

Geometry of the Discriminant Surface for Quadratic Forms

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Abstract

We investigate the manifold \mathcal{M} of (real) quadratic forms in $n > 1$ variables having a multiple eigenvalue. In addition to known facts, we prove that 1) \mathcal{M} is irreducible, 2) in the case of $n = 3$, scalar matrices and only them are singular points on \mathcal{M} .

For $n = 3$, \mathcal{M} is also described as the straight cylinder over \mathcal{M}_0 , where \mathcal{M}_0 is the cone over the orbit of the diagonal matrix $\text{diag}(1, 1, -2)$ by the orthogonal changes of coordinates. We analyze certain properties of this orbit, which occurs a diffeomorphic image of the projective plane. Bibliography: 9 items.

Keywords: quadratic form, discriminant variety, algebraic variety, irreducibility, singular point.

1 Introduction

This paper is an improvement and abbreviation of the manuscript [Me] version of 2010.

Definition

1) $\text{Sym}(n) = \text{Sym}(n, \mathbb{R})$ is the space of symmetric matrices of size n over the field \mathbb{R} of real numbers ($\dim \text{Sym}(n) = n(n+1)/2$).

Symmetric matrices we also call *quadratic forms* or *forms* — presuming that a quadratic form (a bilinear symmetric function) is written in coordinates as a symmetric matrix.

2) $\text{charPol}(X)(\lambda)$ is the characteristic polynomial of a matrix X , $\text{discr}(X)$ is discriminant of $\text{charPol}(X)$.

3) $\mathcal{M} = \{X \in \text{Sym}(n) \mid \text{discr}(X) = 0\}$ is the *surface* in $\text{Sym}(n)$ defined by discriminant. In other words: \mathcal{M} is the surface of all the forms in $\text{Sym}(n)$ that have any multiple eigenvalue.

4) By the word “surface” we mean an open subset in a smooth manifold or in an affine algebraic variety in a finite-dimensional Euclidean space (a surface may have singular points).

5) The word “fibration” means in this paper a fibration over a plane considered a) modulo isometry, b) modulo diffeomorphism.

6) \mathcal{MH} is the set of the forms in $\text{Sym}(n)$ having exactly $n-1$ different eigenvalues.

In the book [Ar1], in Appendix 10, the open area $\text{Sym}_+(n)$ of the positive definite forms in $\text{Sym}(n)$ is represented as the area of ellipsoids. And it is shown that (in our denotations) a) $\text{codim } \mathcal{M} = 2$, b) \mathcal{MH} is a smooth manifold.

Also it simply follows from the theorem and lemma in this paragraph of [Ar1] that \mathcal{MH} is an open in \mathcal{M} and dense in \mathcal{M} set.

Aim of this paper: to answer the questions about

- (1) algebraic irreducibility of \mathcal{M} ,
- (2) location of its singular points, its global structure in the case of $n = 3$,
- (3) algebraic equations for the orbit of a form in \mathcal{M} by the orthogonal coordinate changes, and of its position relatively to the plane of diagonal matrices.

As for $n = 1$ the set \mathcal{M} is empty, the problem is considered only for $n > 1$.

It is known that each matrix in $\text{Sym}(n, \mathbb{R})$ has n real eigenvalues, some of which may coincide (see below Theorem DO).

Definition. We call the *spectrum* or the *multiset of eigenvalues* the set of eigenvalues for a matrix in $\text{Sym}(n)$ — together with their multiplicities, and denote it *spec*.

1.1 About applications

In Appendix 10 of [Ar1] (“Multiplicities of characteristic frequencies and ellipsoids depending on parameters”), it is described the mechanical problem of adjusting of a rigid body by moving of an adjustable mass along an arc rigidly attached to the body. The aim is to make the *ellipsoid of inertia* of the body to become an *ellipsoid of revolution* (=== of rotation). And it turns out that a single adjustable mass (single parameter) is not sufficient in the generic configuration. The reason for this is that codimension of \mathbb{M} occurs not 1 (as one could expect) but 2.

In the section 6 of [Ar2] it is written about the relevance of this problem to the von-Neumann–Wigner theorem about intersection of electronic levels.

In our understanding, from the point of view of mechanics and physics, a quadratic form $X(p)$ depending on a parameter p , represents some physical characteristic of an object, where the eigenvalues are the oscillation frequencies of some parts of an object (construction). Hence the discriminant surface represents a geometrical condition of the resonance effect.

Generally: the discriminant surface is an important object, and the more is known about it the better.

1.2 Acknowledgements

I am grateful to *A. G. Khovanskii* for the reference to Arnold's book (due to which it has been cancelled the large paper part which re-proved the results from this book). I am grateful to *N. V. Ilyushechkin, S. Yu. Orevkov, Yu. L. Sachkov* for certain bits of oral discussion.

