Parametrization of Extremal Trajectories in Sub-Riemannian Problem on Group of Motions of Pseudo Euclidean Plane

Yasir Awais Butt, Yuri L. Sachkov, Aamer Iqbal Bhatti

Abstract We consider the sub-Riemannian length optimization problem on the group of motions of hyperbolic plane i.e. the special hyperbolic group SH(2). The system comprises of left invariant vector fields with 2 dimensional linear control input and energy cost functional. We prove the global controllability of control distribution and use Pontryagin Maximum Principle to obtain the extremal control input and sub-Riemannian geodesics. The abnormal and normal extremal trajectories of the system are analyzed qualitatively and investigated for strict abnormality. A change of coordinates transforms the vertical subssystem of the normal Hamiltonian system into mathematical pendulum. In suitable elliptic coordinates the vertical and horizontal subsystems are integrated such that the resulting extremal trajectories are parametrized by Jacobi elliptic functions.

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1 Introduction

Sub-Riemannian geometry deals with the study of smooth manifolds M that are endowed with a vector distribution Δ and a smoothly varying positive definite quadratic form. The distribution Δ is a subbundle of tangent bundle TM and the quadratic form allows measuring distance between any two points $p_1, p_2 \in M$ [1],[2],[3]. Other names that appear in literature for sub-Riemannian Geometry are Carnot-Carathéodory geometry [4], Non-holonomic Riemannian geometry[5] and Singular Riemannian geometry [6]. The aim of defining and solving a sub-Riemannian problem is to find the optimal curves between two given points p_1, p_2 on the sub-Riemannian manifold M such that sub-Riemannian distance between the points is minimized [2],[3]. Sub-Riemannian problems occur widely in nature [2],[7] and have therefore been extensively studied via geometric control methods on various Lie groups such as the Heisenberg group [8], S^3, SL(2), SU(2) [9], SE(2) [10], Engel group [11], Solvable groups [12], $SOLV^-$ [13], and also in [14],[15],[16],[17]. Few examples of physical systems that describe sub-Riemannian problems and on which Geometric control methods have been successfully applied include parking of cars, rolling bodies on a plane without sliding, motion planning and control of robots, satellites, vision, quantum mechanical systems and even finance [2],[7].

We consider the sub-Riemannian problem on the group of motions of Pseudo-Euclidean plane which is a subspace of pseudo Euclidean space. The pseudo Euclidean space F_m^{n+m} is $(n + m)$-dimensional space defined over field of real numbers $\mathbb{R}$ and endowed with a non-degenerate indefinite quadratic form q [18]:

$$q(x) = (x_1^2 + \cdots + x_n^2) - (x_{n+1}^2 + \cdots + x_{n+m}^2).$$

An important example of pseudo Euclidean Space is the Minkowski space-time that arises in the special theory of relativity. It is essentially a four dimensional pseudo Euclidean space F_1^{1+3}, with three ordinary dimensions of space and another intermingled dimension of time [19]. Minkowski space-time was a refomulation of special theory of relativity and it presented a mathematical setting in which Einstein's theory of relativity and Lorentz geometry could be mathematically formulated.

A pseudo Euclidean plane is a two dimensional subspace F_1^{1+1} with $q(x) = x_1^2 - x_2^2$ [18]. As complex numbers represent vectors on a Cartesian/Euclidean plane, hypercomplex or split complex numbers are used to represent gyrovectors on pseudo Euclidean plane [20]. A pseudo Euclidean plane represents Mikowskian space-time of two dimensions with one spatial variable and one temporal variable [20] and is therefore associated to our understanding of the world. Due to its natural linkage with the Minkowskian space-time plane, formulation of a sub-Riemannian problem on pseudo Euclidean plane can possibly give insight into how nature works.

The motions of pseudo Euclidean plane described in Section 2 form a 3-dimensional Lie group known as special hyperbolic group SH(2) [21]. The optimal control problem comprises a system of left invariant vector fields with 2 dimensional linear control input and energy cost functional. The group SH(2) gives one of the Thurston's three-dimensional geometries [22] and the study of Sub-Riemannian problem on SH(2) bears significance in the program of complete study of all left-invariant Sub Riemannian problems on three dimensional Lie groups following the classification in terms of the basic differential invariants [23].

Notice that an equivalent sub-Riemannian problem was considered in [13] on Lie group $SOLV^-$. However, the parametrization of sub-Riemannian geodesics obtained in [13] is far from complete. This paper seeks rigorous scheme of analysis developed in [10],[11],[16] for parametrization and qualitative analysis of the extremal trajectories. The corresponding results are insightful, in simpler form owing primarily to the use of simpler elliptic coordinates and allow further analysis on global and local optimality of geodesics. The paper is organized as follows. We begin with a brief description of SH(2) in Section 2 and introduce the concepts related to sub-Riemannian geometry in Section 3. Section 4 and 5 contain the main results of this research. In Section 4 we state the sub-Riemannian problem on SH(2) and investigate the global controllability of the control distribution. We apply the Pontryagin Maximum Principle on SH(2) and discuss the abnormal and normal trajectories. Section 5 is a detailed description of the integration of the vertical and horizontal subsystem in Elliptic coordinates. In Section 6 we present the qualitative analysis of projections of extremal trajectories on xy-plane. Section 7 and 8 pertain to future work and conclusion respectively.

2 The Group SH(2) of Motions of Pseudo Euclidean Plane

Following exposition is motivated from [21] and is presented here for the sake of completeness.

2.1 Pseudo Euclidean Plane

A pseudo Euclidean plane is a 2-dimensional real linear space endowed with an indefinite bilinear form given as:

$$[\mathbf{a}, \mathbf{b}] = a_1 b_1 - a_2 b_2.$$

The distance r between a point $\mathbf{a}(a_1, a_2)$ and another point $\mathbf{b}(b_1, b_2)$ on the plane is given by the formula:

$$r^2 = (a_1 - b_1)^2 - (a_2 - b_2)^2 \equiv [\mathbf{a} - \mathbf{b}, \mathbf{a} - \mathbf{b}].$$

Unlike the Euclidean plane, the distance in pseudo Euclidean plane may be real as well as purely imaginary. The distance between two distinct points can also be zero provided $a_1 - b_1 = \pm(a_2 - b_2)$. Geometrically, pseudo Euclidean plane is represented by unit hyperbola $a_1^2 - a_2^2 = 1$ that also depicts Minkowski space-time [20], see Figure 1. The asymptotic lines $a_1 = a_2$ and $a_1 = -a_2$ segregate the plane into four distinct sectors known as Right (RS), Left (LS), Up (US) and Down (DS) sectors. We will consider our problem only on RS where $-a_1 < a_2 < a_1$ and distance between points $r > 0$ because the motions of pseudo Euclidean plane we consider are sector preserving maps. As the points in a Euclidean plane are transformed into polar coordinates via trignometric functions, the points in RS may be represented in polar coordinates by hyperbolic functions as follows[20],[21]:

