

Geometry of the Variety of Real Symmetric Matrices with Multiple Eigenvalues

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Abstract

We investigate the manifold \mathcal{M} of real symmetric $n \times n$ matrices having a multiple eigenvalue. We present an algorithm to derive a minimal-degree equation system for \mathcal{M} , and give its result equations for $n = 3$. We prove that 1) \mathcal{M} is prime and has co-dimension 2, 2) each matrix in \mathcal{M} having $n - 1$ of different eigenvalues is a regular point on the surface \mathcal{M} , 3) in the case $n = 3$, the set of singular points on \mathcal{M} is the set of scalar matrices. We give a geometric description of \mathcal{M} in a neighborhood of each regular point: a fibration over a plane with the fiber being an orbit by conjugations by $SO(n)$. For $n = 3$, \mathcal{M} is also described as the straight cylinder over \mathcal{M}_0 , where \mathcal{M}_0 is the cone over a diffeomorphic image of torus. These results simplify, generalize and complete the results given in some previous works on this subject.

Keywords: symmetric matrix, orthogonal change, discriminant, dimension of manifold, primality of manifold, singular point.

1 Introduction

Disclaimer

- 1) So far (on July 2009) nobody checked this paper for mathematical errors.
- 2) This paper is translated from Russian by the author, and the author is not a native English speaker. We hope, the mathematical contents of the below text is still understandable.

About the word “fibration”

- We use it in the two meanings: 1) fibration over a plane considered modulo isometry,
- 2) fibration over a plane considered modulo diffeomorphism.

Let us denote:

$\text{Sym}(n) = \text{Sym}(n, \mathbb{R})$ is the space of symmetric matrices of size $n \times n$ over the field \mathbb{R} of real numbers ($\dim \text{Sym}(n) = n(n + 1)/2$),

$\text{charPol}(X)(\lambda)$ is the characteristic polynomial of a matrix X of $\text{Sym}(n)$,

$\mathcal{M} = \{X \in \text{Sym}(n) \mid \text{discr } X = 0\}$ is the surface in $\text{Sym}(n)$ defined by discriminant of $\text{charPol } X$. In other words: \mathcal{M} is the variety (manifold, surface) of all matrices in $\text{Sym}(n)$ that have any multiple eigenvalue.

The problem is: to describe the geometry of the algebraic variety \mathcal{M} .

About generality: our conclusions are valid for each dimension n . But for $n < 3$, \mathcal{M} is the line of scalar matrices, and all the below constructs occur trivial for this case. Therefore we consider below only the case of $n \geq 3$.

It is known that each matrix in $\text{Sym}(n, \mathbb{R})$ has n real eigenvalues, some of which coincide.

We call the *multi-set of eigenvalues* the set of eigenvalues for a matrix in $\text{Sym}(n)$ — together with their multiplicities.

The surface \mathcal{M} is given by a single equation $\text{discr}(X) = 0$, where the polynomial $\text{discr}(X)$ has the matrix elements $x_{i,j}$ as variables, has a large degree and many monomials — even for $n = 3$. A knowledge about the surface \mathcal{M} has an important application, especially for the case of $n = 3$. Therefore there were done several investigations for finding various equation systems for \mathcal{M} . Thus, for $n = 3$, it occurs [Il1] that $\text{discr}(X)$ is the sum of squares of several simpler polynomials. This explains why the co-dimension of \mathcal{M} may be greater than one.

In the papers [Ik, D:I]

- 1) it is written that Wigner and von Neumann considered the variety \mathcal{M} (for $n = 3$) and have provided certain in-formal reasons of why its co-dimension should be two,
- 2) it is written that in [Ik] there is proved (for $n = 3$) that this co-dimension is not less than two, and this proof uses the result of the paper [Il1] about decomposing discriminant into a sum of squares,
- 3) there are given (in [Ik, D:I]) the reasons of why in the case of $n = 3$ the dimension of \mathcal{M} is 4, and also there is derived certain conclusion about irreducible components of \mathcal{M} .

And these considerations include various transformations with explicit equation systems for \mathcal{M} .

In this our paper we also deal with the case of arbitrary n , and apply a different approach: the classical method with a linear Lie group of symmetries.

In brief, our approach is as follows. All symmetric matrices are produced by conjugating diagonal matrices with operators from $SO(n)$. The motion in the plane of diagonal matrices is orthogonal to the motion along this conjugation orbit. This provides a smooth parameterization for \mathcal{M} , except certain particular points. And it remains to describe the orbit for a diagonal matrix, with considering the two cases for the number of its different eigenvalues: $n - 1$ or less than $n - 1$.

As to finding for \mathcal{M} of a minimal (in total degree) system, this problem is solved by a simple and generic tool: combining of the Gröbner algorithm for a polynomial ideal with diagonalization by the action of the $O(n)$ group.

The results of this paper are as follows.

- We simplify the conclusions and discourse.
- We prove that \mathcal{M} is prime and has co-dimension is 2 (Sections 2, 5, 5.4) (the decomposition to primes suggested in [D:I] is erroneous).

- We prove that the matrices in \mathcal{M} having the number $n-1$ of different eigenvalues are regular points on \mathcal{M} (Section 5.4).
For $n = 3$ we prove that scalar matrices and only them are singular points on \mathcal{M} (Section 5.6).
- We describe the global structure of the variety \mathcal{M} in the $n(n+1)/2$ – dimensional space. For a neighborhood of a matrix of maximal spectrum, we describe a diffeomorphic parameterization as a fibration over a plane.
For $n = 3$, we also provide a more definite description for \mathcal{M} : the straight cylinder over \mathcal{M}_0 , where \mathcal{M}_0 is the cone over the so-called “d-torus” (Sections 4, 5.6).
- We give a simple algorithm for deriving a minimal (in total degree) equation system for \mathcal{M} , — and also for any separate orbit, — and provide its output for the case of $n = 3$ (Sections 3, 5.1).

So far, we leave un-solved the following problems.

- 1) Find whether it is true for $n > 3$ that each matrix in \mathcal{M} having narrowed spectrum is a singular point.
- 2) For $n = 4$, find minimal-degree equations for \mathcal{M} (the algorithm given below is rather expensive in computation for $n > 3$).
- 3) The structure of the surface \mathcal{M} depends mainly on the structure of an orbit $O(A, n)$ for a matrix A of maximal spectrum in \mathcal{M} . Similarly as for $n = 3$, it would have sense to find for $n = 4, 5$ some global description for $O(A, n)$, more definite than the one that we give in the sequel.

To reduce the main text volume, some of not so interesting proofs in this paper are moved to Application, and the main part has a few references to Application.

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2 Several definitions and preliminary constructs

1. Dg denotes the space of diagonal matrices (it has dimension n),
 $Scal$ denotes the line of scalar matrices. \mathcal{M} is a conic set containing $Scal$.
2. For a diagonal matrix D , denote $D(i) = D(i, i)$.
 $diag(a_1, \dots, a_n)$ denotes the diagonal matrix D having $D(1) = a_1, \dots, D(n) = a_n$.
A diagonal-ordered matrix is a diagonal matrix D in which $D(1) \leq \dots \leq D(n)$.
3. $O(n)$ is (an algebraic and linear) Lie group of orthogonal operators in \mathbb{R}^n .
 $SO(n)$ is (irreducible and connected) Lie subgroup of rotations in $O(n)$ (determinant = 1), it has dimension $\dim SO(n) = n(n-1)/2$.
4. $X \rightarrow g \cdot X \cdot g^{-1}$ is the action of the coordinate (basis) *change* by an orthogonal operator g in a matrix (operator) X .

We shall shortly call this conjugation action “change by g ”, “changes by operators from $SO(3)$ ”, “changes from $SO(3)$ ”, and such.

And for an orthogonal operator g , there holds $g^{-1} = g^*$ (transposed matrix), and the expression $g \cdot X \cdot g^*$ represents the basis change in a symmetric bilinear form X .

For g in $O(n)$, we denote the action of the basis change as

$$g^c X = g^c(X) = g \cdot X \cdot g^*.$$

Further: g^{-c} denotes $(g^{-1})^c$, and $l^\phi = T(l, \phi)^c$ denotes (for $n = 3$) the operator of the change by the rotation operator $T(l, \phi)$ for the axis l and angle ϕ .

In the case of $n = 3$, we call 1-orbit of a matrix M by the axis l the orbit of M under the changes by the rotation operators around l .

5. Denote \mathcal{MD} the (conic) set of diagonal matrices in \mathcal{M} ;
for $1 \leq i < j \leq n$ denote $\Pi_{i,j}$ the plane in Dg defined by the equation $D(i) = D(j)$. Evidently, \mathcal{MD} is union of the number C_n^2 of $(n - 1)$ -dimensional planes $\Pi_{i,j}$. For example, for $n = 3$, $\Pi = \Pi_{1,2}$ is all the matrices of the kind $\text{diag}(\lambda, \lambda, \mu)$, while $\Pi_{1,3}$ and $\Pi_{2,3}$ are obtained from Π by permutational changes from $SO(3)$.

6. Below, the word “orbit” means: the orbit of a matrix in $\text{Sym}(n)$ under the action of changes by the operators from $SO(n)$; $\text{Orbit}(A)$ denotes the orbit of a matrix A .

7. Fix an orthonormal basis $\text{Bas} = \{e_1, \dots, e_n\}$ in \mathbb{R}^n , and let us consider the matrices in $\text{Sym}(n)$ as representations in Bas of symmetric bilinear forms. We also speak of operators from $SO(n)$ given by permutations on the set Bas , or, for example (in the case of $n = 3$), being rotations around the axis e_i .

8. $\text{trace}(X)$ for a matrix X is the sum of the elements on its main diagonal.

Generally, our simple discourse bases on 1) known facts of the linear Lie group theory, 2) classical

Theorem (DO): for any real symmetric bilinear form there exists an orthogonal change which brings this form (matrix) to a diagonal matrix

(see, for example, [VW], paragraph 90).

Lemma. For such a diagonalization there are sufficient the operators from $SO(n)$.

Indeed, if g^c diagonalizes X , and $\det g = -1$, then the composition g_2 of g with the permutational operator for (e_1, e_2) belongs to $SO(n)$, and g_2^c diagonalizes X .

Several more definitions

1) The stabilizer $\text{St}(A)$ of a matrix A from $\text{Sym}(n)$ is the set of operators g , which changes preserve A . It is a smooth subgroup.