1.3 Case $n = 2$

For the quadratic form

$$X = \begin{pmatrix} x & y \\ y & z \end{pmatrix}$$

we have $\text{charPol}(X)(\lambda) = (x - \lambda)(z - \lambda) - y^2 = \lambda^2 - (x + z)\lambda + xz - y^2$.

Hence discriminant of this form (as a polynomial in the elements of X) is

$$(x + z)^2 - 4(xz - y^2) = x^2 + 2xz + z^2 - 4xz + 4y^2 = (x - z)^2 + (2y)^2.$$

It follows from this decomposition into a sum of squares that \mathcal{M} is defined by the equations $\{x = z, y = 0\}$ and is the *line of scalar matrices*: an irreducible surface of codimension 2 in $\text{Sym}(2)$. All the points are regular in this case. So, for the case of $n = 2$ the problem is solved. *In the sequel, it is considered only the case of $n > 2$.*

1.4 About equations for \mathcal{M}

For $n = 3$, $\text{discr}(X)$ is a homogeneous form of degree 6 in 6 variables $x_{i,j}$, consisting of 123 non-zero monomials. How to derive from this expression a knowledge about dimension of \mathcal{M} , its irreducible components, singular points? A naive consideration “a single equation, therefore $\text{codim}(\mathcal{M}) = 1$ ” is erroneous.

The quadratic form discriminant is a sum of squares of several real polynomials. This has been independently proved by several authors, starting from Borhardt in 1846 up to N. V. Ilyushechkin [II] in 1985. Further, in the paper [II2] there are derived for the case of $n = 3$ certain relations between the members of this decomposition, and by this, it derives for the case of $n = 3$ certain four cubic homogeneous forms defining the discriminant surface.

And this our paper completes the above results with various important knowledge about the discriminant surface.

1.5 Our approach

This is the classical method of a linear Lie group of symmetries. All symmetric matrices are produced by conjugating diagonal matrices with operators from $\text{SO}(n)$. A displace in the plane of diagonal matrices is orthogonal (in a certain Euclidean metric) to a displace along the conjugation orbit. This provides a smooth parameterization for \mathcal{M}

— except certain singular points. It remains to describe the orbit for a diagonal matrix, with considering its spectrum width: $n - 1$ or less than $n - 1$.

Definition

- * The metric on the surface \mathcal{M} is taken from the space $\text{Sym}(n)$.
- * The spectrum width for a form from $\text{Sym}(n)$ is the number of its different eigenvalues.

For a set \mathcal{X} in $\text{Sym}(n)$, we call a *form of maximal spectrum* in \mathcal{X} any form from \mathcal{X} having maximal in \mathcal{X} spectrum width, and all other forms are called the forms of *narrowed spectrum*.

- * \mathcal{MH} denotes the set of forms of maximal spectrum in \mathcal{M} — that is of the spectrum width $n - 1$.

1.6 Results

- * It is proved that \mathcal{M} is an irreducible algebraic surface (Section 5).
- * There are presented algebraic equations for the orbit of a form in \mathcal{MH} under the orthogonal coordinate changes (Section 7.1).
It is proved that this orbit intersects orthogonally the plane of diagonal matrices.
- * For $n = 3$ it is proved that the singular points on \mathcal{M} are scalar matrices and only them (Section 7.3).
- * For $n = 3$ we provide a more definite description for \mathcal{M} : the straight cylinder over \mathcal{M}_0 , where \mathcal{M}_0 is the cone over a diffeomorphic image of the projective plane $\mathbb{R}P(2)$ (Lemma Orbit1, Sections 7.2, 7.3).
- * For $n = 3$ there are analyzed certain additional properties of the orbit by orthogonal changes for a form in \mathcal{MH} : its certain circumference family having a common point, and the minimal dimension of an embracing plane (Section 7.2).

The next questions can be as follows.

- 1) For $n > 3$, is it true that each matrix in \mathcal{M} having narrowed spectrum is a singular point?
- 2) The structure of \mathcal{M} depends mainly on the structure of an orbit $\text{Orbit}(A, n)$ for $A \in \mathcal{M}$. Similarly as for $n = 3$, it would have sense to find for $n = 4, 5$ some global description for $\text{Orbit}(A, n)$, more definite than the one given in the sequel.

Agreement. 1. To reduce the main text volume, some routine proofs are moved to Appendix, and the main part contains references to Appendix. And *the proofs in Appendix do not use the statements being proved in the main part of the paper.*

2. The labels of theorems and lemmata are unique, their scope is the whole text, including Appendix.

2 Results from Arnold's book

Theorem (Ar).

- (1) \mathcal{M} is a finite union of smooth manifolds of codimension 2 and higher in the space $\text{Sym}(n)$.
- (2) \mathcal{MH} is a smooth manifold of codimension 2 in $\text{Sym}(n)$.
- (3) \mathcal{MH} is a set open in \mathcal{M} and dense in \mathcal{M} .