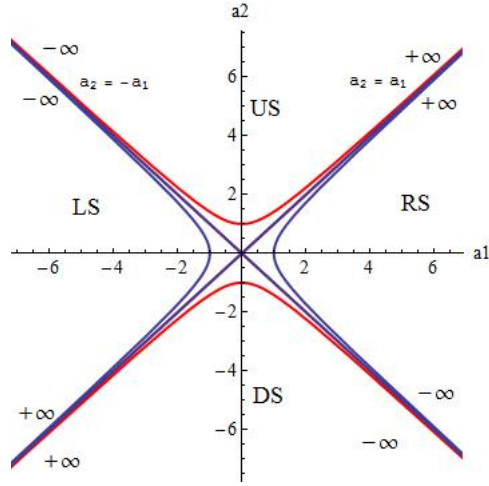$$a_1 = r \cosh \varphi,$$
$$a_2 = r \sinh \varphi,$$

Fig. 1 Pseudo Euclidean plane represented by unit hyperbola

where $r \in \mathbb{R}^+$ is the length and $\varphi \in \mathbb{R}$ is the hyperbolic angle of rotation of the gyrovector.

2.2 Group SH(2) of Motions of Pseudo Euclidean Plane

The motions we consider are non homogeneous, linear, distance, orientation and sector preserving maps of points in Pseudo Euclidean plane. Specifically the motions comprise translations and hyperbolic rotations given as:

$$b_1 = a_1 \cosh z + a_2 \sinh z + x,$$
$$b_2 = a_1 \sinh z + a_2 \cosh z + y,$$

where $x, y, z \in \mathbb{R}$.

Thus motion m of pseudo Euclidean plane is completely parametrized by $x, y, z \in \mathbb{R}$. Composition of two motions $m_1(x_1, y_1, z_1)$ and $m_2(x_2, y_2, z_2)$ is another motion $m_3(x_3, y_3, z_3)$ given as:

$$m_3(x_3, y_3, z_3) = m_1(x_1, y_1, z_1).m_2(x_2, y_2, z_2),$$

where,

$$x_3 = x_2 \cosh z_1 + y_2 \sinh z_1 + x_1,$$
$$y_3 = x_2 \sinh z_1 + y_2 \cosh z_1 + y_1,$$
$$z_3 = z_1 + z_2.$$

The identity motion m_{Id} is given by $x = y = z = 0$, and inverse of a motion $m(x, y, z)$ is given by $m^{-1}(x^1, y^1, z^1)$ where,

$$x^1 = -x \cosh z + y \sinh z,$$
$$y^1 = x \sinh z - y \cosh z,$$
$$z^1 = -z.$$

The composition of motion m with m^{-1} is given as $m^{-1}m = m_{Id}$.

2.3 Lie Group and Lie Algebra Representation

The group SH(2) can be represented by third order matrices:

$$M = \text{SH}(2) = \left\{ \begin{pmatrix} \cosh z & \sinh z & x \\ \sinh z & \cosh z & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\}.$$

The Lie group SH(2) comprises three basis one-parameter subgroups given as:

$$w_1(t) = \begin{pmatrix} \cosh t & \sinh t & 0 \\ \sinh t & \cosh t & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad w_2(t) = \begin{pmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad w_3(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix},$$

whereas basis for Lie algebra are the tangent matrices $A_i = \frac{dw_i(t)}{dt} \mid_{t=0}$ to the subgroups of Lie group SH(2). A_i are given as:

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

The Lie algebra is thus:

$$\mathcal{L} = T_{Id}M = \text{sh}(2) = \text{span}\{A_1, A_2, A_3\}.$$

The multiplication rule for $\mathcal{L}$ is $[A, B] = AB - BA$. Therefore, the Lie bracket for sh(2) is given as $[A_1, A_2] = A_3$, $[A_1, A_3] = A_2$ and $[A_2, A_3] = 0$.

3 Sub-Riemannian Geometry

3.1 Riemannian Manifold

Smooth Riemannian space/manifold (M, g) is a real smooth manifold M on which an inner product g_p can be defined on each tangent space T_pM, $\forall p \in M$. The inner product varies smoothly everywhere on M such that for any vector fields X and Y on M and $p \in M$, $p \mapsto g_p(X(p), Y(p))$ is a smooth function [2],[3]. The family of all inner products g_p on M allows to define various geometric constructs such as length of curves, distance between points and angles on Riemannian manifold exactly like the scalar product defines such notions on Euclidean space. This family of inner products is termed as Riemannian metric (tensor)[24].

3.2 Sub-Riemannian Manifold

Sub-Riemannian space/manifold is a variation of the Riemannian manifold. It comprises a manifold M of dimension n, a smooth vector distribution Δ whose rank m is constant such that $m \leq n$, and g is a Riemannian metric on Δ. It is denoted as a triple (M, Δ, g). The distribution Δ on M is a smooth linear subbundle of the tangent bundle TM i.e. $\Delta \subset TM$. Intuitively, on sub-Riemannian manifold, the motion is restricted along paths that are tangent to horizontal subspaces or the admissible directions of motion are constrained to horizontal subspaces Δ_q, $q \in M$ [25],[26]. Sub-Riemannian manifolds naturally arise in such diverse areas as non-holonomic systems in classical mechanics, image reconstruction, image inpainting etc see e.g. [2],[7],[27],[28].

3.3 Sub-Riemannian Distance

Consider a sub-Riemannian manifold (M, Δ, g) and a Lipschitzian horizontal curve $\gamma : I \subset \mathbb{R} \to M; \dot{\gamma}(t) \in \Delta_{\gamma(t)}$ for almost all $t \in I$. The length of γ is given as:

$$length(\gamma) = \int\limits_I \sqrt{g_{\gamma(t)}(\dot{\gamma}(t))}dt,$$

where $g_{\gamma(t)}$ is the inner product in $\Delta_{\gamma(t)}$[3]. The sub-Riemannian distance between two points $p, q \in M$ is length of the shortest curve joining p to q:

$$d(p;q) = inf \left\{ length(\gamma) : \begin{matrix} \gamma \ is \ horizontal \ curve \\ \gamma \ joins \ p \ to \ q \end{matrix} \right\}.$$

3.4 Sub-Riemannian Problem

Consider a driftless dynamical system on sub-Riemannian manifold (M, Δ, g):

$$\dot{q} = \sum_{i=1}^{m} u_i(t)f_i(q), \qquad (u_1, \cdots, u_m) \in \mathbb{R}^m.$$