2) The width of the spectrum for a matrix from $\text{Sym}(n)$ is the number of its different eigenvalues.

For a set \mathcal{X} in $\text{Sym}(n)$, any matrix from \mathcal{X} having maximal in \mathcal{X} spectrum width we call a *matrix of maximal spectrum* (in \mathcal{X}),

and all the rest are called the matrices of *narrowed spectrum*.

3) Denote \mathcal{MH} the set of matrices of maximal spectrum in \mathcal{M} — that is having the spectrum width $n - 1$. This is a topologically open set in \mathcal{M} .

4) Denote $\Pi_{i,j}^h = \Pi_{i,j} \cap \mathcal{MH}$.

The following three statements are evident.

(1) The spectrum width of a matrix in \mathcal{M} is the spectrum width of its diagonal form.

(2) For a matrix in $\text{Sym}(n)$ having all different eigenvalues, the orbit dimension is $\dim \text{SO}(n)$,

matrices of maximal spectrum in \mathcal{M} have orbits of maximal (for \mathcal{M}) dimension, equal to $\dim\text{SO}(n) - 1$.

(3) If a matrix in $\text{Sym}(n)$ has the spectrum width $n - 1$, then its stabilizer is a single-parameter subgroup of rotations of the plane $L(u, v)$ — for each two non-collinear vectors u and v belonging to the same multiple eigenvalue.

Example: for $n = 3$, the diagonal matrices in \mathcal{MH} are exactly the matrices of kind $\text{diag}(\lambda, \lambda, \mu)$ with $\lambda \neq \mu$, and also the two families obtained from this one by permutations on the main diagonal.

s-metric

Define an Euclidean metric on the space $\text{Sym}(n)$: sum of squares of the elements of a matrix X . We denote this quadratic form $\text{sQuad}(X)$ and call it s-metric.

In this paper, the expressions “distance”, “angle”, “orthogonality”, “isometry”, “orthogonal projection”, “circumference”, “sphere”, “circumference center”, “radius”, “surface diameter”, “straight cylinder” applied to points and subsets in $\text{Sym}(n)$ — are understood in the sense of s-metric.

Lemma S (probably, known).

For any matrix X from $L(n)$ and any orthogonal operator g , it holds the equation $\text{sQuad}(X) = \text{sQuad}(g \cdot X) = \text{sQuad}(g^c X)$.

That is: the changes from $O(n)$ preserve s-distance and s-angles in $L(n)$.

Proof. An orthogonal operator preserves s-square for each column-vector in X . Hence, $\text{sQuad}(gX) = \text{sQuad}(X)$. Similarly, the right-hand side multiplication by g^* preserves s-square of each row. Therefore $\text{sQuad}(g^c X) = \text{sQuad}(X)$.

Lemma 1.

- (1) Any change from $SO(n)$ maps \mathcal{M} on itself.
- (2) Scalar matrices are the only fixed points of the action of $SO(n)$ on $\text{Sym}(n)$.
- (3) \mathcal{M} is the union of mutually non-intersecting orbits.
- (4) Each diagonal matrix in \mathcal{M} is brought by some change from $SO(n)$ to an ordered diagonal. (that is all the results for permutations applied to the main diagonal of a matrix in \mathcal{MD} are represented by some changes from $SO(n)$).
- (5) Two matrices in \mathcal{M} belong to the same orbit if and only if they have the same eigenvalue multi-set.

(6) For each matrix in \mathcal{M} , its orbit has exactly one diagonal-ordered matrix.

(7) 7.1. For each real number t and symmetric matrix M , it holds $\text{Orbit}(t \cdot M) = t \cdot \text{Orbit}(M)$.

7.2. For $n = 3$, orbits of any two matrices in \mathcal{MH} differ from each other in a shift at some scalar matrix and a homothety by some non-zero factor (and we use that this transformation preserves angles).

(8) 8.1. Discriminant of a symmetric matrix does not change with adding to this matrix of any scalar matrix. Any shift by a scalar matrix maps the surface \mathcal{M} on itself.

8.2. For each real number s , intersection \mathcal{M}_s of \mathcal{M} with the hyper-plane $\text{trace}(X) = 0$ is a surface mapped on itself by the changes from $SO(n)$.

\mathcal{M} is a straight cylinder over \mathcal{M}_0 having the line Scal as element.

Proof. The statements (1), (2) and (3) are known and evident.

Let us prove (4). Let the searched permutation on the main diagonal is presented by an orthogonal operator g . D has at least two equal elements on the main diagonal. Hence, the permutational operator g_2 for the corresponding basis vector pair has determinant -1 . Therefore, either g belongs to $SO(n)$ or the composition $g \cdot g_2$ belongs to $SO(n)$ and its change brings D to a diagonal-ordered form.

Let us prove the statements (5) and (6). The change action preserves the coefficients of the characteristic polynomial. Hence, it remains only to prove the second part of the statement (5). Let A and B have the same eigenvalue multi-set. By Theorem DO and its additional lemma, A and B are diagonalized by some changes from $SO(n)$. By the statement (4), they are brought further to a diagonal-ordered matrices. As these diagonal-ordered matrices represent the same multi-set, they are equal.

This also proves the statement (6).

Let us prove the statement (7). The statement (7.1) is evident. As to (7.2), it is, evidently, sufficient to prove it for a pair of diagonal matrices in \mathcal{MH} . And hence, it is sufficient to prove it for a pair of matrices (D, D_2) , where $D = \text{diag}(0, 0, 1)$, and D_2 is any matrix of kind $\text{diag}(\lambda, \lambda, \mu)$ with $\lambda \neq \mu$.

D_2 is obtained from D by adding of some scalar matrix $\text{Sc} = \text{diag}(a, a, a)$ and by homothety by some non-zero number factor b : $(D + \text{Sc}) \cdot b = D_2$. For such representation, it is sufficient to put $a = \lambda/(\mu - \lambda)$, $b = \mu - \lambda$. Each change operator g^c is an isomorphism on the algebra of square matrices. In particular, for each square matrix M there hold the equations

$$g^c(M + \text{Sc}) = (g^c M) + (g^c \text{Sc}) = (g^c M) + \text{Sc}, \quad g^c(b \cdot M) = b \cdot g^c M.$$

Therefore with adding of a scalar matrix Sc to any symmetric matrix, the orbit shifts at the vector Sc , and with multiplying M by a coefficient b , the orbit is homothetically multiplied by b .

Let us prove the statement (8). 8.1: Adding a scalar operator $\mu \cdot E$ to a symmetric operator M shifts the spectrum of M at the number μ . This preserves discriminant, because the discriminant depends only on the differences of eigenvalues.

8.2: For each real number s , denote Π_s^* the (five-dimensional) plane in $\text{Sym}(3)$, defined by the equation $\text{trace}(X) = s$. Denote $\mathcal{M}_s = \mathcal{M} \cap \Pi_s^*$. Any basis change operator preserves the matrix trace. Therefore each such plane is mapped on itself by the changes from $SO(n)$ — as well as each (three-dimensional) restriction \mathcal{M}_s .

Further, the surface \mathcal{M} is union of the restrictions \mathcal{M}_s for all s . Adding of any scalar matrix $s \cdot E$ shifts the surface \mathcal{M} along itself — because (by the statement (8.1)) this adding preserves discriminant. This shift maps the plane Π_0^* on the plane Π_s^* , and maps the restriction \mathcal{M}_0 on \mathcal{M}_s . Hence, \mathcal{M} is the cylinder over \mathcal{M}_0 having the line Scal as element. The line Scal is orthogonal to the plane Π_0^* , as one can see from the s-product of a scalar matrix by a matrix of zero trace. Hence, this cylinder is straight.

The lemma is proved.

Lemma A0 (probably, known).

Commutator of a diagonal and an anti-symmetric square matrices (of the same size) is a symmetric matrix having zero diagonal.

Lemma A1 (known). The counter-image \mathcal{M}' of an algebraic set \mathcal{M} for a polynomial map $F : R^n \rightarrow R^m$ is an algebraic set.

Lemma A2 (known). For a surjective polynomial map from an algebraic set \mathcal{M} onto an algebraic set \mathcal{M}' , if \mathcal{M} is prime, then \mathcal{M}' is prime.

Simple proofs for the Lemmata A0, A1, A2 are given in Application.

Theorem “Primality”.

The discriminant surface \mathcal{M} is algebraically prime (that is it is not a union of any two algebraic sets which do not contain each other).

Proof. Consider the map $DO : \Pi \times SO(n) \rightarrow \mathcal{M}$,

$$DO((\lambda, \mu_1, \dots, \mu_{n-2}), g) = g^c \text{diag}(\lambda, \lambda, \mu_1, \dots, \mu_{n-2})$$

(no restriction for the numbers in the first argument). This is a surjective polynomial map of algebraic surfaces, where the support domain has one dimension more than the image \mathcal{M} . And it is surjective. The algebraic surface $\Pi \times SO(n)$ is prime. Hence, by Lemma A2, \mathcal{M} is prime.

To be fully rigorous, we need to ensure that the image of DO is \mathcal{M} . By Theorem (DO) and its additional lemma, replacing Π with \mathcal{MD} makes this map surjective. By Lemma 1 (4), all \mathcal{MD} is obtained from Π by changes from $SO(n)$.

The theorem is proved.

3 General algorithm for minimal–degree equations for \mathcal{M}

Initially, \mathcal{M} is defined by a single homogeneous form discr of degree $2n$. And it is natural to search for a smaller–degree equation system for this surface. There exists a recent work by N. V. Ilyushechkin which represents for the case of $n = 3$ this discriminant as a sum of *squares of four cubic forms*, and displays these forms explicitly. Denote this polynomial set $\text{IEs}(3)$. Evidently, for $n = 3$, \mathcal{M} is defined by the system $\text{IEs}(3)$.

Now, we formulate a generic algorithm for a complete equation system search for the discriminant surface.

The surface \mathcal{M} is defined by the ideal $I(\mathcal{M})$ of all polynomials in the variables X_s which are zero on \mathcal{M} . Let us call this ideal a *complete system* for \mathcal{M} .