Proof.

This theorem immediately follows from the material in Appendix 10 of the book [Ar1]. The points (1) and (2) is our reformulation, for in this paragraph of [Ar1] there are considered only positive definite forms in $\text{Sym}(n)$.

This theorem is derived from [Ar1] as follows.

Definition.

- 1) $\text{Sym}_+(n)$ is the set of positive definite forms in $\text{Sym}(n)$. It is defined by the condition of positiveness of the eigenvalues and is an open area in $\text{Sym}(n)$.
- 2) $\mathcal{M}_+ = \mathcal{M} \cap \text{Sym}_+(n)$, $\mathcal{MH}_+ = \mathcal{MH} \cap \text{Sym}_+(n)$.

Each of the manifolds $\text{Sym}_+(n)$, \mathcal{M}_+ , \mathcal{MH}_+ is isometrically mapped on itself by the coordinate changes from $\text{SO}(n)$ (see below Lemma S). Further, adding of a scalar matrix (applying a *scalar shift*) isometrically maps each of the manifolds $\text{Sym}(n)$, \mathcal{M} , \mathcal{MH} on itself (see below Lemma Orbit1). And for each form in $\text{Sym}(n)$ some its neighbourhood is mapped by some scalar shift into $\text{Sym}(n)_+$. Therefore, the local metrical properties of the surface \mathcal{M}_+ are the same as of \mathcal{M} . This way (by a scalar shift), the theorem in [Ar1] for ellipsoids is generalized to the case of the surface \mathcal{M} in $\text{Sym}(n)$.

In the book [Ar1], in the Section (A) of Appendix 10, it is described a diffeomorphism from \mathcal{M}_+ onto the manifold of ellipsoids in the space $V = \mathbb{R}^n$ having center in 0: $A \mapsto \{v \in V \mid vAv^* = 1\}$ — a quadratic form is mapped to the set — ellipsoid of all the vectors at which this form takes the value 1. Let us explain in what sense ellipsoids form a smooth manifold. The manifold of ellipsoids is identified with the manifold $\text{SO}(n) \times R_+^n$ by parameterizing it by a pair (g, spe) , $g \in \text{SO}(n)$, $\text{spe} \in R_+^n$. Here g is the operator from $\text{SO}(n)$ which maps the fixed orthonormal basis (e_1, \dots, e_n) to the basis of the major axes of the ellipsoid, and the tuple spe of positive real numbers defines the length of each major axis of the ellipsoid. The number of these parameters is $\dim \text{SO}(n) + n = (n(n-1)/2) + n = n(n+1)/2$.

In this representation, \mathcal{M}_+ is the manifold of all the *ellipsoids of rotation*.

Theorem 1 from Appendix 10 of [Ar1] asserts (in our denotation):

- (1) \mathcal{M}_+ is a finite union of smooth manifolds of codimension 2 and higher in the manifold Sym_+ ,

- (2) \mathcal{MH}_+ is a smooth manifold of codimension 2 in Sym_+

(the statement (2) is not in the formulation, but it is in the initial part of the proof of Theorem 1 in [Ar1]).

Further, by the lemma in the above Theorem 1 in [Ar1], the set $\mathcal{M}_+ \setminus \mathcal{MH}_+$ is

a finite union of smooth manifolds in Sym_+ of codimension more than 2. Due to this, \mathcal{MH}_+ is a set open in \mathcal{M}_+ and dense in \mathcal{M}_+ .

And by the described above symmetry of scalar shifts, this theorem is also true for the surfaces \mathcal{M} and \mathcal{MH} in $\text{Sym}(n)$.

The theorem is proved.

3 Several definitions and preliminary constructs

Definition (Basic).

- 0. $V = \mathbb{R}^n$ is the Euclidean space.
 $\mathcal{L}(v_1, \dots, v_m)$ denotes the subspace in V spanned by the vectors v_1, \dots, v_m .
 $\mathcal{L}(V) = \mathcal{L}(n)$ denotes the algebra of the linear operators in V
(or — real square matrices of size n).
 E denotes the identity operator in V .
- 1. We call an irreducible surface an irreducible (prime) affine algebraic manifold over any field ([Sha], Book 1, Chapter I, Section 3.1). In this paper, all the spaces are over \mathbb{R} .
Although in [Sha] all the results are given for an algebraically closed field, our two references to [Sha] are so that the corresponding proofs from [Sha] are also valid for the real space.
- 2. Dg denotes the space of diagonal matrices (of dimension n),
 Scal denotes the line of scalar matrices.
Evidently, \mathcal{M} is a conic set containing Scal .
The word “diagonal” means the main diagonal of a square matrix.
- 3. $A(i, j) = A_{i,j}$ are the denotations for the element $a_{i,j}$ of a matrix A .
For a diagonal matrix D , denote $D(i) = D(i, i)$.
 $\text{diag}(a_1, \dots, a_n)$ denotes the diagonal matrix D having $D(1) = a_1, \dots, D(n) = a_n$.
A diagonal-ordered matrix is a diagonal matrix D in which $D(1) \leq \dots \leq D(n)$.
- 4. $O(V) = O(n)$ is (an algebraic, linear) Lie group of orthogonal operators in V .
 $\text{SO}(V) = \text{SO}(n)$ is the Lie subgroup in $O(V)$ defined by the condition determinant = 1. It has dimension $\dim \text{SO}(n) = n(n-1)/2$.