The problem of finding horizontal curves γ from initial state q_0 to final state q_1 with shortest sub-Riemannian distance $d(q_0;q_1)$ and tangent to a given distribution $\Delta_q \subset T_qM$ is called a sub-Riemannian problem [3],[27]. Intuitively there exists a set of vector fields f_i whose values $\forall q \in M$ form a local orthonormal frame of the sub-Riemannian structure $(\Delta; g)$. The horizontal curves $\gamma : I \subset \mathbb{R} \to M$ are the solutions of the following optimal control problem in M:

$$\dot{q} = \sum_{i=1}^{m} u_i(t)X_i(q)), \qquad q \in M, \qquad (u_1, \cdots, u_m) \in \mathbb{R}^m,$$

$$q(0) = Id, \qquad q(t_1) = q_1,$$

$$J = \int_0^{t_1} \sqrt{\sum_{i=1}^{m} u_i^2}dt \to \min.$$

4 Sub-Riemannian Problem on SH(2)

Consider following driftless control system on SH(2):

$$\dot{q} = u_1 f_1(q) + u_2 f_2(q), \ q \in M = \text{SH}(2), \ (u_1, u_2) \in \mathbb{R}^2, \tag{1}$$

$$q(0) = Id, \qquad q(t_1) = q_1, \tag{2}$$

$$l = \int_0^{t_1} \sqrt{u_1^2 + u_2^2} \, dt \to \min, \tag{3}$$

$$f_1(q) = qA_2, \qquad f_2(q) = qA_1. \tag{4}$$

Here (1) represents the dynamical system with control inputs u_i and control distribution $\Delta = \text{span}\{f_1, f_2\}$. In (2), $q(0)$ is the intital state at time $t = 0$ and $q(t_1)$ represents the final state to be reached at time t_1 whereas l is the sub-Riemannian distance (length functional) to be minimized. Canonical frame on M in terms of [23] is given as:

$$f_1(q), \ f_2(q), \ f_0(q) = qA_3,$$

$$[f_1, f_0] = 0, \qquad [f_2, f_0] = f_1, \qquad [f_2, f_1] = f_0. \tag{5}$$

By [23], the sub-Riemannian structure:

$$(M, \Delta, g), \qquad \Delta = span\{f_1, f_2\}, \qquad g(f_i, f_j) = \delta_{ij},$$

is unique upto rescaling, left invariant contact sub-Riemannian structure on SH(2). Here δ_{ij} is the Kronecker delta.

Hypothetically, the system represents a Reeds-Shepp car moving on a hyperbolic plane. The car is a rigid body and is allowed to move forward, backward and rotate about a vertical axis passing through its centre of mass. Being point mass, the position and orientation of car on hyperbolic plane are the coordinates (x, y) of its centre of mass and the angle z respectively. Given an initial state q_0 and a final state q_1, the objective is to determine a horizontal curve $\gamma \subset \text{SH}(2)$ between the initial and final state that minimizes the sub-Riemannian length functional (3). The admissible trajectories are assumed Lipschitz and admissible control inputs u_i are assumed bounded.

4.1 Complete Controllability

Theorem 1 *The control system (1) is completely controllable. Moreover, the optimal control problem (1)-(4) has solutions.*

Proof Consider the family of vector fields $\Delta \subset Vec(\text{SH}(2))$, given by (1), where $Vec(\text{SH}(2))$ is the set of all vector fields on SH(2). Since $[f_1(q), f_2(q)] = -[f_2(q), f_1(q)] = -f_0(q)$, the system is full rank because the distribution Δ satisfies the bracket generating condition (also known as *Hörmander condition*). Hence,

$$\mathcal{L}_q\Delta = span\{f_1(q), f_2(q), -f_0(q)\} = \text{T}_q\text{SH}(2) \qquad \forall q \in M.$$

By Rachevsky-Chow's Theorem [29],[30], for a connected manifold and corresponding bracket generating control distribution, the system is completely controllable. From the definition of SH(2) it is connected and Δ is bracket generating, hence the system (1) is completely controllable.

Existence of optimal trajectories for problem (1)-(4) follows from Filippov's theorem [25]. □

4.2 Pontryagin's Maximum Principle for Sub-Riemannian Problem on SH(2)

In coordinates (x, y, z) the basis vector field are given as:

$$f_1(q) = \cosh z \frac{\partial}{\partial x} + \sinh z \frac{\partial}{\partial y},$$

and

$$f_2(q) = \frac{\partial}{\partial z}.$$

Therefore (1) may be written as:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} \cosh z \\ \sinh z \\ 0 \end{pmatrix} u_1 + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u_2. \tag{6}$$

By Cauchy Schwarz inequality,

$$(l(u))^2 = \left(\int_0^{t_1} \sqrt{u_1^2 + u_2^2} dt \right)^2 \leq t_1 \int_0^{t_1} (u_1^2 + u_2^2) dt.$$

Thus sub-Riemannian length functional minimization problem (3) is equivalent to the problem of minimizing the following energy functional with fixed t_1[26]:

$$J = \frac{1}{2} \int_0^{t_1} (u_1^2 + u_2^2) dt \rightarrow min. \tag{7}$$

We write the PMP form for (1),(2),(7) using coordinate free approach described in [25]. Consider control dependent Hamiltonian for PMP corresponding to vector fields $f_1(q)$ and $f_2(q)$:

$$h_u^\nu(\lambda) = \langle \lambda, f_u(q) \rangle + \frac{\nu}{2}(u_1^2 + u_2^2), \quad q = \pi(\lambda), \quad \lambda \in T^*M. \tag{8}$$

Let $h_i(\lambda) = \langle \lambda, f_i(q) \rangle$ be the Hamiltonians corresponding to basis vector fields f_i. Then (8) can be written as:

$$h_u^\nu(\lambda) = u_1 h_1(\lambda) + u_2 h_2(\lambda) + \frac{\nu}{2}(u_1^2 + u_2^2), \qquad u \in \mathbb{R}^2. \tag{9}$$

Now PMP for optimal control problem is given by using Theorem 12.3 [25] as:

Theorem 2 *Let $\tilde{u}(t)$ be optimal control and $\tilde{q}(t)$ be optimal trajectory for $t \in [0, t_1]$ and $h_u^\nu(\lambda)$ given by (9) be the Hamiltonian function for (1),(2),(7). Then, there exists a nontrivial pair:*

$$(\nu, \lambda_t) \neq 0, \qquad \nu \in \mathbb{R}, \quad \lambda_t \in T^*_{\tilde{q}(t)}M, \quad \pi(\lambda_t) = \tilde{q}(t).$$

where λ_t is a Lipschitzian curve and $\nu \in \{-1, 0\}$ is a number, for which following conditions hold for almost all time $t \in [0, t_1]$:

$$\dot{\lambda}_t = \overrightarrow{h}^\nu_{\tilde{u}(t)}(\lambda_t), \tag{10}$$

$$h^\nu_{\tilde{u}(t)}(\lambda_t) = \max_{u \in \mathbb{R}^2} h^\nu_u(\lambda_t). \tag{11}$$

where $\overrightarrow{h}^\nu_{\tilde{u}(t)}(\lambda_t)$ is the Hamiltonian vector field corresponding to the Hamiltonian function $h^\nu_{\tilde{u}(t)}$.