Fix the *grading by the total degree* on this polynomial algebra. There is known the algorithm of the *Gröbner basis* [Bu]. Given any finite basis for an ideal in a polynomial ring (over an Euclidean coefficient ring), this algorithm produces a finite basis for this ideal minimal by the given grading, and, in a certain sense, canonical. In our case, denote this Gröbner basis gs . The property of the method is so that if the ideal $I(\mathcal{M})$ contains any polynomial of total degree less than d , then gs contains a polynomial of total degree less than d . Therefore, the problem of defining of the surface \mathcal{M} by equations of possibly small degree has its generic and algorithmic solution — if only there is found any finite basis for a complete system for the surface. For example, $\{\text{discr}\}$ is not a complete basis in our case, as one can see from the cubic forms $\text{IEs}(3)$.

In general, our method is as follows. If a map $F : \mathbb{R}^m \rightarrow \mathcal{M}$ is surjective, then this defines the maps $X(i, j)$ from \mathbb{R}^m for each position (i, j) in the matrix X . Then, the ideal Rel of all the algebraic relations between the maps $\{X(i, j) | 1 \leq i \leq j \leq n\}$ is just a complete equation system for \mathcal{M} . It remains to formulate the case when a

parameterization F (not necessary injective, or smooth) allows an algorithm for finding of a finite basis for Rel.

In our case, we provide as F a certain *polynomial parameterization*. Namely, build a diagonal matrix

$$D = \text{diag}(\lambda, \lambda, \mu_1, \dots, \mu_{n-2})$$

and consider its diagonal elements as *variables* for polynomials with coefficients in \mathbb{R} . Denote Eigs the set of these variables. Further, let the matrix Y of size $n \times n$ consist of the elements

$$Ys = \{y_{i,j} \mid 1 \leq i, j \leq n\}.$$

We denote the variable set: $Zs = Ys \cup \text{Eigs}$, and we shall say that the variable set consists of n^2 of o-variables and $n - 1$ of eg-variables. Consider the matrix product $X = Y \cdot D \cdot Y^*$, where the factors are considered as matrices over $\mathbb{R}[Zs]$.

Evidently, a *generic diagonal matrix* of \mathcal{M} is presented by the above D and also by the matrices made from D by permutations on the main diagonal. By Theorem DO, each matrix in \mathcal{M} is diagonalized by some orthogonal change. This gives us the following generic matrix for \mathcal{M} :

$$X = Y \cdot D \cdot Y^*.$$

Here the elements of these matrices are considered as elements of the quotient-ring QZ of the algebra $\mathbb{R}[Zs]$ by the orthogonality conditions for the operator Y . These conditions form the set

$$\text{OrtEs} = \{\text{row}(i, Y) \cdot \text{row}(j, Y) = \delta(i, j) \mid 1 \leq i \leq j \leq n\}$$

— number $n(n + 1)/2$ of quadratic equations. Here the expression $\text{row}(i, Y)$ denotes taking the row No i from a matrix, and the symbol “.” denotes scalar product of two rows.

The above generic matrix X uses only one matrix from the set of diagonal-permutation results for D , because all the diagonal permutation results in D are expressed by changes from $SO(n)$.

The generic matrix X (over QZ) is symmetric. The elements $X(i, j)$ (conjugation classes) have representatives as polynomials in $\mathbb{R}[Zs]$. Now we see that the ideal Rel of algebraic relations for $X(i, j)$ is a complete equation system for \mathcal{M} (this ideal consists of polynomials in $\mathbb{R}[Xs]$ in the variables $Xs = \{x_{i,j} \mid 1 \leq i \leq j \leq n\}$).

Further, there is known a simple algorithm for finding of a canonical basis Rels for the ideal Rel. It is described in [GTZ] (Corollary 3.2), and it bases, in its turn, on the Gröbner basis method. This algorithm produces such a basis for the ideal of algebraic relations that is minimal in the grading and canonical. For example: if there exists a polynomial of total degree less than three which is zero on the surface \mathcal{M} , then the returned basis Rels also must have such a polynomial.

About the coefficient domain The problem formulation mentions the surface in a real space and its equations over the real number field. And even for a matrix in \mathcal{M} having rational elements, its eigenvalues are, generally, algebraic numbers.

Nevertheless: discriminant in our problem has integer coefficients, for natural reasons, the obtained system for \mathcal{M} has integer coefficients, and all the intermediate computations need only to operate with polynomials having rational coefficients.

About computer calculation

The algorithm presented above is important independently on a computer device. For example, it can be evaluated by hand for each given value of n (though, it is difficult to compute for $n > 3$). On the other hand, computer evaluation is also interesting and helpful. We applied for this our common system for computer algebra called DoCon.

Case $n = 3$

For this argument, we have obtained the result of the above algorithm. It also can (with some effort) be evaluated by hand, though we skip this exercise. The result is the system Rels for the surface \mathcal{M} containing seven homogeneous cubic equations and one more equation of degree four.

Note: Rels does not contain any polynomial of total degree less than three. Hence, the surface \mathcal{M} is essentially non-quadratic. This is due to the following property of a Gröbner basis: if a non-zero polynomial is zero when restricted to \mathcal{M} (belongs to $I(\mathcal{M})$), then its leading monomial is a multiple of leading monomial of some polynomial in the basis Rels .

The relation of the system Rels to the equations $\text{IEs}(3)$ is that

- 1) the ideal $I(\text{Rels})$ contains $\text{IEs}(3)$,
- 2) square of each polynomial of Rels belongs to $I(\text{IEs}(3))$.

This membership relation is checked by a certain known simple algorithm. Further, if Rels has any polynomial that has some power belonging to the ideal of the rest of the system, then this polynomial can be removed. And let us repeat this simplification while such a polynomial is found in the current system. There exists a certain simple algorithm which detects the above relation between a polynomial and a finite set of polynomials. This process produces a reduced system defining the same surface. In our example, it shows, again, four homogeneous forms (it prints the left-hand sides of equations, while the right-hand sides are zero):

$\text{RelsS} =$

$$\begin{aligned} &\{x^{12}x^{23} - x^{12}x^{13}x^{22} + x^{12}x^{13}x^{33} - x^{13}x^{23}, \\ &x^{12}x^{23}x^{11} - x^{12}x^{23}x^{22} - x^{13}x^3 + x^{13}x^{23}x^2 - x^{13}x^{11}x^{22} + x^{13}x^{11}x^{33} \\ &\quad + x^{13}x^{22}x^2 - x^{13}x^{22}x^{33}, \\ &x^{12}x^{13}x^{11} - x^{12}x^{13}x^{22} - x^{13}x^{23}x^{23} + x^{23}x^3 - x^{23}x^{11}x^2 + x^{23}x^{11}x^{22} \\ &\quad + x^{23}x^{11}x^{33} - x^{23}x^{22}x^{33}, \\ &x^{12}x^{23}x^{11} - x^{12}x^{23}x^{22} - x^{13}x^{23}x^{11} + x^{13}x^{23}x^{33} + x^{23}x^{23}x^{22} - x^{23}x^{23}x^{33} - x^{11}x^{23}x^{22} \\ &\quad + x^{11}x^{23}x^{33} + x^{11}x^{22}x^2 - x^{11}x^{33}x^2 - x^{22}x^{23}x^{33} + x^{22}x^{33}x^2, \\ &x^{12}x^3 - x^{12}x^{23}x^2 - x^{12}x^{11}x^{22} + x^{12}x^{11}x^{33} + x^{12}x^{22}x^{33} - x^{12}x^{33}x^2 \\ &\quad - x^{13}x^{23}x^{11} + x^{13}x^{23}x^{33}\} \end{aligned}$$

Conclusion We

- 1) present a generic algorithm for minimal-degree equations for \mathcal{M} ,
- 2) give its computer result (Rels , RelsS) for $n = 3$,
- 3) use the equations $\text{IEs}(3)$ (derived manually and proved by N. V. Ilyushechkin) to verify our result for $n = 3$.

Equations for \mathcal{M}_0

By Lemma 1, \mathcal{M} is a straight cylinder over \mathcal{M}_0 . Hence, it is sufficient to describe the surface \mathcal{M}_0 — the restriction to the plane $x^{11} + x^{22} + x^{33} = 0$. x^{11} is eliminated, and this results into the system

M0eqs =

$$\{x_{12}x_{23}x_{22} + (1/2)x_{12}x_{23}x_{33} + (1/2)x_{13}^3 - (1/2)x_{13}x_{23}^2 - x_{13}x_{22}^2 \\ + (1/2)x_{13}x_{22}x_{33} + (1/2)x_{13}x_{33}^2, \\ x_{12}^2x_{22} + (1/2)x_{12}^2x_{33} - (1/2)x_{13}^2x_{22} - x_{13}^2x_{33} - (1/2)x_{23}^2x_{22} \\ + (1/2)x_{23}^2x_{33} + x_{22}^3 + (3/2)x_{22}^2x_{33} - (3/2)x_{22}x_{33}^2 - x_{33}^3, \\ x_{12}^2x_{23} + (3/2)x_{12}x_{13}x_{33} - (1/2)x_{13}^2x_{23} - (1/2)x_{23}^3 + x_{23}x_{22}^2 \\ + (5/2)x_{23}x_{22}x_{33} + x_{23}x_{33}^2, \\ x_{12}^3 - x_{12}x_{23}^2 + x_{12}x_{22}^2 + x_{12}x_{22}x_{33} - 2x_{12}x_{33}^2 + x_{13}x_{23}x_{22} + 2x_{13}x_{23}x_{33}\}$$

— again, four cubic forms, but the variable x_{11} is eliminated.

Checking computer evaluation

How reliable is computer evaluation for this task? As a rule, large programs have errors, the same is with the very electronic schemes of computers.

In our example, there are possible, at least, the following relatively easy checks.

1. Discriminant must reduce to zero by the basis Rels (belong to $I(\text{Rels})$).

Each member of Rels has some degree which is a multiple of discriminant (we expect, square, or degree four is sufficient).

2. Substituting into Rels of the values $x_{i,j}$ for the matrix $D = \text{diag}(\lambda, \lambda, \mu_1, \dots, \mu_{n-2})$ must produce zero polynomial in the variables $\lambda, \mu_1, \dots, \mu_{n-2}$ — this is easy to check.

3. The matrix X (over the ring QZ) must be symmetric,

its trace must be $2\lambda + \mu_1 + \dots + \mu_{n-2}$,

its determinant must be $\lambda^2 \cdot \mu_1 \cdot \dots \cdot \mu_{n-2}$.

4 Geometry of discriminant surface in $\text{Sym}(3)$

In the case of $n = 3$, we have a more definite description of this surface.