We use certain properties of $\text{SO}(n)$, like connectedness, algebraic irreducibility, compactness, smoothness. They are explained in the textbooks on the linear (and/or algebraic) Lie groups. For the Russian language, we give a reference [VinO] for this. But for English, I know only [Bo] (which is of a rather abstract style). Irreducibility of $\text{SO}(n)$ follows from that it is algebraic and connected: [Bo], head I, Section 1.2, first Proposition.

- 5. $X \mapsto g X g^{-1}$ is the action of the coordinate (basis) *change* by an invertible operator g in a matrix (operator) X . We shortly call this conjugation action “change by g ”, “changes by operators from the group G ”, “changes from G ”.

And for an orthogonal operator g , there holds $g^{-1} = g^*$ (transposed matrix), and the expression $g X g^*$ represents the basis change in a symmetric bilinear form X .

For $g \in O(n)$, we denote the action of the basis change as

$$g^c X = g^c(X) = g X g^*,$$

and remember that g^c is an operator in $\text{Sym}(n)$. g^{-c} denotes $(g^{-1})^c$.

In the case of dimension 3 ($n = 3$), 1) $\text{RtA}(l, \phi)$ denotes the operator of rotation about an axis l by an angle ϕ ,

2) l^ϕ denotes the change $\text{RtA}(l, \phi)^c$,

3) 1-orbit of a form M by an axis l means the orbit of M under the changes by the rotation operators about l .

6. Denote \mathcal{MD} the set of diagonal matrices in \mathcal{M} , $\mathcal{MDH} = \mathcal{MD} \cap \mathcal{MH}$.

For $1 \leq i < j \leq n$ denote $\Pi_{i,j}$ the plane in Dg defined by the equation $D(i) = D(j)$.

For example, for $n = 3$, $\Pi = \Pi_{1,2}$ is all the matrices of the kind $\text{diag}(\lambda, \lambda, \mu)$, while $\Pi_{1,3}$ and $\Pi_{2,3}$ are obtained from Π by permutational changes from $\text{SO}(3)$.

Denote $\Pi_{i,j}^h = \Pi_{i,j} \cap \mathcal{MH}$.

7. In the sequel the word “orbit” means: the orbit $\text{Orbit}(A)$ of a form $A \in \text{Sym}(n)$ under the action of changes from $\text{SO}(n)$.

8. Fix a scalar product (Euclidean metric) scp on the space $V = \mathbb{R}^n$, an orthonormal by scp basis $\text{Bas} = \{e_1, \dots, e_n\}$ in V , and let the matrices in $\mathcal{L}(n)$ represent the linear operators in V written in Bas . Also let the matrices in $\text{Sym}(n)$ represent the symmetric bilinear forms on V written in Bas .

We also speak of operators from $O(n)$ given by permutations on the set Bas , or, for example (in the case of $n = 3$), being rotations about an axis e_i .

9. $\text{trace}(X)$ denotes the sum of the diagonal elements of a matrix X .

10. The stabilizer $\text{St}(A)$ of a form $A \in \text{Sym}(n)$ is the subgroup of operators g in a group ($O(n)$ or $\text{SO}(n)$) which change preserves A (that is $gA = Ag$, $g^c A = A$).

11. The eigenspace of a given eigenvalue λ of a linear operator A in a space V is the subspace of all the vectors v such that $Av = \lambda v$.

12. For a commutative ring R and a set $S \subset R^n$, a map $P : S \rightarrow R^m$ is called *polynomial*, if each coordinate of its image is represented as a polynomial in the coordinates $\{x_1, \dots, x_n\}$ of the support domain.

For example, the map $P : \text{Dg} \times \text{SO}(n) \rightarrow \text{Sym}(n)$ given by the formula $P(D, g) = g^c D$ is *polynomial*, because each element of the result matrix is expressed as a polynomial in the elements of the matrices D and g .

Example: for $n = 3$, the diagonal matrices in \mathcal{MH} are exactly the matrices of the kind $\text{diag}(\lambda, \lambda, \mu)$ with $\lambda \neq \mu$, and also the two families obtained from this one by permutations on the diagonal.