4.3 Abnormal Trajectories

Abnormal trajectories correspond to the case $\nu = 0$. The Hamiltonian (9) in this case can be written as:

$$h_u^0(\lambda) = u_1 h_1(\lambda) + u_2 h_2(\lambda). \tag{12}$$

Theorem 3 *All abnormal extremal trajectories for problem (1),(2),(7) are constant.*

Proof Let λ_t be the abnormal extremal, then the maximization condition of PMP yields:

$$h_1(\lambda_t) = h_2(\lambda_t) \equiv 0. \tag{13}$$

Differentiating (13) w.r.t. Hamiltonian vector field and noting that Poisson bracket follows the same multiplication rule as that of Lie bracket of $\mathrm{sh}(2)$:

$$\dot{h}_1 = \left\{ h_u^0, h_1 \right\} = \{u_1 h_1 + u_2 h_2, h_1\} = u_1 \{h_1, h_1\} + u_2 \{h_2, h_1\} = u_2 h_0,$$

$$\dot{h}_2 = \left\{ h_u^0, h_2 \right\} = \{u_1 h_1 + u_2 h_2, h_2\} = u_1 \{h_1, h_2\} + u_2 \{h_2, h_2\} = -u_1 h_0.$$

Therefore,

$$u_2(t) h_0(\lambda_t) = u_1(t) h_0(\lambda_t) \equiv 0,$$
$$\implies u_1^2(t) h_0^2 + u_2^2(t) h_0^2 = 0.$$

If $h_0(\lambda_t) = 0$ for some λ_t, then $h_1(\lambda_t) = h_2(\lambda_t) = h_0(\lambda_t) \equiv 0$ which means $\lambda_t = 0$. This is impossible since $\nu = 0$. Therefore $u_1^2(t) + u_2^2(t) = 0 \implies u_1 = u_2 = 0$ and hence the abnormal extremal trajectories are constant. $\qquad \square$

4.4 Normal Trajectories

Normal trajectories correspond to the case $\nu = -1$. The Hamiltonian (9) in this case can be written as:

$$H = h_u^{-1}(\lambda) = u_1 h_1(\lambda) + u_2 h_2(\lambda) - \frac{1}{2} \left(u_1^2 + u_2^2 \right), \qquad u \in \mathbb{R}^2. \tag{14}$$

Using the maximization condition of PMP, the trajectories of the normal Hamiltonian satisfy the equalities:

$$\frac{\partial H}{\partial u} = \begin{pmatrix} h_1 - u_1 \\ h_2 - u_2 \end{pmatrix} = 0,$$
$$\implies u_1 = h_1, \quad u_2 = h_2. \tag{15}$$

Note that if $u_i = 0$, then normal extremal trajectories are constant. Therefore the abnormal trajectories are not strictly abnormal. The normal extremals are the trajectories of Hamiltonian system $\dot{\lambda} = \vec{H}(\lambda)$, $\lambda \in T^* M$ with the maximized Hamiltonian $H = \frac{1}{2} \left(h_1^2 + h_2^2 \right) \geq 0$. Specifically, for non-constant normal extremals $H > 0$. Note that Hamiltonian function in normal case is homogeneous w.r.t. h_1, h_2 and therefore we consider its trajectories for the level surface $H = \frac{1}{2}$. The phase cylinder containing the initial covector λ in this case is:

$$C = T_{q_0}^* M \cap \left\{ H(\lambda) = \frac{1}{2} \right\} = \left\{ (h_1, h_2, h_0) \in \mathbb{R}^3 | h_1^2 + h_2^2 = 1 \right\}. \tag{16}$$

Differentiating (14) w.r.t. Hamiltonian vector field we get:

$$\dot{h}_1 = \{H, h_1\} = \left\{ \frac{1}{2} \left(h_1^2 + h_2^2 \right), h_1 \right\} = h_2 \{h_2, h_1\} = h_2 h_0,$$

$$\dot{h}_2 = \{H, h_2\} = \left\{ \frac{1}{2} \left(h_1^2 + h_2^2 \right), h_2 \right\} = h_1 \{h_1, h_2\} = -h_1 h_0,$$

$$\dot{h}_0 = \{H, h_0\} = \left\{ \frac{1}{2} \left(h_1^2 + h_2^2 \right), h_0 \right\} = h_1 \{h_1, h_0\} + h_2 \{h_2, h_0\} = h_1 h_2.$$

Hence, complete Hamiltonian system in normal case is given as:

$$
\begin{pmatrix} \dot{h}_1 \\ \dot{h}_2 \\ \dot{h}_0 \\ \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} h_2 h_0 \\ -h_1 h_0 \\ h_1 h_2 \\ h_1 \cosh z \\ h_1 \sinh z \\ h_2 \end{pmatrix}.
\tag{17}
$$

Theorem 4 *Vertical subsystem of the Hamiltonian system (17) in normal case is a mathematical pendulum.*

Proof Introduce following coordinates transformation:

$$h_1 = \cos \alpha, \quad h_2 = \sin \alpha. \tag{18}$$

Thus,

$$\dot{h}_1 = -\sin \alpha \dot{\alpha} = \sin \alpha . h_0,$$
$$\dot{\alpha} = -h_0. \tag{19}$$

Similarly,

$$\dot{h}_0 = \cos \alpha \sin \alpha = \frac{1}{2} \sin 2\alpha. \tag{20}$$

Let us introduce another change of coordinates:

$$\gamma = 2\alpha \in 2S^1 = \mathbb{R}/4\pi\mathbb{Z}, \quad c = -2h_0 \in \mathbb{R}, \tag{21}$$

$$\implies \dot{\gamma} = 2\dot{\alpha} = -2h_0 = c,$$

and

$$\dot{c} = -2\dot{h}_0 = -2h_1 h_2 = -2\cos\alpha\sin\alpha = -\sin 2\alpha = -\sin\gamma.$$

Thus,

$$\begin{pmatrix} \dot{\gamma} \\ \dot{c} \end{pmatrix} = \begin{pmatrix} c \\ -\sin \gamma \end{pmatrix}. \tag{22}$$

It can be easily seen that (22) represents a mathematical pendulum.   □

5 Parametrization of Extremal Trajectories

5.1 Hamiltonian System

Hamiltonian system for normal trajectories was given in (17). Under the transformations introduced in (18),(21), the horizontal subsystem can be written as:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} h_1 \cosh z \\ h_1 \sinh z \\ h_2 \end{pmatrix} = \begin{pmatrix} \cos \frac{\gamma}{2} \cosh z \\ \cos \frac{\gamma}{2} \sinh z \\ \sin \frac{\gamma}{2} \end{pmatrix}. \tag{23}$$