(DS3) of the structure of the discriminant surface \mathcal{M} in $\text{Sym}(3, \mathbb{R})$.

(1) \mathcal{M} is a *prime* algebraic variety of co-dimension 2.

Its singular points are scalar matrices and only them.

\mathcal{M} is defined by four cubic forms, and there does not exist a non-zero polynomial of total degree less than three which is zero on \mathcal{M} .

\mathcal{M} is union of restrictions \mathcal{M}_s , where the restriction for s is expressed by the equation $\text{trace}(X) = s$.

\mathcal{M} is the straight cylinder over \mathcal{M}_0 having the line parallel to Scal as element.

Therefore, it is sufficient to describe the surface \mathcal{M}_0 — and its description is as follows.

(2) A three-dimensional surface \mathcal{M}_0 in a five-dimensional space has the following structure.

2.1. \mathcal{M}_0 is the cone, with zero as vertex, over the d-torus; the d-torus is the orbit of the diagonal matrix $\text{diag}(1, 1, -2)$; this two-dimensional orbit resides on the four-dimensional sphere having center in zero.

2.2. The orbit of each non-scalar matrix in \mathcal{M} is a two-dimensional and algebraically-quadratic surface residing on a four-dimensional sphere; it is diffeomorphic to torus as a manifold defined by two charts. We call this surface *d-torus*.

The orbit orthogonally intersects the plane Dg of diagonal matrices, and it has exactly three common points with Dg .

It is not contained in any four-dimensional plane.

Also this orbit is defined by four quadratic equations.

2.3. The diameter of the orbit of any non-scalar matrix in \mathcal{M} having eigenvalues λ and μ is $|\lambda - \mu|$.

(3) The map $DO : \Pi \times SO(3) \rightarrow \mathcal{M}$, $DO((\lambda, \mu), g) = g^c \text{diag}(\lambda, \lambda, \mu)$ (with no restrictions on λ and μ) is a surjective polynomial map of algebraic surfaces — from five-dimensional one onto four-dimensional.

(4) A natural smooth parameterization for the surface $\mathcal{M}_0 \setminus \{0\}$ can be presented as a direct product of the motion along the element line of this cone by smooth parameterization for the orbit of the intersection point of this line with the sphere.

The orbit of each matrix in \mathcal{M} is obtained from the orbit of $D = \text{diag}(1, 1, -2)$ by shift at some scalar matrix and by multiplying by some real coefficient.

(5) Intersection of \mathcal{M}_0 with the plane of diagonal matrixes consists of the three different lines intersecting in zero. These lines have the following parameterization: $\text{diag}(\lambda, \lambda, -2\lambda)$, $\text{diag}(\lambda, -2\lambda, \lambda)$, $\text{diag}(-2\lambda, \lambda, \lambda)$.

Half of the proof for this theorem is given by the general theorem of the next Section.

5 Geometry of discriminant surface in $\text{Sym}(n)$

Below, Π denotes the plane $\Pi_{1,2}$ of dimension $n - 1$ in the surface \mathcal{MD} .

Theorem (DS) on the structure of discriminant surface \mathcal{M} in $\text{Sym}(n, \mathbb{R})$.

(1) \mathcal{M} is a *prime* algebraic variety of co-dimension 2.

\mathcal{M} is union of the restrictions \mathcal{M}_s , where the restriction for s is defined by the equation $\text{trace}(X) = s$.

\mathcal{M} is the straight cylinder over \mathcal{M}_0 having a line parallel to Scal as element.

Matrices in \mathcal{M} having $n - 1$ different eigenvalues are regular points on \mathcal{M} . These matrices form an open and everywhere dense set \mathcal{MH} in \mathcal{M} .

We give a simple algorithm (expensive to perform for $n > 3$) for finding for each given n of polynomial equations for \mathcal{M} having minimal total degree.

(2) Orbits for matrices in \mathcal{M} are classified by the eigenvalue multi-set, or — by a unique diagonal-ordered matrix in the orbit.

(3) \mathcal{MH} has the following structure.

3.1. \mathcal{MH} is a smooth fibration over the $(n - 1)$ -dimensional plane Π^h , with the fiber being an orbit — smooth compact surface of dimension $\dim \text{SO}(n) - 1$.

Each orbit in \mathcal{MH} orthogonally intersects the plane Dg of diagonal matrices and has exactly one common point with each sub-plane in Dg , conjugated with Π by changes from $SO(n)$ (there are number of C_n^2 of such sub-planes).

Orbit of a scalar matrix is a single point.

3.2. Orbit of each matrix in \mathcal{MH} is a smooth, compact, algebraic, connected surface of dimension $\dim \text{SO}(n) - 1 = (n(n - 1)/2) - 1$. It resides at the sphere in the hyper-plane orthogonal to the line of scalar matrices. We give a simple algorithm (expensive to perform for $n > 3$) deriving from the given eigenvalues of a matrix A in \mathcal{MH} equations for the orbit of A having minimal total degree.

3.3. Diameter dm of the orbit of a matrix in \mathcal{MH} having the eigenvalue multi-set $\{\lambda_1, \dots, \lambda_n\}$, is bounded as $\max_{i,j} |\lambda_i - \lambda_j| \leq \text{dm} \leq 2d(D, \text{Scal})$, where $d(D, \text{Scal})$ is the distance from the matrix $D = \text{diag}(\lambda_1, \dots, \lambda_n)$ to the line of scalar matrices.

(4) The map $DO : \Pi \times SO(n) \rightarrow \mathcal{M}$,

$$DO((\lambda, \mu_1, \dots, \mu_{n-2}), g) = g^c \text{diag}(\lambda, \lambda, \mu_1, \dots, \mu_{n-2})$$

(no restriction for the numbers in the first argument) is a surjective polynomial map of algebraic surfaces, where the support domain has one dimension more than the image \mathcal{M} .

(5) A smooth atlas on \mathcal{MH} can be naturally built as follows.

From each point $D = \text{diag}(\lambda, \lambda, \mu_1, \dots, \mu_{n-2})$ in \mathcal{MH} there flows out the orbit $\text{Orbit}(D)$ of dimension $\dim SO(n) - 1$. A smooth parameterization for this orbit in the neighborhood of D is given by “quotientation” of the action of $SO(n)$. Namely, the stabilizer $\text{St}(D)$ is a single-parameter subgroup of rotations of the bi-dimensional plane $L(e_1, e_2)$. This orbit is locally diffeomorphic to the homogeneous space SO' of conjugation classes by $\text{St}(D)$.

A smooth parameterization for a neighborhood of each point of this orbit is by translation to this neighborhood by some change from $SO(n)$ of the parameterization near D . A total smooth parameterization for \mathcal{MH} is the product of the parameterization for the orbit of D and motion along a certain plane Π in \mathcal{MD} .

Now, let us proceed with the proof.

First, recall that the statement (4) and the primality statement are proved earlier, in Theorem “Primality”.

5.1 Investigating an orbit by orthogonal changes

Lemma 2.

(1) If A is a non-scalar matrix in $\text{Sym}(n)$, then its orbit intersects orthogonally the plane of diagonal matrices.

(2) Orbit of any matrix in $\text{Sym}(n)$ resides in the hyper-plane orthogonal to the line of scalar matrices.

(3) Orbit of any matrix in \mathcal{MH} is a smooth, compact, algebraic, connected surface of dimension $\dim SO(n) - 1 = (n(n-1)/2) - 1$.

Dimension of an orbit for a matrix A in \mathcal{M} is 0 for a scalar A , $\dim SO(n) - 1$ — for a matrix of maximal spectrum, a number $0 < d < \dim SO(n) - 1$ in other cases.

Proof.

(1): Due to the symmetry by the group action and due to diagonalization, it is sufficient to prove this statement for an orbit of a diagonal matrix D . Also it is sufficient to prove the statement (A): each tangent vector for the orbit of D in D is some matrix M having zero diagonal. Then, by definition of the quadratic form $sQuad$, it will follow that $sQuad(D_1, M) = 0$ for each diagonal matrix D_1 .

Now, prove (A). The tangent space T to the orbit of D in D consists of all the commutators of D with operators from the Lie algebra $so(n)$, and this algebra consists of anti-symmetric matrices. Therefore, by Lemma A0, each matrix in T has zero diagonal. The statement (1) is proved.

(2): suppose the matrix A in $\text{Sym}(n)$ is not scalar, and Π^* is the hyper-plane orthogonal to the line Scal and containing A . Let A_0 be the base of this projection of A . By Lemma S, orthogonal changes preserve angles and distances in $\text{Sym}(n)$. Each point in Scal is a fixed point with respect to these changes. Therefore Π^* is mapped by these changes onto itself, the orbit of A resides in Π^* , and its points are equally-distant from A_0 .

(3): orbit of each matrix A in \mathcal{M} is an algebraic surface. Because let $\text{EV}(A)$ be the eigenvalue multi-set for A . By Lemma 1 of Section 2, the orbit for A consists exactly of the matrices in \mathcal{M} which eigenvalue multi-set is $\text{EV}(A)$. And this latter condition on a matrix X is equivalent to the following system of $n + 1$ equations on X :

$$\begin{aligned} \text{discr}(X) &= 0, \\ c_i(X) &= \text{sig}(i) \cdot \text{elSym}_i(\text{ev}(A)), \quad 0 \leq i \leq n - 1. \end{aligned}$$

These equations, — except the first one, — are the Viète expressions for a polynomial coefficients via its roots, in which there are substituted the values for the roots. Here

$c_i(X)$ is the coefficient of degree i in the characteristic polynomial for X — a polynomial in the variables—elements of X ,

$\text{ev}(A)$ is the number sequence made of the values of the multi-set $\text{EV}(A)$, with repetitions of multiple values,

elSym_i is the elementary symmetric polynomial of degree $n - i$,

$\text{sig}(i)$ is the appropriate sign: plus or minus.

Minimal equations algorithm for orbit of a matrix in \mathcal{M}

When given the value n and the values a, b_1, \dots, b_{n-2} for the variables $\lambda, \mu_1, \dots, \mu_{n-2}$, a minimal system for the orbit of a matrix in \mathcal{M} having this eigenvalue multi-set is obtained as follows. In the algorithm of the Section 3, add the following linear equations to the orthogonality conditions on the matrix Y : $\lambda = a, \mu_1 = b_1, \dots, \mu_{n-2} = b_{n-2}$. And further, it is applied the same algorithm for finding relations. Evidently, this produces a degree—minimal equation system for the orbit.