Our discourse bases on known facts of the linear Lie group theory and the following classical

Theorem DO. (1) For any (real) quadratic form M there exists a change $g \in O(n)$ which brings M to a (real) diagonal matrix having on the diagonal all the eigenvalues of M .

Orthogonality in this theorem is considered with respect to the scalar product scp introduced in the above definition (Basic).

(2) Each eigenvalue λ for M defines the non-zero eigenspace of λ for M .

(3) Eigenspaces belonging to different eigenvalues of M are orthogonal to each other.

(4) (our addition) For bringing M to such a diagonal matrix there are sufficient the changes from $\text{SO}(n)$.

This theorem is proved, for example, in [Wa], Head XII, §90. The statement (1) is by the theorem in the end of §90 about a pair of quadratic forms, and we put one of these forms to be scp . The proof for the points (2) and (3) is contained in the discourse in the whole §90.

But it remains to add the proof for the point (4). Let h be the operator in $O(n)$ defined by transposition of the vectors e_1 and e_2 , and let g bring M to a diagonal matrix. If $\det(g) = 1$, then the statement is true. Otherwise, $\det(g) = -1$, and the composition hg belongs to $\text{SO}(n)$ and still brings M to a diagonal form.

Remark. As $\text{charPol}(A)$ does not change with any invertible coordinate change in A , and due to Theorem DO, the spectrum width for a matrix in $\text{Sym}(n)$ is the spectrum width of its diagonal form.

Lemma Stab. Let $M \in \mathcal{MH}$.

1) The set V_1 of the vectors belonging to the multiple eigenvalue for M is a subspace of dimension 2.

2) The connected component G_e of unity in the stabilizer $\text{St}(M)$ in $\text{SO}(n)$ is a single-parameter subgroup of the operators which restriction on V_1 are rotations of this 2-dimensional space about 0, and which restriction to the orthogonal complement to V_1 is identity.

Appendix Stab provides a simple proof for this.

Definition s-metric. Define a scalar product \langle_s, \rangle on the space $\mathcal{L}(n)$:

$$\langle_s X, Y \rangle = \sum_{1 \leq i, j \leq n} x_{i,j} y_{i,j},$$

and call it s -metric.

It is an Euclidean metric: a positive definite bilinear function.

Denote $s\text{Quad}(X)$ its corresponding quadratic form — sum of squares of the elements of a matrix X .

Note that s -metric defines the same topology in $\mathcal{L}(n)$ as the usual linear operator norm.

Agreement

In this paper, the expressions “distance”, “angle”, “orthogonality”, “isometry”, “orthogonal projection”, “straight cylinder”, “circumference”, “sphere”, “circumference center”, “radius” applied to points and subsets in $\mathcal{L}(n)$ — are understood in the sense of s -metric

(do not confuse this with the metric scp of the space V on which the operators from $\mathcal{L}(n), \text{Sym}(n)$ act!).

Lemma S (probably, known).

For any $X \in \mathcal{L}(n)$ and any orthogonal operator g , it holds the law

$$\text{sQuad}(X) = \text{sQuad}(gX) = \text{sQuad}(g^c X).$$

In particular, any change from $O(n)$ is an isometry of $\mathcal{L}(n)$ in the s-metric.

Proof. By the definition of an orthogonal operator g , such an operator preserves the scalar square for each column-vector C_j in X : $\text{sQuad}(gC_j) = \text{sQuad}(C_j)$. As the $\text{sQuad}(X)$ is sum of the scalar squares of the columns in X , it holds $\text{sQuad}(gX) = \text{sQuad}(X)$. Similarly, the right-hand side multiplication by g^* preserves the scalar square of each row, and hence the s-square of X . Therefore $\text{sQuad}(g^c X) = \text{sQuad}(X)$.

The Lemma is proved.

Definition.

- 1) A *shift* at a matrix A is the map of adding of A in the affine space $\mathcal{L}(n)$.
- 2) A *scalar shift* is a shift at a scalar matrix.

We use that a shift at a matrix is an isometry of the affine space of $\mathcal{L}(n)$. This is because a shift at a vector in an affine space preserves the vector defined by any pair of points.

4 Preliminary notions about orbit

Lemma Orbit1. (1) Any change from $\text{SO}(n)$ maps \mathcal{M} on itself.

(2) Scalar matrices are the only fixed points of the action of $\text{SO}(n)$ by changes on $\text{Sym}(n)$.

(3) \mathcal{M} is union of mutually non-intersecting orbits.

(4) Each matrix in \mathcal{MD} is brought by some change from $\text{SO}(n)$ to an ordered diagonal. In particular, each permutation at the diagonal of a matrix in \mathcal{MD} is represented by some change from $\text{SO}(n)$.

(5) Two forms in \mathcal{M} belong to the same orbit if and only if they have the same spectrum.