5.2 Decomposition of the Initial Phase Cylinder

Following the techniques employed in [10], the decomposition of phase cylinder C proceeds as follows. The total energy integral of the pendulum obtained in (22) is given as:

$$E = \frac{c^2}{2} - \cos \gamma = 2h_0^2 - h_1^2 + h_2^2, \qquad E \in [-1, +\infty). \tag{24}$$

The initial phase cylinder (16) may be decomposed into following subsets based upon the pendulum energy that correspond to various pendulum trajectories:

$$C = \bigcup_{i=1}^{5} C_i,$$

where

$$C_1 = \{\lambda \in C | E \in (-1, 1)\},$$
$$C_2 = \{\lambda \in C | E \in (1, \infty)\},$$
$$C_3 = \{\lambda \in C | E = 1, c \neq 0\},$$
$$C_4 = \{\lambda \in C | E = -1\} = \{(\gamma, c) \in C | \gamma = 2\pi n, c = 0\}\}, \quad n \in \mathbb{N},$$
$$C_5 = \{\lambda \in C | E = 1\} = \{(\gamma, c) \in C | \gamma = 2\pi n + \pi, c = 0\}\}, \quad n \in \mathbb{N}.$$

Continuing the approach taken in [10] the subsets C_i may be further decomposed as:

$$C_1 = \cup_{i=0}^{1} C_1^i, \quad C_1^i = \{(\gamma, c) \in C_1 | sgn(\cos(\gamma/2)) = (-1)^i\},$$
$$C_2 = C_2^+ \cup C_2^-, \quad C_2^{\pm} = \{(\gamma, c) \in C_2 | sgn\, c = \pm 1\},$$
$$C_3 = \cup_{i=0}^{1}(C_3^{i+} \cup C_3^{i-}), \quad C_3^{i\pm} = \{(\gamma, c) \in C_3 | sgn(\cos(\gamma/2)) = (-1)^i, sgn\, c = \pm 1\},$$
$$C_4 = \cup_{i=0}^{1} C_4^i, \quad C_4^i = \{(\gamma, c) \in C | \gamma = 2\pi i, c = 0\},$$
$$C_5 = \cup_{i=0}^{1} C_5^i, \quad C_5^i = \{(\gamma, c) \in C | \gamma = 2\pi i + \pi, c = 0\}.$$

In all of the above $i = 0, 1$. The phase portrait of the pendulum and corresponding decomposition of initial phase cylinder C is depicted in Figure 2.
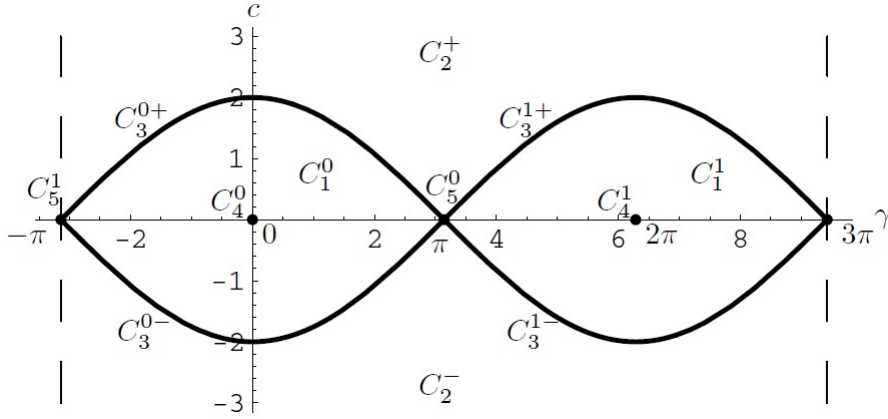
Fig. 2 Decomposition of the Phase Cylinder and the Connected Subsets

5.3 Elliptic Coordinates

Employing the approach developed in [10],[11] we transform the system in terms of elliptic coordinates (φ, k) on the domain $\cup_{i=1}^{3} C_i \subset C$. Note that φ is the reparametrized time of motion and k is the reparametrized energy of the pendulum. Correspondingly, we describe the system and the extremal trajectories in terms of Jacobi elliptic functions $\mathrm{sn}(\varphi, k)$, $\mathrm{cn}(\varphi, k)$, $\mathrm{dn}(\varphi, k)$, $\mathrm{am}(\varphi, k)$, and $\mathrm{E}(\varphi, k) = \int_0^{\varphi} \mathrm{dn}^2(t, k) dt$. Detailed description of Jacobi elliptic functions may be found in [31].

5.3.1 Case 1 : $\lambda = (\varphi, k) \in C_1$

$$k = \sqrt{\frac{E+1}{2}} = \sqrt{\sin^2 \frac{\gamma}{2} + \frac{c^2}{4}} \in (0, 1), \tag{25}$$

$$\sin \frac{\gamma}{2} = s_1 k \, \mathrm{sn}(\varphi, k), \quad s_1 = sgn\left(\cos \frac{\gamma}{2}\right), \tag{26}$$

$$\cos \frac{\gamma}{2} = s_1 \mathrm{dn}(\varphi, k), \tag{27}$$

$$\frac{c}{2} = k \, \mathrm{cn}(\varphi, k), \quad \varphi \in [0, 4K(k)]. \tag{28}$$

Proposition 1 *In elliptic coordinates the flow of vertical subsystem rectifies.*

Proof Using (25)

$$k^2 = \sin^2 \frac{\gamma}{2} + \frac{c^2}{4}. \tag{29}$$

Taking the time derivative of (29),

$$2k\dot{k} = 2 \sin \frac{\gamma}{2} \cos \frac{\gamma}{2} \frac{\dot{\gamma}}{2} + \frac{c\dot{c}}{2} \tag{30}$$

Using (22),

$$2k\dot{k} = \frac{\dot{\gamma}}{2} \sin \gamma - \frac{\dot{\gamma}}{2} \sin \gamma = 0.$$

Now either $k = 0$ or $\dot{k} = 0$. Since $k \in (0, 1)$, hence it cannot be zero and therefore:

$$\dot{k} = 0. \tag{31}$$

Using (28) and the derivatives of elliptic functions defined in [31],

$$\frac{d}{dt}\left(\frac{c}{2}\right) = \frac{d}{dt}k\mathrm{cn}(\varphi, k),$$

$$\frac{\dot{c}}{2} = k\frac{d}{d\varphi}\mathrm{cn}(\varphi, k).\frac{d\varphi}{dt} + k\frac{d}{dk}\mathrm{cn}(\varphi, k).\frac{dk}{dt} + \mathrm{cn}(\varphi, k).\frac{dk}{dt},$$

$$-\sin\gamma = -2k\mathrm{sn}(\varphi, k)\mathrm{dn}(\varphi, k)\dot{\varphi}.$$

because $\frac{dk}{dt} = 0$. Now,

$$\dot{\varphi} = \frac{\sin\gamma}{2k\,\mathrm{sn}(\varphi, k).\mathrm{dn}(\varphi, k)}. \tag{32}$$