Further, the orbit of A is a prime, connected and compact algebraic surface. Because the map

$$DO_A : SO(n) \rightarrow \text{Orbit}(A), \quad DO_A(g) = g^c A$$

is a polynomial map of algebraic surfaces. By definition of an orbit, its image is $\text{Orbit}(A)$, and the group $SO(n)$ is compact and connected. Hence, primality of $\text{Orbit}(A)$ follows from a) that $SO(n)$ is prime, b) that DO_A is a polynomial map onto whole $\text{Orbit}(A)$, c) Lemma A2.

So: the orbit for A is an algebraic, prime, compact, connected surface contained in some sphere inside a hyper-plane Π^* . The minimal dimension for an embracing plane depends on the multiplicities in the spectrum of A .

The further discourse in this lemma (on a smooth parameterization of an orbit) we provide only for the case of a matrix A of a maximal spectrum in \mathcal{M} . This is sufficient for the main two theorems in this paper.

Let $\text{Bas} = \{e_1, \dots, e_n\}$ be the basis consisting of the eigenvectors of A , and the vectors e_1 and e_2 belong to the same eigenvalue. Then the stabilizer $\text{St}(A)$ is a single-parameter subgroup of rotations in the two-dimensional plane $L(e_1, e_2)$. It is known from the Lie group theory that in this case the orbit of A is locally diffeomorphic to

a smooth surface — homogeneous space SO' of conjugated classes in $SO(n)$ modulo $St(D)$, having dimension $\dim SO(n) - 1$.

Due to the action of $SO(n)$, small neighborhoods of all points of the orbit are isometric. This provides an atlas for a smooth parameterization of the orbit.

Although a smooth parameterization of the orbit is proved, — by a reference to the general Lie group theory, — still we provide in Application a certain more definite description of parameterization for the case of $n = 3$.

5.2 Orbit structure details in the case of $n = 3$

Two definitions:

1) for a point M in an affine space and for a vector subspace L , $\Pi(M, L)$ is the plane with the subspace L , containing the point M .

2) The expression “subspace of 1-orbit” means the vector subspace of the plane of 1-orbit.

Lemma ST.

In the case of $n = 3$, the following holds for orbit of any non-scalar matrix A in \mathcal{M} .

(1) This orbit resides on the four-dimensional sphere (in a five-dimensional plane) and is diffeomorphic to a torus as a manifold of two charts.

This is a bi-dimensional, algebraically-quadratic surface. It resides in the hyperplane orthogonal to the line of scalar matrices and it is not contained in any four-dimensional plane.

This orbit is union of a smooth single-parameter family of circumferences of the same radius having a common point A .

The orbit orthogonally intersects the plane Dg of diagonal matrices, and it has exactly three common points with Dg .

(2) This orbit is described by four quadratic equations in the variables $x_{2,2}, x_{3,3}, x_{1,2}, x_{1,3}, x_{2,3}$ of the elements of a matrix X in $\text{Sym}(3)$.

(3) The orbit diameter for a matrix in \mathcal{M} having the eigenvalues (λ, λ, μ) is $|\lambda - \mu|$.

We call this orbit a d -torus.

Proof. Due to the group action and by the statement (7) of Lemma 1, it is sufficient to prove Lemma ST for the diagonal matrix $D = \text{diag}(1, 1, -2)$.

Let us prove the statement (1).

Consider first the 1-orbit of D by the axis e_1 . This is a circumference (residing in its bi-dimensional plane), which we denote Cir_1 . This is visible from the explicit formulae for a change in D by a rotation around e_1 . The general matrix of this 1-orbit is

$$C1G = e_1^\phi D =$$

$$\begin{bmatrix} [1, & 0, & & 0 & &] \\ [0, & c^2 - 2*b^2, & & -3*c*b & &] \\ [0, & -3*c*b, & & -2*c^2 + b^2] \end{bmatrix},$$

where $c = \cos \phi$, $s = \sin \phi$. The only diagonal matrices in this circumference are D and $D_2 = \text{diag}(1, -2, 1)$. When the angle ϕ changes from zero to $\pi/2$, the matrix $C1G$ passes (without repetition) the arc from D to D_2 . At this stage, the matrix

element $C1G(2, 3)$ first changes monotonously from zero to $-3/2$, and then changes monotonously back to zero. When the angle ϕ changes from $\pi/2$ to π , the matrix $C1G$ passes (without repetition) the arc from D_2 to D ; this is the “lower” half of the 1-orbit, it differs from the upper half only in the sign of the element $C1G(2, 3)$.

The subspace L_1 of the 1-orbit Cir_1 is linearly generated by the vector $D_2 - D$ and the tangent vector to Cir_1 in the point D . After norming by the factor -3 , these two vectors produce the following basis for the space L_1 :

$$\begin{aligned} M1 = \text{diag}(0, 1, -1), \quad M2 = & [0, 0, 0] \\ & [0, 0, 1] \\ & [0, 1, 0]. \end{aligned}$$

Now, let l be the axis in the space $L(e_1, e_2)$ obtained from e_1 by rotation at an angle ψ around e_3 , let h be the operator of this rotation. Then the 1-orbit $Cir(\psi)$ for D by l is a circumference containing D and isometric to Cir_1 ; this isometry is the operator h^c (in the space $\text{Sym}(3)$), and it also maps the plane L_1 to the plane $L(\psi)$ of the 1-orbit $Cir(\psi)$. This property holds due to that the operator h^c conjugates the subgroups $G(e_1)$ and $G(l)$. Therefore, for these two 1-orbits it holds the equation

$$\text{Orbit}(l)D = h^c \text{Orbit}(e_1)(h^{-c}D).$$

But in this case, h is a rotation around e_3 , and $D = \text{diag}(1, 1, -2)$. Hence h commutes with the operator D , and the relation between these two 1-orbits is simplified: $\text{Orbit}(l)D = h^c \text{Orbit}(e_1)D$. That is these 1-orbits are conjugated by h .

So, it appears that the orbit for D contains the union of a smooth single-parameter family of circumferences of the same radius, all containing the point D . Let us also note that the plane $\Pi(\psi)$ of 1-orbit in this family turns dependently on ψ in various directions in a five-dimensional space. We shall see later that $\text{Orbit}(D)$ is union of these circumferences.

Now, let us parameterize the standard *torus* as usual, with the angles $0 \leq \phi, \psi < 2\pi$. And consider the map

$$\text{AxR}(\phi, \psi) = l(\psi/2)^{\phi/2}D$$

from this torus to the $\text{Orbit}(D)$. Here $l(\alpha) = e_3^\alpha(e_1)$ is the axis obtained from e_1 by rotation around e_3 . Evidently, AxR is a smooth map. And the circumference $\phi = 0$ is a particular subset for this map: it is mapped to the point D .

Let us call the *half-torus* this torus minus the circumference of $\phi = 0$. Evidently, half-torus is diffeomorphic to a cylinder.

With changing the angle ψ in the torus parameterization, the axis l takes the following remarkable values. $\psi = 0$ corresponds $l = e_1$ and the 1-orbit Cir_1 containing the matrices D and $D_2 = \text{diag}(1, -2, 1)$.

$\psi = \pi$ corresponds to the axis $l(\pi/2) = e_2$ and 1-orbit Cir_2 containing the matrices D and $D_3 = \text{diag}(-2, 1, 1)$.

In the Lemma *ATor in Application*, it is proved that 1) $\text{rank AxR} = 2$ everywhere on the half-torus, 2) AxR maps bijectively the half-torus on $\text{Orbit}(D) \setminus \{D\}$.

On the other hand, we suggest to take these two statements as evident.

This provides the first chart for a diffeomorphism from the torus onto the orbit. The second chart is presented by the map

$$\text{El}(\phi, \psi) = e_2^{\psi/2} e_1^\phi D, \quad -\pi/4 < \phi < \pi/4, \quad 0 \leq \psi < 2\pi$$

— composition of the change by a rotation around e_1 and the change by a rotation around e_2 . This is a smooth map onto a neighborhood of the point D in the orbit from the second half-torus. This second half-torus contains the circumference $\phi = 0$. This is not difficult to find the tangent operator for this map in the point D and to see that it has rank two on the circumference $\phi = 0$ on the torus (and hence, on some neighborhood of this circumference). Therefore these two overlapping charts (which union is the whole torus) define a diffeomorphism from the torus to the manifold of the orbit of the matrix D .

But the metrical properties of the orbit are more complex.

For example, the orbit of the above matrix D is not contained in any four-dimensional plane.

The simplest way to see this is to consider an embracing plane for the orbit of $D = \text{diag}(0, 0, 1)$. Because by the statement (7) of Lemma 1, the latter orbit differs metrically from the former only by a homothety by a non-zero factor.

It is easy to compute that the tangent space to the orbit in the point D is generated by the vectors

$$T1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad T2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

These are the results of changes in D by infinitely small rotation around e_1 and around e_2 respectively, they are obtained by commuting the matrix D with two (anti-symmetric) matrices from a basis of the Lie algebra $so(3)$.

Also the orbit contains the matrices $D_2 = \text{diag}(0, 1, 0)$ and $D_3 = \text{diag}(1, 0, 0)$. Respectively, the tangent space in D_3 contains the vector

$$B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Denote $D_2' = D_2 - D$, $D_3' = D_3 - D$.

If a plane L contains the orbit for D , then L contains the point D and also the directions $T1$, $T2$, B , $D_2' = \text{diag}(0, 1, -1)$, $D_3' = \text{diag}(1, 0, -1)$. The vectors $T1$, $T2$ and B are mutually orthogonal (recall the s-metric). The vector D_i' is orthogonal to $T1$, $T2$ and B for $i = 2, 3$, and the vectors D_2' and D_3' constitute a staircase matrix of rank two. So: the above five directions are linearly independent and belong to each plane containing the Orbit(D).

This conclusion on an embracing plane dimension shows, in particular, that the orbit is not isometric, for example, neither to sphere nor to standard torus.

Let us prove the Lemma statement about quadratic equations.

A general algorithm for finding minimal equations for the orbit of D is described earlier, in the Section (3) and in the proof for Lemma 2. In the case of $n = 3$ the computation is not expensive. Let us print its result for the matrix

$$D = \text{diag}(1, 1, -2).$$

For any other matrix in \mathcal{MH} , the orbit geometry differs only in homothety by a non-zero factor (by the statement (7) of Lemma 1).