(6) For each matrix in \mathcal{M} , its orbit contains exactly one diagonal-ordered matrix.

(7) 7.1. For each real number t and a form $A \in \text{Sym}(n)$, it holds $\text{Orbit}(t \cdot A) = t \cdot \text{Orbit}(A)$.

7.2. For $n = 3$, the orbits of any two forms in \mathcal{MH} are obtained from each other by some scalar shift and a homothety by some non-zero factor (and we use that this transformation preserves the angle between any two straight lines in $\text{Sym}(n)$).

(8) 8.1. Discriminant of a matrix in $\mathcal{L}(n)$ does not change under a scalar shift. Any scalar shift maps the surface \mathcal{M} on itself.

8.2. For each real number s , intersection \mathcal{M}_s of \mathcal{M} with the hyper-plane $\text{trace}(X) = s$ is a surface mapped on itself by any change from $\text{SO}(n)$.

\mathcal{M} is a straight cylinder over \mathcal{M}_0 having the line Scal as element.

Remark.

In the statements (4), (5), (6) it is essential that a matrix has a multiple eigenvalue.

Proof. The statements (1), (2) and (3) are known and evident.

Proving (4) D has at least two equal elements on the main diagonal, let them correspond to the basis vectors u and v respectively. Then the operator g_2 which transposes u and v and is identical on the rest of the basis, has determinant -1. Some permutation σ on the basis brings the main diagonal to an *ordered diagonal* (definition (Basic) in Section 3) and is represented by some operator $g \in O(n)$. If σ is even, then g satisfies the requirement of the point (4). Otherwise, the composition $g g_2$ belongs to $SO(n)$, and its change brings D to a diagonal-ordered form. *The point (4) is proved.*

Proving (5) and (6). The change action preserves the coefficients of the characteristic polynomial, so, it preserves the spectrum. Let us prove the reverse part of the statement (5). Let A and B have the same spectrum. By Theorem DO, A and B are diagonalized by some changes from $SO(n)$. By the statement (4), they are brought further to diagonal-ordered matrices. As these diagonal-ordered matrices represent the same multi-set, they are equal. *And this also proves the statement (6).*

Proving (7) The statement (7.1) is evident. Let us prove (7.2).

By Theorem DO, the orbit of any matrix in \mathcal{MH} coincides with the orbit of some diagonal matrix in \mathcal{MH} , and by the statement (5), it is the orbit of some matrix D_1 of the kind $\text{diag}(\lambda, \lambda, \mu)$ with $\lambda \neq \mu$. Therefore, it is sufficient to prove (7.2) for any diagonal matrices $D_1, D_2 \in \mathcal{MH}$, where D_1 is given above, $D_2 = \text{diag}(\lambda_2, \lambda_2, \mu_2)$, ($\lambda_2 \neq \mu_2$).

The matrix $D = \text{diag}(0, 0, 1)$ is transformed to D_1 by a shift at some matrix of the kind $C = \text{diag}(a, a, a)$ and by a homothety by some non-zero factor b : $(D + C) \cdot b = D_1$. For such representation, it is sufficient to put $a = \lambda/(\mu - \lambda)$, $b = \mu - \lambda$. Each change g^c is an isomorphism on the algebra of square matrices. In particular, for each square matrix X there hold the equations

$$g^c(X + C) = (g^c X) + (g^c C) = (g^c X) + C, \quad g^c(b \cdot X) = b \cdot g^c X.$$

Therefore, with adding a scalar matrix C to any symmetric matrix X , the orbit shifts at C , and with multiplying X by a coefficient b , the orbit is homothetically multiplied by b . Therefore, $\text{Orbit}(D_1)$ is obtained from $\text{Orbit}(D)$ by an appropriate scalar shift and a homothety required in the point (7.2).

In the same way, by a scalar shift and an invertible homothety, D_2 is converted to D . The composition of these shifts and homotheties gives the transition $D_2 \mapsto D \mapsto D_1$. And such a transformation equals to composition of a single scalar shift and a single invertible homothety — due to the equations

$$\begin{aligned} (((X + a \cdot E) \cdot b) + a_2 \cdot E) \cdot b_2 &= b b_2 ((X + a \cdot E) + (a_2/b) \cdot E) = \\ &= b b_2 (X + (a + a_2/b) \cdot E). \end{aligned}$$

The statement (7) is proved.

Proving (8) **8.1:** adding of a scalar matrix $c \cdot E$ shifts the spectrum of any matrix A in $\mathcal{L}(n)$ at the number c . This follows from the definition of a vector belonging to an

eigenvalue λ : $Av = \lambda v \implies (A + cE)v = \lambda v + cv = (\lambda + c)v$ (note that for a non-symmetric matrix the eigenvalues and their vectors may be not real).

This shift preserves discriminant, because (by the discriminant definition via the eigenvalues) it depends only on the pairwise differences of eigenvalues.