Now using (26),(27):

$$\sin\frac{\gamma}{2}\cos\frac{\gamma}{2} = s_1 k\,\mathrm{sn}(\varphi, k).s_1\mathrm{dn}(\varphi, k),$$

$$2\sin\frac{\gamma}{2}\cos\frac{\gamma}{2} = 2s_1^2 k\,\mathrm{sn}(\varphi, k).\mathrm{dn}(\varphi, k),$$

$$\sin\gamma = 2k\,\mathrm{sn}(\varphi, k).\mathrm{dn}(\varphi, k).$$

Thus (32) becomes:

$$\dot{\varphi} = 1. \tag{33}$$

$\square$

5.3.2 Case 2 : $\lambda = (\varphi, k) \in C_2$

$$k = \sqrt{\frac{2}{E+1}} = \sqrt{\frac{1}{\sin^2\frac{\gamma}{2} + \frac{c^2}{4}}} \in (0, 1), \tag{34}$$

$$\sin\frac{\gamma}{2} = s_2\mathrm{sn}\left(\frac{\varphi}{k}, k\right), \quad s_2 = \mathrm{sgn}(c), \tag{35}$$

$$\cos\frac{\gamma}{2} = \mathrm{cn}\left(\frac{\varphi}{k}, k\right), \tag{36}$$

$$\frac{c}{2} = \frac{s_2}{k}\mathrm{dn}\left(\frac{\varphi}{k}, k\right), \quad \varphi \in [0, 4kK(k)]. \tag{37}$$

5.3.3 Case 3 : $\lambda = (\varphi, k) \in C_3$

$$k = 1, \tag{38}$$

$$\sin\frac{\gamma}{2} = s_1 s_2 \tanh\varphi, \quad s_1 = \mathrm{sgn}\left(\cos\frac{\gamma}{2}\right), \quad s_2 = \mathrm{sgn}(c), \tag{39}$$

$$\cos\frac{\gamma}{2} = s_1/\cosh\varphi, \tag{40}$$

$$\frac{c}{2} = s_2/\cosh\varphi, \quad \varphi \in (-\infty, \infty). \tag{41}$$

Using the procedure outlined for Case 1, it can be proved that the flow of the pendulum rectifies for cases 2 and 3 as well.

5.4 Integration of Vertical Subsystem

Since the flow of vertical subsystem rectifies in elliptic coordinates, therefore, the vertical subsystem is trivially integrated as $\varphi = t + \varphi_0$ and $k = constant$, where φ_0 is the value of φ at $t = 0$.

5.5 Integration of Horizontal Subsystem

In the following we consider integration of horizontal subsystem (23) for cases 1-3 noted above. Assuming zero initial state i.e. $x(0) = y(0) = z(0) = 0$.

5.5.1 Case 1 : $\lambda = (\varphi, k) \in C_1$

Theorem 5 *In case 1 extremal trajectories are parametrized as follows:*

$$
\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{s_1}{2} \left[\left(w + \frac{1}{w(1-k^2)} \right) [\mathrm{E}(\varphi) - \mathrm{E}(\varphi_0)] + \left(\frac{k}{w(1-k^2)} - kw \right) [\mathrm{sn}\,\varphi - \mathrm{sn}\,\varphi_0] \right] \\ \frac{1}{2} \left[\left(w - \frac{1}{w(1-k^2)} \right) [\mathrm{E}(\varphi) - \mathrm{E}(\varphi_0)] - \left(\frac{k}{w(1-k^2)} + kw \right) [\mathrm{sn}\,\varphi - \mathrm{sn}\,\varphi_0] \right] \\ s_1 \ln\left[(\mathrm{dn}\,\varphi - k\mathrm{cn}\,\varphi).w \right] \end{pmatrix}
\tag{42}
$$

where $w = \frac{1}{\mathrm{dn}\varphi_0 - k\mathrm{cn}\varphi_0}$.

Proof From (23) consider $\dot{z} = \sin\frac{\gamma}{2} = s_1 k \,\mathrm{sn}(\varphi, k)$. The solution to this ODE can be written as:

$$
\int_0^z dz = \int_{\varphi_0}^{\varphi} s_1 k \,\mathrm{sn}\,\varphi d\varphi.
\tag{43}
$$

Using [32], eq(43) becomes:

$$
z = s_1 \ln(\mathrm{dn}\,\varphi - k\mathrm{cn}\,\varphi) - s_1 \ln(\mathrm{dn}\,\varphi_0 - k\mathrm{cn}\,\varphi_0).
\tag{44}
$$

Let $\ln w = -\ln(\mathrm{dn}\,\varphi_0 - k\mathrm{cn}\,\varphi_0)$, $w = \frac{1}{\mathrm{dn}\,\varphi_0 - k\mathrm{cn}\,\varphi_0}$. Then (44) becomes

$$
z = s_1 \ln[(\mathrm{dn}\,\varphi - k\mathrm{cn}\,\varphi).w].
\tag{45}
$$

From (23) now consider,

$$
\dot{x} = \cos\frac{\gamma}{2}\cosh z = s_1 \mathrm{dn}\,\varphi \cosh\left(s_1 \ln\left[(\mathrm{dn}\,\varphi - k\mathrm{cn}\,\varphi).w \right] \right),
\tag{46}
$$

$$
\dot{x} = \frac{s_1}{2}\left(w.\mathrm{dn}^2\varphi - kw.\mathrm{dn}\,\varphi\mathrm{cn}\,\varphi + \frac{\mathrm{dn}\,\varphi}{(\mathrm{dn}\,\varphi - k\mathrm{cn}\,\varphi).w} \right).
$$

This can be integrated as:

$$
x = \frac{s_1}{2}\left[w\int_{\varphi_0}^{\varphi}\mathrm{dn}^2\varphi d\varphi - kw\int_{\varphi_0}^{\varphi}\mathrm{dn}\,\varphi\mathrm{cn}\,\varphi d\varphi + \frac{1}{w}\int_{\varphi_0}^{\varphi}\frac{\mathrm{dn}^2\varphi + k\mathrm{cn}\,\varphi\mathrm{dn}\,\varphi}{\mathrm{dn}^2\varphi - k^2\mathrm{cn}^2\varphi}d\varphi \right].
\tag{47}
$$

Now using the standard identities of elliptic functions result of integration of (47) can be written as:

$$x = \frac{s_1}{2}\left[\left(w + \frac{1}{w\left(1-k^2\right)}\right)[\mathrm{E}(\varphi) - \mathrm{E}(\varphi_0)] + \left(\frac{k}{w\left(1-k^2\right)} - kw\right)[\mathrm{sn}\,\varphi - \mathrm{sn}\,\varphi_0]\right].$$
(48)