In this example, the orbit resides in the hyper-plane Π^* , defined by the equation $x_{11} + x_{22} + x_{33} = 0$ (and orthogonal to the line Scal), and it resides on the four-dimensional sphere having its center in zero. The algorithm derives the following equations for this orbit:

```
orbitEqs = {x11 + x22 + x33,          -- x11 is eliminated
            x23^2 - x22*x33 + x22 + x33 - 1,
            x13^2 + x22*x33 + x33^2 - x22 - 1,
            x12*x33 - x13*x23 - x12,
            x12^2 + x22^2 + x22*x33 - x33 - 1}
```

— four quadratic equations for the variables $x_{22}, x_{33}, x_{12}, x_{13}, x_{23}$ defining a bi-dimensional smooth surface in a five-dimensional plane.

The lemma is proved.

5.3 Parametric equations for 1-orbit.

Analyzing and checking the orbit equations

Bringing to main axes in some of the above equations and summing some of them shows that the orbit is intersection of 1) cylinder (with the element space $L(x_{12}, x_{13})$) over a straight circular cone, 2) cylinder over an ellipsoid, and so on.

Our impression is that, first, it is difficult to understand the orbit structure in this way. Second, it is remarkable that the equations in the system are quadratic. Third, this orbit is the image of a certain complex injection of a torus. Therefore, to verify this computer evaluation result, — and also for the algorithm demonstration, — let us apply certain interesting additional checks and computations.

In passing, we also derive a generic parametric equation system for an 1-orbit.

First, find, in what way the equations `orbitEqs` contain the circumferences in the d -torus, having a common point $D = \text{diag}(1, 1, -2)$ and being the 1-orbits by the axes l — as it is described in Lemma ST.

Define an axis l by a real number k , putting that the vector v of this axis is $v = e_1 + ke_2$. So: 1) the axis passes through the right-hand half-plane, 2) $k = 0$ corresponds to $l = e_1$, 3) $k = \infty$ corresponds to $l = e_2$.

Further, for a negative k , the orbit by the axis $l(k)$ differs from the orbit by $l(-k)$ only in the sign of the element $X(1, 3)$ of the current matrix. This will be visible from the parametric formula for an 1-orbit derived below.

The orbit equations are derived by applying the algorithm of the Section 3. Further, to obtain equations for the 1-orbit by l , it suffices to add the fixed-axis condition to the orthogonality conditions on the operator Y in the algorithm. The fixed-axis condition is $Y(e_1 + ke_2) = e_1 + ke_2$ — three linear equations. Here k is a parameter, and the

algorithm operates with polynomials having *rational functions in k* as coefficients. And the result equations depend explicitly on the parameter *k*.

In this example, the algorithm produces a minimal-degree basis for the ideal of relations for the elements $X(i, j)$ of the matrix. And by Lemma ST, we expect to see a relation basis defining a circumference in the plane $L(l)$. The algorithm prints the following equations for an 1-orbit:

$$\begin{aligned} \text{1-orbitEqs} = \{ & x_{11} + x_{33}k^2/(k^2+1) + (k^2-1)/(k^2+1), \\ & x_{22} + x_{33}/(k^2+1) + (-k^2+1)/(k^2+1), \\ & x_{13} + kx_{23}, \\ & x_{12} - x_{33}k/(k^2+1) - 2k/(k^2+1), \\ & x_{23}^2 + x_{33}^2/(k^2+1) + x_{33}/(k^2+1) - 2/(k^2+1) \} \end{aligned}$$

Hence, the general matrix of 1-orbit is

$$\begin{bmatrix} [-x_{33}k^2/(k^2+1) + (1-k^2)/(k^2+1), & x_{33}k/(k^2+1) + 2k/(k^2+1), & -kx_{23}] \\ [-- & -x_{33}/(k^2+1) + (k^2-1)/(k^2+1), & x_{23}] \\ [-- & -- & x_{33}] \end{bmatrix}$$

Here wild-card denotes the elements under the main diagonal in this symmetric matrix. And separating of a complete square in the last equation shows that the variables x_{23} and x_{33} are bound with the ellipse:

$$(k^2+1)x_{23}^2 + (x_{33}+1/2)^2 = 9/4 \quad (\text{E}).$$

The elements $X(i, j)$ are expressed linearly though x_{23} and x_{33} . So, we see that each 1-orbit is an ellipse which is the projection of the ellipse (E) to the corresponding plane. This agrees with the statement of the Lemma ST about the circumference family. One could continue this test and make sure that this ellipse is a circumference of the same radius square (9/4), as Cir_1 from the proof of Lemma ST.

Further, for the equations **1-orbitEqs** there are the two remarkable values for k : zero and infinity. $k = 0$ must produce the equations for the circumference Cir_1 . This substitution yields the matrix

$$\begin{bmatrix} [1, 0, & 0] \\ [0, -x_{33}-1, & x_{23}] \\ [0, x_{23} & x_{33}], \end{bmatrix} \quad x_{23}^2 + (x_{33}+1/2)^2 = 9/4 \quad (\text{E1}).$$

This coincides with the parameterization for the circumference Cir_1 from Lemma ST. And the equation (E1) must express the equality $\det X = -2 = \det D$. To check this, compute $\det(X) + 2$ modulo (E1). Indeed, its result is zero.

With k approaching infinity, l has the limit e_2 , and there must appear the equations on the 1-orbit of D by e_2 , and this orbit must be the circumference Cir_2 , of the same radius. The equations **1-orbitEqs** contain the relation $x_{13} + kx_{23} = 0$. For infinite k , let us choose x_{13} as the independent variable, and express x_{23} as $-x_{13}/k$. With this, substituting in the equations **1-orbitEqs** of the limit $k = +\infty$ produces the generic matrix

$$X = \begin{bmatrix} [-x_{33}-1, 0, & x_{13}] \\ [0, & 1, & 0] \\ [x_{13}, & 0, & x_{33}], \end{bmatrix}$$

with the condition $((k^2+1)/k^2)x_{13}^2 + (x_{33}+1/2)^2 = 9/4$. This condition is equivalent (under the given limit for k) to the equation $x_{13}^2 + (x_{33}+1/2)^2 = 9/4$. And this coincides with the parameterization for the circumference Cir_2 in Lemma ST.

5.4 Smooth parameterization for the surface \mathcal{MH}

Let us prove the statement (5) of Theorem DS.

Due to the action of the group $SO(n)$, it is sufficient to build a smooth parameterization of \mathcal{M} for a neighborhood of a diagonal matrix D in \mathcal{MH} having $D(1) = D(2)$. Recall that $\Pi = \Pi(1, 2)$ is the plane of diagonal matrices D in \mathcal{M} satisfying the condition $D(1) = D(2)$.

Let us parameterize a neighborhood U of a matrix D in the plane Π in \mathcal{MD} by the vector of n eigenvalues of *spectrum width* $n - 1$: $\text{diag}(\lambda, \lambda, \mu_1, \dots, \mu_{n-2})$. Consider the polynomial map

$$DO : \Pi \times SO(n) \rightarrow \mathcal{MH},$$

$$DO((\lambda, \mu_1, \dots, \mu_{n-2}), g) = g^c \text{diag}(\lambda, \lambda, \mu_1, \dots, \mu_{n-2}).$$

By the statement (4) of Theorem DS proved earlier, $\text{Orbit}(D)$ is a smooth surface of dimension $\dim SO(n) - 1$. And its proof includes a local parameterization of this orbit by the conjugation classes by the stabilizer $\text{St}(D)$, where this stabilizer consists of rotations of the bi-dimensional plane $L(e_1, e_2)$. Consider the map DO_q which differs from DO in that operator g in the argument is taken not from $SO(3)$ but from the space of the conjugation classes by the stabilizer. Fixing the first argument in DO_q to correspond to D , we obtain a local parameterization for $\text{Orbit}(D)$.

In the neighborhood U the values in the vector $(\lambda, \mu_1, \dots, \mu_{n-2})$ remain different. Hence, in U all the matrices have the same vector of eigenvalue multiplicities as D : $(2, 1, \dots, 1)$. Hence, all the matrices in this neighborhood (in Π) have the same stabilizer in $SO(n)$. And it was proved earlier that the orbit intersection with the plane of diagonal matrices is orthogonal. Due to all this, it is evident that DO_q is a smooth parameterization of a neighborhood of the matrix D in \mathcal{M} (this is the same as in \mathcal{MH}).

The statement (5) is proved.

Corollary about dimension of \mathcal{M}

In particular, dimension of \mathcal{M} is $\dim(\Pi) + \dim(\text{Orbit}(D)) =$

$$n - 1 + \dim SO(n) - 1 = n + (n(n - 1)/2) - 2 = n(n + 1)/2 - 2 = \dim(\text{Sym}(n)) - 2.$$

Therefore (for each n), co-dimension of \mathcal{M} is 2.

So, \mathcal{MH} is represented as a smooth fibration of co-dimension 2 over a $(n - 1)$ -dimensional plane with the fiber being an orbit and a smooth, algebraic, compact, connected surface of dimension $\dim(SO(n)) - 1$.

5.5 Bounds on an orbit diameter

Let us prove the statement (3.3) of Theorem DS. We need to derive bounds on the orbit diameter for a matrix in \mathcal{MH} . Such a matrix is diagonalized by some change in $SO(n)$, and this change preserves distances. Therefore it is sufficient to prove bounds for a diagonal matrix $D = \text{diag}(\lambda_1, \dots, \lambda_n)$ in \mathcal{MH} . Let i and j be so that the value $|\lambda_i - \lambda_j|$ is maximal for all the pairs of the eigenvalues of D . $\text{Orbit}(D)$ contains all the results of changes in D by rotations of the plane $L(e_i, e_j)$. With these changes, the result matrix passes through a circumference of the radius $|\lambda_i - \lambda_j|/2$. This proves the first inequality of the statement (3.3).

The upper bound of this statement follows from that (by Lemma 2), the orbit resides on the hyper-sphere with the center in the scalar matrix, being the projection of D to the scalar line. Hence, the diameter of this sphere is an upper bound on the orbit diameter. This proves the statement (3.3).