The statement (8.1) is proved.

8.2: recall that the surface \mathcal{M} resides in $\text{Sym}(n)$ and contains the line Scal of scalar matrices. For each real number s , denote $\Pi(s)$ the hyper-plane in $\text{Sym}(n)$ defined by the equation $\text{trace}(X) = s$, $\mathcal{M}_s = \mathcal{M} \cap \Pi(s)$. So, $\mathcal{M} = \cup \{\mathcal{M}_s \mid s \in \mathbb{R}\}$.

Any change from $\text{SO}(n)$ preserves the matrix trace, as well as the discriminant value. Therefore, such a change maps each plane $\Pi(s)$ on itself and maps each restriction \mathcal{M}_s on itself. So, for the point 8.2, it only remains to prove the statement about the straight cylinder.

Parameterize the line Scal by the value s of a matrix trace: $s \mapsto C(s)$. Use also that $\text{Scal} = L(E)$. It follows from the s-metric definition that for all X $\langle_s X, E \rangle = \text{trace}(X)$. For each X from $\Pi(s)$ X and $C(s)$ have the same trace, hence the vector $\overline{C(s)X}$ is a matrix of zero trace. Hence $\langle_s \overline{C(s)X}, E \rangle = 0$ for all $X \in \Pi(s)$, and the hyper-plane $\Pi(s)$ is orthogonal to Scal . As the surface \mathcal{M}_s is in $\Pi(s)$, the line Scal orthogonally intersects \mathcal{M}_s (in the point $C(s)$).

Each fiber \mathcal{M}_s is obtained from \mathcal{M}_0 by the scalar shift at $C(s)$, in particular, all these fibers are isometrical to \mathcal{M}_0 . Summing up the discourse, conclude that 1) \mathcal{M} contains the line Scal , 2) \mathcal{M} is union of the fibers, 3) each fiber is obtained from \mathcal{M}_0 by the scalar shift at $C(s)$, intersects with Scal in $C(s)$, and is orthogonal to Scal .

This means that \mathcal{M} is the straight cylinder over \mathcal{M}_0 having the line Scal as element.
The lemma is proved.

Lemma A0 (probably, known).

Commutator of a diagonal and an anti-symmetric square matrices (of the same size) is a symmetric matrix having zero diagonal.

Lemma A1 (known). The counter-image \mathcal{M}' of an algebraic set \mathcal{M} for a polynomial map $F : R^n \rightarrow R^m$ is an algebraic set.

Lemma A2 (probably, known). For a surjective polynomial map from an algebraic set \mathcal{M} onto an algebraic set \mathcal{M}' , if \mathcal{M} is irreducible, then \mathcal{M}' is irreducible.

Simple proofs for lemmata A0, A1, A2 are given in Appendix.

Remark. The lemmata A1 and A2 are related to the topology of the algebraically closed sets in an affine space. The corresponding notions are given, for example, in [Sha], Head I, §2, §3.

5 Theorem of irreducibility

Theorem “Irreducibility”.

The discriminant surface \mathcal{M} is algebraically irreducible

(that is it is not a union of any two algebraic sets different from \mathcal{M}).

Proof.

Let $\Pi = \mathbb{R}^{n-1}$ be the plane parameterized by the tuple $(\lambda, \mu_1, \dots, \mu_{n-2})$ of real numbers. Consider the map $\text{DO} : \Pi \times \text{SO}(n) \rightarrow \mathcal{M}$,

$$\text{DO}((\lambda, \mu_1, \dots, \mu_{n-2}), g) = g^c \text{diag}(\lambda, \lambda, \mu_1, \dots, \mu_{n-2}).$$

This is a polynomial map of algebraic surfaces (where the support domain has one dimension more than the image).

Let us prove that this map is surjective. Let $A \in \mathcal{M}$. By Theorem DO, the orbit of A contains some diagonal matrix D . By the statement 5 of Lemma Orbit1, there exists a multiple eigenvalue in the spectrum of D (and A), denote it λ . By the statement 4 of Lemma Orbit1, each permutation on the diagonal of D is represented by some change from $\text{SO}(n)$. And some permutation produces a matrix of the kind $\text{diag}(\lambda, \lambda, \mu_1, \dots, \mu_{n-2})$. Hence, $A = \text{DO}((\lambda, \mu_1, \dots, \mu_{n-2}), g)$ for some $g \in \text{SO}(n)$. *The surjectiveness is proved.*

As it is mentioned in the beginning of Section 3, $\text{SO}(n)$ is an irreducible algebraic surface. The plane Π is also irreducible.

Direct product of any two irreducible algebraic sets is irreducible. To prove this can be an exercise. Also the proof is given, for example, in [Sha], Head I, §3.1. By this theorem, the algebraic surface $\Pi \times \text{SO}(n)$ is irreducible. Hence, by Lemma A2, \mathcal{M} is irreducible.