From (23) now consider,

$$\dot{y} = \cos\frac{\gamma}{2}\sinh z = s_1\mathrm{dn}\,\varphi\sinh(s_1\ln[(\mathrm{dn}\,\varphi - k\mathrm{cn}\,\varphi).w]),$$
$$\dot{y} = s_1^2\mathrm{dn}\,\varphi\sinh(\ln[(\mathrm{dn}\,\varphi - k\mathrm{cn}\,\varphi).w]),$$
$$\dot{y} = \mathrm{dn}\,\varphi\sinh(\ln[(\mathrm{dn}\,\varphi - k\mathrm{cn}\,\varphi).w]).$$
(49)

The integration follows the same pattern as described above and hence final result of integration of (49) can be written as:

$$y = \frac{1}{2}\left[\left(w - \frac{1}{w\left(1-k^2\right)}\right)[\mathrm{E}(\varphi) - \mathrm{E}(\varphi_0)] - \left(\frac{k}{w\left(1-k^2\right)} + kw\right)[\mathrm{sn}\,\varphi - \mathrm{sn}\,\varphi_0]\right].$$
(50)

$\square$

5.5.2 Case 2 : $\lambda = (\varphi, k) \in C_2$

Theorem 6 *Consider horizontal system (23) for case 2 (34)-(37) and substitute $\psi_0 = \frac{\varphi_0}{k}$ and $\psi = \frac{\varphi}{k} = \psi_0 + \frac{t}{k}$, the integration results can be summarized as:*

$$x = \frac{1}{2}\left(\frac{1}{w(1-k^2)} - w\right)\left[\mathrm{E}(\psi) - \mathrm{E}(\psi_0) - k'^2(\psi - \psi_0)\right]$$
$$+ \frac{1}{2}\left(kw + \frac{k}{w(1-k^2)}\right)[\mathrm{sn}\,\psi - \mathrm{sn}\,\psi_0],$$
$$y = -\frac{s_2}{2}\left(\frac{1}{w(1-k^2)} + w\right)\left[\mathrm{E}(\psi) - \mathrm{E}(\psi_0) - k'^2(\psi - \psi_0)\right]$$
$$+ \frac{s_2}{2}\left(kw - \frac{k}{w(1-k^2)}\right)[\mathrm{sn}\,\psi - \mathrm{sn}\,\psi_0],$$
$$z = s_2\ln[(\mathrm{dn}\,\psi - k\mathrm{cn}\,\psi).w],$$
(51)

where $w = \frac{1}{\mathrm{dn}\,\psi_0 - k\mathrm{cn}\,\psi_0}$.

Proof Proof follows from the procedure outlined in case 1. $\square$

5.5.3 Case 3 : $\lambda = (\varphi, k) \in C_3$

Theorem 7 *In this case, $k = 1$. Integration results are summarized as:*

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{s_1}{2}\left[\frac{1}{w}(\varphi - \varphi_0) + w(\tanh\varphi - \tanh\varphi_0)\right] \\ \frac{s_2}{2}\left[\frac{1}{w}(\varphi - \varphi_0) - w(\tanh\varphi - \tanh\varphi_0)\right] \\ -s_1s_2\ln[w\,sech\,\varphi] \end{pmatrix}$$
(52)

and $w = \cosh\varphi_0$.

Proof Consider horizontal system (23) for case 3 (38)-(41):

$$\dot{z} = \sin\frac{\gamma}{2} = s_1 s_2 \tanh\varphi,$$

$$z = -s_1 s_2 [\ln(\operatorname{sech}\varphi) - \ln(\operatorname{sech}\varphi_0)].$$

Let $-\ln(\operatorname{sech}\varphi_0) = \ln w$, $w = \cosh\varphi_0$, then:

$$z = -s_1 s_2 \ln[w\operatorname{sech}\varphi]. \tag{53}$$

From (23) now consider,

$$\dot{x} = \cos\frac{\gamma}{2}\cosh z = s_1\operatorname{sech}\varphi\cosh\left(-s_1 s_2 \ln[w\operatorname{sech}\varphi]\right),$$

$$\dot{x} = \frac{s_1\operatorname{sech}\varphi}{2}\left[e^{\ln[w\operatorname{sech}\varphi]} + e^{-\ln[w\operatorname{sech}\varphi]}\right],$$

$$dx = \frac{s_1\operatorname{sech}\varphi}{2}\left[\frac{1 + w^2\operatorname{sech}^2\varphi}{w\operatorname{sech}\varphi}\right]d\varphi,$$

$$x = \frac{s_1}{2}\left[\frac{1}{w}(\varphi - \varphi_0) + w(\tanh\varphi - \tanh\varphi_0)\right]. \tag{54}$$

From (23) now consider,

$$\dot{y} = \cos\frac{\gamma}{2}\sinh z = s_1\operatorname{sech}\varphi\sinh\left(-s_1 s_2 \ln[w\operatorname{sech}\varphi]\right),$$

$$\dot{y} = \frac{-s_2\operatorname{sech}\varphi}{2}[e^{\ln[w\operatorname{sech}\varphi]} - e^{-\ln[w\operatorname{sech}\varphi]}],$$

$$dy = \frac{-s_2\operatorname{sech}\varphi}{2}\left[w\operatorname{sech}\varphi - [w\operatorname{sech}\varphi]^{-1}\right]d\varphi,$$

$$y = \frac{s_2}{2}\left[\frac{1}{w}(\varphi - \varphi_0) - w(\tanh\varphi - \tanh\varphi_0)\right]. \tag{55}$$

$$\square$$

5.6 Integration of Horizontal Subsystem - Degenerate Cases

In the following we present the integration of horizontal subsystem in degenerate cases i.e. $\lambda \in C_4$ and $\lambda \in C_5$.

5.6.1 Case 4 : $\lambda \in C_4$

Theorem 8 *Integration results in case 4 are summarized as follows:*

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} sgn\left(\cos\frac{\gamma}{2}\right)t \\ 0 \\ 0 \end{pmatrix}. \tag{56}$$

Proof

$$\dot{z} = \sin\frac{\gamma}{2} = \sin\left(\frac{2n\pi}{2}\right) = 0.$$

Since $z(0) = 0$,

$$z = 0. \tag{57}$$

Therefore,

$$\dot{x} = \cos\frac{\gamma}{2}\cosh z = \cos\left(\frac{2n\pi}{2}\right),$$

$$\dot{x} = sgn\left(\cos\frac{\gamma}{2}\right),$$

$$x = sgn\left(\cos\frac{\gamma}{2}\right)t + W_x,$$

$$x = sgn\left(\cos\frac{\gamma}{2}\right)t, \tag{58}$$

where $W_x = 0$ becasue $x(0) = 0$. Now,

$$\dot{y} = \cos\frac{\gamma}{2}\sinh z = \cos\left(\frac{2n\pi}{2}\right)\sinh(0) = 0,$$

$$y = W_y$$

$$y = 0, \tag{59}$$

where $W_y = 0$ because $y(0) = 0$. □

5.6.2 Case 5 : $\lambda \in C_5$

Theorem 9 *Integration results is case 5 are summarized as follows:*

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ sgn\left(\sin\frac{\gamma}{2}\right)t \end{pmatrix}. \tag{60}$$