A detail: the diameter square of the above sphere is expressed via the eigenvalues as $(\lambda_1 - m)^2 + \dots + (\lambda_n - m)^2$, where m is the arithmetical mean value for the numbers $\lambda_1, \dots, \lambda_n$.

5.6 Proof for the structure of \mathcal{M} for $n = 3$

By statement (8) of Lemma 1, \mathcal{M} is a straight cylinder over the surface restriction \mathcal{M}_0 . So, it remains to find the geometric structure of \mathcal{M}_0 . The eigenvalue multi-set of each matrix in \mathcal{M}_0 is of the kind $\{\lambda, \lambda, -2\lambda\}$. By statement (7.2) of Lemma 1, the orbit of such a matrix is obtained from the orbit of the matrix $D = \text{diag}(1, 1, -2)$ by the homothety by the factor λ . Therefore, \mathcal{M}_0 is the cone with its vertex in zero and its base being a d-torus — $\text{Orbit}(D)$. This (bi-dimensional) d-torus (Section 5, Lemma ST) resides on a four-dimensional sphere S_4 with center in zero.

Any diagonal matrix belonging to \mathcal{M}_0 is of one of the following kinds: $D(\lambda) = \text{diag}(\lambda, \lambda, -2\lambda)$ — and also the families $D_2(\lambda)$ and $D_3(\lambda)$ obtained from $D(\lambda)$ by permutations on the main diagonal.

Remarks: 1) For each λ , the matrices $D(\lambda), D_2(\lambda)$ and $D_3(\lambda)$ belong to the same orbit; 2) $D(1)$ and $D(-1)$ are central-symmetric points on the sphere S_4 , but they belong to different orbits; 3) Intersection of \mathcal{M}_0 with the sphere S_4 is union of $\text{Orbit}(D(1))$ and $\text{Orbit}(D(-1))$.

Theorem.

Zero matrix is the only singular point on the surface \mathcal{M}_0 .

Proof. Regularity of any non-zero matrix in \mathcal{M}_0 is proved earlier, in Subsection 5.4. And zero is singular because 1) \mathcal{M}_0 is a cone with vertex in zero, 2) we present four (straight) lines, intersecting in zero, belonging to \mathcal{M}_0 , and having linearly independent directions. The two of them are the lines, connecting the point 0 with the matrices $D = \text{diag}(1, 1, -2)$ and $D_2 = \text{diag}(1, -2, 1)$ respectively. Further, the orbit of the matrix D contains the 1-orbit for D by the axis e_1 . This is a circumference containing the matrices D and D_2 , so that D corresponds to the angle 0, and D_2 corresponds the angle $\pi/2$, and these two are the only diagonal matrices in this 1-orbit. For the proof, we also need some non-diagonal matrices. For example, $e_1^{\pi/4} D$ is the matrix

$$\mathbf{M1} = \begin{bmatrix} 1 & 0 & 0 &] \\ 0 & a & b &] \\ 0 & b & -a-1] \end{bmatrix}, \quad a = -1/2, \quad b = -3/2.$$

Similarly, the 1-orbit for D by e_2 contains the matrix

$$\mathbf{M2} = \begin{bmatrix} a & 0 & b &] \\ 0 & 1 & 0 &] \\ b & 0 & -a-1] \end{bmatrix}$$

— with the same a and b . It is easier to see linear independence of the symmetric matrices $D, D_2, \mathbf{M1}, \mathbf{M2}$ if we represent each of them in the row-vector form: $[X(i, j) | 1 \leq i \leq j \leq 3]$ — with skipping the sub-diagonal part of the matrix. This produces the matrix of the four rows:

$$\begin{bmatrix}
1, & 0, & 0, & 1, & 0, & -2 \\
1, & 0, & 0, & 0, & -2, & 1 \\
1, & 0, & 0, & a, & b, & -a-1 \\
a, & 0, & b, & 1, & 0, & -a-1
\end{bmatrix}.$$

The first loop of clearing of the first column in the staircase form algorithm produces the matrix

$$\begin{bmatrix}
1, & 0, & 0, & 1, & 0, & -2 \\
0, & 0, & 0, & -1, & -2, & 3 \\
0, & 0, & 0, & a-1, & b, & -a+1 \\
0, & 0, & b, & 1-a, & 0, & a-1
\end{bmatrix}$$

Further, the fourth row is moved to the second place, and to the third row it is added the second row multiplied by $a-1$. This produces the matrix

$$\begin{bmatrix}
1, & 0, & 0, & 1, & 0, & -2 \\
0, & 0, & b, & 1-a, & 0, & a-1 \\
0, & 0, & 0, & -1, & -2, & 3 \\
0, & 0, & 0, & 0, & b2 & a2
\end{bmatrix}.$$

Here $b \neq 0$, $b2 = b - 2(a-1) = -3/2 - 2(-3/2) \neq 0$. Hence, the matrices $D, D_2, M1, M2$ are not linearly dependent. So, we have found such four straight lines in the surface \mathcal{M}_0 which are linearly independent and intersect in zero. Therefore, zero is a singular point in \mathcal{M}_0 . Let us explain why it is singular. All other points of this surface are regular, and the tangent space in each of non-zero points has dimension three. To prove by contradiction, suppose that zero is regular. Then, by the definition of a regular point, some neighborhood of zero in \mathcal{M}_0 is diffeomorphic to a ball in \mathbb{R}^3 . Such a ball cannot contain four linearly independent directions which we have found. This contradiction shows that zero is a singular point in \mathcal{M}_0 .

Theorem DS3 is proved.

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6 Application

6.1 Several lemmata

Lemma A0 (probably, known).

For a diagonal quadratic matrix D and an anti-symmetric matrix A of the same size, the commutator $[D, A]$ is a symmetric matrix having zero main diagonal.

Proof. Symmetry of the result matrix is proved by the following equations:

$$(DA - AD)^* = (DA)^* - (AD)^* = A^*D^* - D^*A^* = (-A)D - D(-A) = DA - AD.$$

Further, denote D_i the diagonal matrix in which $D(j) = \delta_{i,j}$ for all j . Each diagonal matrix is a linear combination of the matrices D_i . And as commutator is a bi-linear map, it is sufficient to prove the goal for the commutator $[A, D_i]$ for each i . The given matrix A has zero main diagonal, and it is visible that $A \cdot D_i$ and $D_i \cdot A$ have zero main diagonal. Therefore, their sum has zero main diagonal.

Example:

$$\begin{array}{ccc} |0 & 1| * |a & 0| & - & |a & 0| * |0 & 1| & = & |0 & b| & - & |0 & a| & = & |0 & b-a| \\ |-1 & 0| & |0 & b| & & |0 & b| & |-1 & 0| & & |-a & 0| & & |-b & 0| & & |b-a & 0 & |. \end{array}$$

Lemma A1 (known).

For any polynomial map $F : R^n \rightarrow R^m$, counter-image \mathcal{M}' of an algebraic set \mathcal{M} is an algebraic set.

Proof. Let \mathcal{M} be the set of zeroes of polynomials $\{p_1, \dots, p_k\}$. Then \mathcal{M}' is the set of zeroes for the set of compositions $\{p_1(F(X)), \dots, p_k(F(X))\}$. Indeed, if A belongs to \mathcal{M}' , then $F(A)$ belongs to \mathcal{M} ; then $p_i(F(A)) = 0$ for all i .

Conversely: if $p_i(F(A)) = 0$ for all i , then $F(A)$ belongs to \mathcal{M} , hence A belongs to \mathcal{M}' .

Lemma A2 (known).

For a surjective polynomial map from an algebraic set \mathcal{M} onto an algebraic set \mathcal{M}' , if \mathcal{M} is prime, then \mathcal{M}' is prime.

To prove by contradiction, suppose that \mathcal{M}' is union of algebraic sets M_1 and M_2 , none of which contains another. By Lemma A1, the counter-images T_1 and T_2 of M_1 and of M_2 respectively are algebraic sets. Also it holds $\mathcal{M} = T_1 \cup T_2$. And as none of M_1 and M_2 contains another, their counter-images are in the same relation. This proves the lemma.

6.2 Addition to Lemma 2.

a) $\text{St}(A)$ is not a normal subgroup, and a linear complement in the Lie algebra to the tangent space for this subgroup is not closed by commutator.

b) Although a smooth parameterization is already proved (by a general reference to the Lie group theory), still let us describe a local smooth parameterization by a more definite construct, and for simplicity, consider only the case of $n = 3$.

Due to translations by $SO(3)$ on the orbit, it is sufficient to define a smooth parameterization for an orbit of a diagonal matrix $D = \text{diag}(a, a, b)$, with $a \neq b$, in a neighborhood of D .

In this case, the stabilizer $\text{St}(D)$ is the subgroup of rotations around e_3 . Let χ_3 be the tangent vector in unity to $\text{St}(D)$, let χ_1 and χ_2 be the tangent vectors in unity to the (uni-parametric) subgroups of rotations around e_1 and e_2 respectively. The sub-space $V = L(\chi_1, \chi_2)$ does not contain the vector χ_3 (but it is not closed by commutator).

Let \exp_V be the map \exp of the operator exponent considered in restriction to V . \exp_V maps V to a bi-dimensional surface inside $SO(3)$. There is also the map $C : SO(3) \rightarrow \text{Sym}(3)$ of the group action: $C(g) = g^c D$.

The composition $\text{CE}(A) = C(\exp_V(A))$ is a map from V to $\text{Sym}(3)$, having image inside \mathcal{M} . It also is a composition of a pair of smooth maps and has rank two in zero. The latter holds due to the following two reasons.

1) The infinitely small rotation around e_1 affects only the rows No 2 and 3 in the matrix D , and in the tangent map, the image element at the position $(1, 3)$ is zero, and the element at $(2, 3)$ is $b - a$.

2) The infinitely small rotation around e_2 affects only the rows No 1 and 3 in the matrix D , and in the tangent map, the image element at the position $(1, 3)$ is $b - a$, and the element at $(2, 3)$ is zero.

So, this produces a diffeomorphism from a neighborhood V_e in a bi-dimensional plane onto a neighborhood in the orbit of D .

For the case of arbitrary n , — in a similar way, — the tangent vector to the stabilizer has a linear complement V of dimension $\dim SO(n) - 1$, and there is considered the parameterization map $A \rightarrow T(\exp_V(A))$.

6.3 Lemma ATor

The orbit of the matrix $D = \text{diag}(1, 1, -2)$ by changes of $SO(3)$ is diffeomorphic to the (bi-dimensional) torus.

