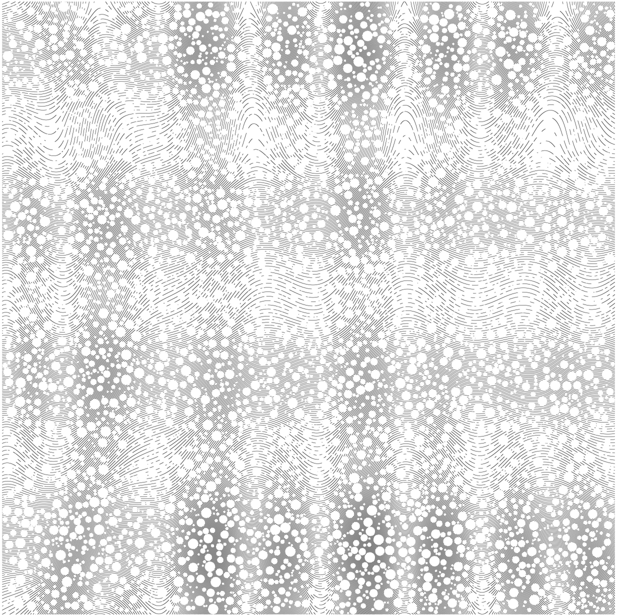
```
Image recovered
```

```
Domains Constructed = 183
Original Creating Time = 92 sec
Task Creating Time = 8 sec
Solve Time =  3.80 sec
Recovery Time =  86.72 sec
Tasks = 681
Light Tasks = 143
Heavy Tasks = 538
```

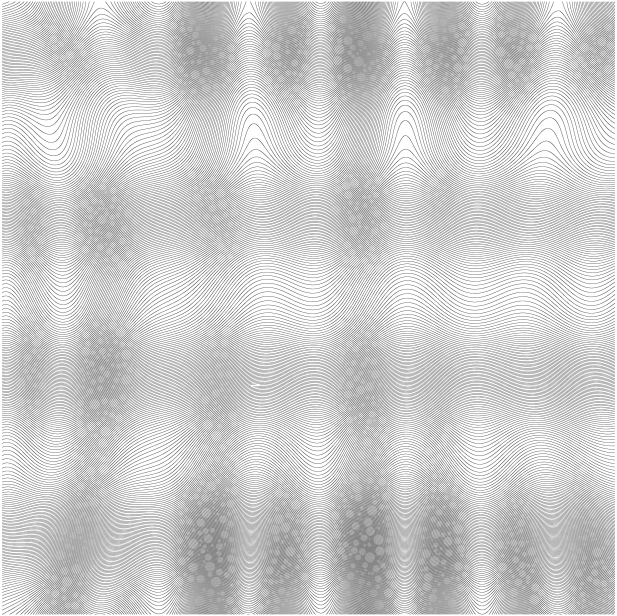
Initial image



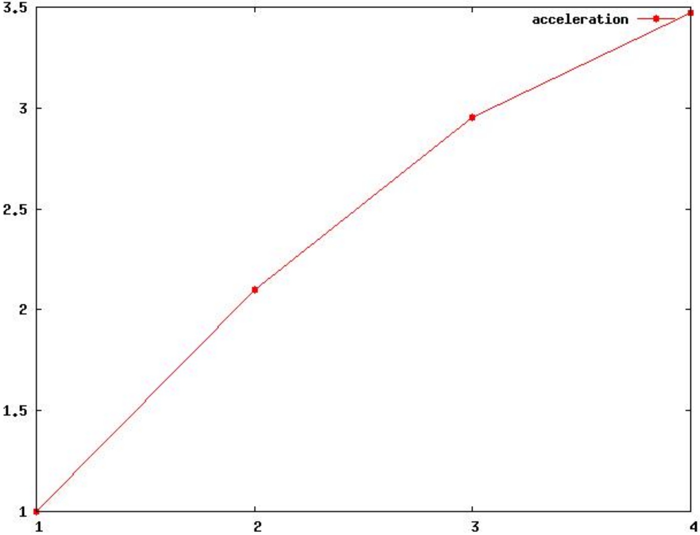
Corrupted image



Restored image



Perfect parallelization



Summing up:

Left-invariant sub-Riemannian problem on SE(2)

- Continuous symmetries
- Optimal trajectories exist
- Extremal trajectories parameterised by Jacobi's functions
- Discrete symmetries
- Maxwell points and time corresponding to symmetries
- Bounds on the first conjugate time
- Global structure of the exponential mapping
- Cut time and cut points
- Maxwell strata
- Cut locus
- Optimal synthesis
- Software for computation of optimal controls and trajectories
- Applications: robotics, vision

Publications

http://www.botik.ru/PSI/CPRC/sachkov/

[1] I. Moiseev, Yu. L. Sachkov, Maxwell strata in sub-Riemannian problem on the group of motions of a plane, *ESAIM: COCV*, 16 (2010), 380–399, available at arXiv:0807.4731 [math.OC].

[2] Yu. L. Sachkov, Conjugate and cut time in sub-Riemannian problem on the group of motions of a plane, *ESAIM: COCV*, 16 (2010), 1018–1039, available at arXiv:0903.0727 [math.OC].

[3] Yu. L. Sachkov, Cut locus and optimal synthesis in the sub-Riemannian problem on the group of motions of a plane, *ESAIM: COCV*, 17 (2011) *accepted*, available at arXiv:0903.0727 [math.OC].

Classification of invariant sub-Riemannian structures on 3D Lie groups (A.Agrachev)