Proof

$$\dot{z} = \sin\frac{\gamma}{2} = \sin\left(\frac{\pi + 2n\pi}{2}\right) = sgn\left(\sin\frac{\gamma}{2}\right).$$

Thus,

$$z = sgn\left(\sin\frac{\gamma}{2}\right)t + W_z,$$

$$z = sgn\left(\sin\frac{\gamma}{2}\right)t, \tag{61}$$

where $W_z{=}0$ because $z(0) = 0$. Now,

$$\dot{x} = \cos\frac{\gamma}{2}\cosh z = \cos\left(\frac{\pi + 2n\pi}{2}\right)\cosh z = 0,$$

$$x = 0, \tag{62}$$

because $x(0) = 0$. Now,

$$\dot{y} = \cos\frac{\gamma}{2}\sinh z = \cos\left(\frac{\pi + 2n\pi}{2}\right)\sinh z = 0,$$

$$y = 0, \tag{63}$$

because $y(0) = 0$. □

6 Qualitative Analysis of Projections of Extremal Trajectoris on xy-Plane

The standard formula for the curvature of a plane curve $(x(t), y(t))$ is given as [33]:

$$\kappa = \frac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{(\dot{x}^2 + \dot{y}^2)^{\frac{3}{2}}}. \tag{64}$$

Using (17),(64) curvature of projections $(x(t), y(t))$ of extremal trajectories of the Hamiltonian system (17) is given as:

$$\kappa = \frac{-c\sin\frac{\gamma}{2}}{2\cos^2\frac{\gamma}{2}(\cosh 2z)^{\frac{3}{2}}}. \tag{65}$$

The curves have inflection points when $\sin\frac{\gamma}{2} = 0$ and cusps when $\cos\frac{\gamma}{2} = 0$ or $c = 0$. We see that all curves $(x(t), y(t))$ have inflection points for $\lambda \in \cup_{i=1}^{3} C_i$ but only for $\lambda \in C_2$ the curves have cusps. The resulting trajectories $(x(t), y(t))$ are shown in Figures 3, 4, 5. In degenerate case 4 i.e. $\lambda \in C_4$, the extremal trajectories q_t are sub-Riemannian geodesics in the plane $\{z = 0\}$. The curve $(x(t), y(t))$ is a straight line on the x-axis. In case 5 i.e. $\lambda \in C_5$, the curve $(x(t), y(t))$ is just the initial point $(0, 0)$ for $\{x = y = 0\}$. For non-zero initial conditions $x(0) = W_x$, $y(0) = W_y$, the motions of pseudo Euclidean plane are only hyperbolic rotations whereas the translations along x-axis and y-axis are zero. The resulting trajectory is a quarter circle in the RS that approaches the upper and lower arms of the hyperbola in RS of Figure 1 as $t \to \infty$.
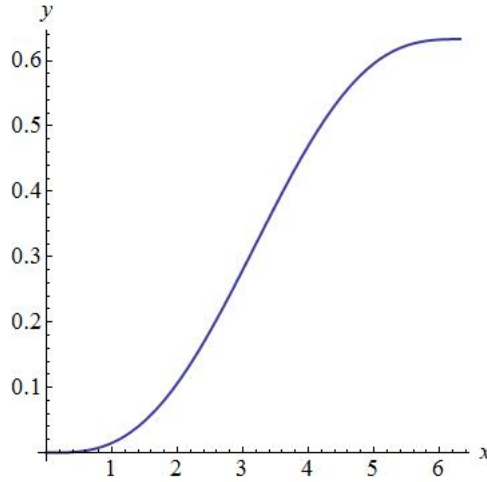

Fig. 3 Inflectional Trajectories $\lambda \in C_1$

7 Future Work

Most natural extension of this work is the computation of Maxwell strata and obtaining the global bound on cut time based on discrete symmetries of the vertical subsystem.
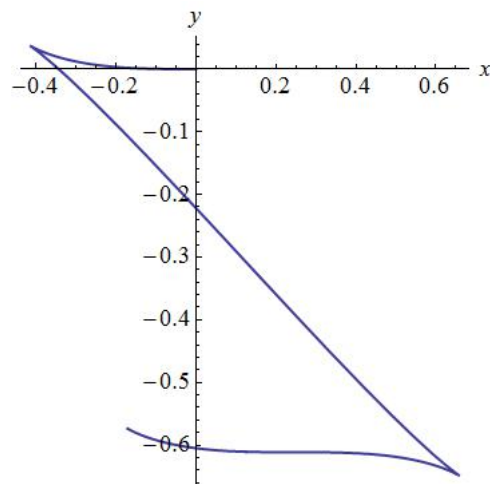
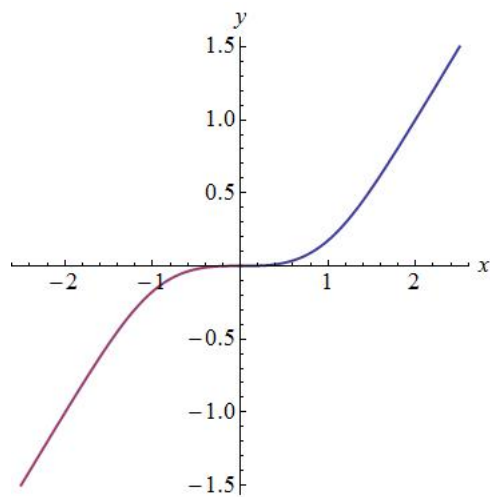**Fig. 4** Inflectional Trajectories $\lambda \in C_2$

Fig. 5 Inflectional Trajectories $\lambda \in C_3$

To this end the methods developed in [10],[17],[16],[14] have been employed and results shall be reported in another paper. Investigation of local and global optimality of extremal trajectories via description of conjugate and cut loci is another exciting and challenging dimension in the problem under consideration. Description of global structure of exponential map and optimal synthesis is the ultimate goal to be addressed in the entire research on SH(2).

8 Conclusion

Sub-Riemannian problem on SH(2) is important from perspective of Mathematics as well as Physics. The direct connection between the pseudo Euclidean plane and

Minkowski space-time geometry suggests that such analysis could potentially lead to better understanding of the Special Theory of Relativity. The Mathematics perspective is also equally significant. The group SH(2) as abstract algebraic structure has its own significance and sub-Riemannian problem on SH(2) is important in the entire program of study of three dimensional Lie groups. In this paper we have obtained the complete parametrization of extremal trajectories in terms of Jacobi elliptic functions and described the nature of projections of extremal trajectories on xy-plane. This shall pave the way for further work on the goals highlighted in Section 7 and is target of our future work on SH(2).

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