Here we need to provide a detailed proof for the statement of Lemma ST about the two charts of diffeomorphism. First, prove that the second chart El is a diffeomorphism from the half-torus to a neighborhood of the point D on the orbit.

It is known that the Lie group $SO(3)$ is parameterized by the Euler angles. So that any operator in $SO(3)$ is $e_{2,\psi} \cdot e_{1,\phi} \cdot e_{3,\chi}$, where l_ϕ denotes the rotation at the angle ϕ around the axis l . Each rotation around e_3 commutes with the operator D . Therefore, $\text{Orbit}(D)$ is parameterized by the formula $e_2^\psi e_1^\phi D$ — in the denotations of the Section 2. It is known that the map of the Euler angles diffeomorphically parameterizes the surface $SO(3)$ — without the pole. In our case, the pole is at the angle $\phi = \pi/2$ in our latter formula with the two rotations. It corresponds to the matrix $D_2 = \text{diag}(1, -2, 1)$ in the orbit, which is a fixed point for rotations around e_2 .

In the chart El, the angle ϕ for rotation around e_1 changes near zero, without reaching the pole. Therefore, the chart El described in Lemma ST is a diffeomorphism from the second half-torus to a neighborhood of the point D on the orbit.

Now, prove that the chart AxR is a diffeomorphism from the first half-torus to the orbit without the point D . This parameterization of the orbit part is a combination of rotation of the axis at the angle $\psi/2$ from e_1 to the axis l and the change by rotation around l at a non-zero angle $\phi/2$. Let us prove that this map has rank two everywhere at the torus where $\phi \neq 0$.

As we have seen in Lemma ST, the changes in D by rotations around e_1 and around $l(\psi)$ respectively are related by the formula of the 1-orbit conjugation: $l(\psi)^\phi D = e_3^\psi e_1^\phi D$. The map $e_3^{-\psi}$ is an isometry on the space $\text{Sym}(3)$, and also it maps isometrically the image of the map AxR onto itself, and with this, the 1-orbit D by $l(\psi)$ maps onto the 1-orbit by e_1 . Therefore, the statement about rank AxR is sufficient to prove for all points M of the 1-orbit of D by e_1 .

This parameterization in the neighborhood of the matrix $M = e_1^\phi D$ is a combination of a small shift along the circumference Cir_1 (of 1-orbit by e_1) and a small rotation of this circumference (together with its plane in \mathbb{R}^5) corresponding to the rotation of the axis.

The partial derivative of M by ϕ is a tangent vector to the circumference Cir_1 , it is a non-zero vector in the subspace L_1 . In the Lemma ST, it is derived a basis for L_1 :

$$M1 = \text{diag}(0, 1, -1), \quad M2 = \begin{bmatrix} 0, & 0, & 0 \\ 0, & 0, & 1 \\ 0, & 1, & 0 \end{bmatrix} \quad (M1M2).$$

It is sufficient to prove that the partial derivative of M by ψ at the point $\psi = 0$ is a non-zero vector orthogonal to the vectors $M1$ and $M2$. This derivative is the change h of infinitely small rotation around e_3 in some matrix in the subspace L_1 . The infinitely small rotation around e_3 is represented by the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The action of h at $M1$ is the commutator

$$C1 = [A, M1] = \begin{bmatrix} 0 & 1, & 0 \\ 1, & 0, & 0 \\ 0, & 0, & 0 \end{bmatrix},$$

and this result is orthogonal to $M1$ and to $M2$. The action of h at $M2$ is the commutator

$$C2 = [A, M2] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

and this result is orthogonal to $M1$ and to $M2$. The partial derivative of AxR by the axis rotation is obtained by applying h to some non-zero vector in the subspace L_1 . Each non-zero vector v in L_1 is a linear combination of $M1$ and $M2$. Respectively, hv is the same linear combination of the vectors $C1$ and $C2$, and it is a non-zero vector orthogonal to the subspace L_1 .

This proves the statement about rank AxR .

Let us prove injectiveness of the map AxR .

Sub-Lemma. For each $0 < \psi < \pi$, intersection of 1-orbit of the matrix D by the axis $l(\psi)$ with the plane $\Pi(D, L_1)$ (of 1-orbit by e_1) consists only of the point D .

Proof. In the Section 5.3 it is shown a generic (symmetric) matrix of this 1-orbit:

$$G0 = \begin{bmatrix} -x33*k^2/(k^2+1) + (1-k^2)/(k^2+1), & x33*k/(k^2+1) + 2*k/(k^2+1), & -k*x23 \\ -- & -x33/(k^2+1) + (k^2-1)/(k^2+1), & x23 \\ -- & -- & x33 \end{bmatrix},$$

$$(k^2+1)*x23^2 + (x33+1/2)^2 = 9/4 \quad (E).$$

And due to the condition on ψ , we have here $k \neq 0$. For L_1 it is known a basis $\{M1, M2\}$ given by the formulae (M1M2). So that the plane $\Pi = \Pi(D, L_1)$ consists of all symmetric matrices of kind

$$GM' = \text{diag}(1, 1, -2) + a \text{diag}(0, 1, -1) + b M2 =$$

$$\begin{bmatrix} 1, & 0 & 0 \\ 0, & 1+a, & b \\ 0, & b, & -2-a \end{bmatrix},$$

where a and b are any real numbers. Therefore, if an instance of the family $G0$ belongs Π , then $x_{23} = 0$, and this instance is

$$G0' = \begin{bmatrix} 1, & 0, & 0 \\ 0, & A(x_{33}), & 0 \\ 0 & 0 & x_{33} \end{bmatrix}, \quad x_{33} + 1/2 = \pm 3/2.$$

The trace must be zero, hence, $A(x_{33}) = -x_{33}-1$, and $G0' = \text{diag}(1, -x_{33}-1, x_{33})$. It remains to verify that among the two values for x_{33} , fits (-2) and only it.

For $x_{33} = -2$, the family GM' is satisfied under $a = b = 0$. The family $G0$ (with substitution of $x_{23} = 0$) produces the expressions in k which simplify to the matrix $\text{diag}(1, 1, -2)$.

Further, the substitution $x_{33} = 1$ (and $x_{23} = 0$) in $G0$ yields the matrix family having at the position $(2, 3)$ the expression $k/(k^2+1) + 2k/(k^2+1)$. As $G0'(1, 2) = 0$, then $k = 0$, and this contradicts the Sub-lemma condition on the angle ψ . Therefore, in the intersection of the orbit with the plane Π , the matrix family instances for $G0$ and GM are equal D .

The Sub-lemma is proved.

Continue the proof of injectiveness of the map AxR . Proving by contradiction, suppose that $AxR(\phi_1, \psi_1) = AxR(\phi_2, \psi_2)$ for some $0 < \phi_1, \phi_2 < 2\pi$, $0 \leq \psi_1 \leq \psi_2 < 2\pi$. In the case of $\psi_1 = \psi_2$, it comes out that the values of the parameters ϕ_1 and ϕ_2 correspond to the same point of the circumference $\text{Cir}(\psi)$, and it follows then $\phi_1 = \phi_2$.

There remains the case of $\psi_1 < \psi_2$. Denote $O_{1,2}$ intersection of 1-orbits of the matrix D by the axes $l_1 = l(\psi_1)$ and $l_2 = l(\psi_2)$ respectively. The change $h = e_3^{-\psi_1}$ is an isometry mapping the 1-orbit by l_1 onto Cir_1 , this isometry also maps $O_{1,2}$ onto intersection of Cir_1 and the image by h of the second 1-orbit. By the Sub-lemma, this intersection consists only of the matrix D . Hence, intersection of the two considered 1-orbits consists only of the matrix D , and the point D in these 1-orbits corresponds only to the value $\phi_1 = \phi_2 = 0$. This contradiction to the condition on the angles proves injectiveness of the map AxR .

It remains to prove surjectiveness of AxR .

By the statement (3) of Lemma 2, $\text{Orbit}(D)$ is a smooth, compact, and connected surface of dimension two. The map $AxR(\phi, \psi) = l(\psi)^\phi D$ smoothly maps the torus onto a compact and connected surface residing in $\text{Orbit}(D)$. This map has rank two everywhere at the torus except the circumference $\phi = 0$ (corresponding the point D of the orbit). Denote this surface (the image of AxR) ImA .

The intuition behind the proof is as follows. If \mathcal{X} and \mathcal{Y} are bi-dimensional, smooth, compact, and connected surfaces, and \mathcal{X} is a subset in \mathcal{Y} , then these surfaces coincide.

Because otherwise, there exists some point X in the border of \mathcal{X} in \mathcal{Y} . There exists a neighborhood of X in \mathcal{X} diffeomorphic to a circle in \mathbb{R}^2 , and this contradicts to the border position of \mathcal{X} .

Now, keeping in mind a particular place of the point D in the map AxR , let us provide a formal proof. We need to prove that the set $\text{Orbit}(D) \setminus \text{ImA}$ is empty. Proving by contradiction, suppose that it contains some point M . The orbit without the point D is a smooth surface. Also it is connected. Because for each pair of different points at this surface there exists a smooth curve (with a non-zero derivative vector in each point) connecting these points and residing in the orbit. Some non-empty neighborhood of the point D on the orbit is diffeomorphic to a circle in \mathbb{R}^2 . So, if the chosen curve contains D , it can be modified by a small change so that it would not contain D .

Therefore, there exists a smooth curve $\gamma(t)$, mapping the segment $[0, 1]$ to the orbit, avoiding the point D , and such that $\gamma(0) = D_2$ and $\gamma(1) = M$ (instead of D_2 , there fits any point in ImA different from D).

ImA is a compact set. Hence, there exist real numbers $t_1 \in [0, 1]$ and $\epsilon > 0$ such that for each $t \leq t_1$ $\gamma(t) \in \text{ImA}$ and for each $t \in (t_1, t_1 + \epsilon)$ $\gamma(t)$ does not belong to ImA . As $M_1 = \gamma(t_1) \neq D$, then some non-empty neighborhood of M_1 in ImA is diffeomorphic to a circle in \mathbb{R}^2 . Also this neighborhood is contained in $\text{Orbit}(D)$. Therefore there exists a non-empty interval around t_1 which is mapped by γ to ImA . This contradiction with the value choice for t_1 and ϵ proves that the map AxR is surjective.

Lemma ATor is proved.