The theorem is proved.

6 Structure of the discriminant surface in $\text{Sym}(n)$

Below, Π denotes the plane $\Pi_{1,2}$ of dimension $n-1$ in the surface \mathcal{MD} — see Definition (Basic). Below we identify it to \mathbb{R}^{n-1} by the parameterization $(\lambda, \mu_1, \dots, \mu_{n-2}) \mapsto \text{diag}(\lambda, \lambda, \mu_1, \dots, \mu_{n-2})$.

Also keep in mind that in the plane Π the matrices from \mathcal{MH} form an open set.

Recall also that the word “orbit” means: the orbit of a quadratic form under the action of changes from $\text{SO}(n)$.

Theorem DS

- (1) \mathcal{M} is an irreducible algebraic variety.
- (2) \mathcal{M} is union of the restrictions \mathcal{M}_s , each defined by the equation $\text{trace}(X) = s$. \mathcal{M} is the straight cylinder over \mathcal{M}_0 having the line Scal as element.
- (3) Orbits for the matrices in \mathcal{M} are classified by spectrum, or — by the unique diagonal-ordered matrix in the orbit.
- (4) \mathcal{MH} has the following structure.

4.1. Locally, \mathcal{MH} is a smooth fibration over the plane Π with the fiber being an orbit of a diagonal matrix from \mathcal{MH} .

The orbit of each matrix in \mathcal{MH} orthogonally intersects the plane Dg of diagonal matrices.

4.2. Orbit of each matrix in \mathcal{MH} is a smooth, compact, algebraic, connected surface of dimension $\dim \text{SO}(n) - 1 = (n(n-1)/2) - 1$. It resides on a sphere in the hyper-plane orthogonal to the line of scalar matrices.

- (5) The map $\text{DO} : \Pi \times \text{SO}(n) \rightarrow \mathcal{M}$,

$$\text{DO}((\lambda, \mu_1, \dots, \mu_{n-2}), g) = g^c \text{diag}(\lambda, \lambda, \mu_1, \dots, \mu_{n-2})$$

is a surjective polynomial map of algebraic surfaces (where the support domain has one dimension more than the image).

Remark.

1. A local parameterization of \mathcal{MH} is the same as a parameterization of \mathcal{M} in some neighbourhood of a point in \mathcal{MH} . Because by Theorem Ar (Section 2), \mathcal{MH} is open in \mathcal{M} .

2. The statement (4.1) does not differ much from Theorem 1 from [Ar1] about smoothness of \mathcal{MH} in the area of ellipsoids (Section 2 of this paper, Theorem Ar, point (2)). Therefore, we skip the proof for this point, instead we provide the following informal description for the local parameterization near a point in \mathcal{MH} . Due to the action of $\text{SO}(n)$, the local structure of \mathcal{MH} near each point is isometrical to a neighbourhood of a diagonal matrix $D_0 \in \mathcal{MH}$. In some neighbourhood U of D_0 , the surface \mathcal{MH} is a smooth fibering to orbits, where each fiber-orbit comes out from $D \in \Pi \cap U$. The orbit of D is locally diffeomorphic to the homogeneous space $\text{SO}(n)/\text{St}(D)$ of the congruence classes by the stabilizer. And by Lemma Stab, the connected component St_1 of unity in the stabilizer $\text{St}(D)$ is a single-parametric subgroup of rotations of the

2-dimensional eigenspace V_1 of the (unique) multiple eigenvalue in D (taking in account that the operators from St_1 are identical on the orthogonal complement to V_1).

The used here notions about a stabilizer, orbit, and a smooth structure on the space of congruence classes, we find in the book [VinO] (in Russian), head I, Theorem 1, and the following exercise; Theorems 3, 4, section 9 about homogeneous spaces.

For English, the corresponding theory is, hopefully, given in [Bo].

In terms of the manifold of ellipsoids ([Ar1], Appendix 10), this statement (4.1) about a smooth fibration is expressed as follows. In the parameterization of a neighbourhood of a matrix $D \in \text{MDH}$ by the pair (g, spe) (see Section 2), the part spe corresponds to the plane Π in the space of diagonal matrices. And under infinitesimal rotations g (close to E) of the basis of the major ellipsoid axes, each axis turns in a direction orthogonal to this axis.

7 Addition to the theorem for the case of $n = 3$

Recall that $\mathcal{M}_s = \{A \in \mathcal{M} \mid \text{trace}(A) = s\}$.

Theorem DS3.

(1) Singular points on \mathcal{M} are the scalar matrices and only them.

(2) \mathcal{M} is the straight cylinder over \mathcal{M}_0 (Theorem DS (2)). Therefore, it is sufficient to describe the surface \mathcal{M}_0 . This 3-dimensional surface resides in a 5-dimensional space and has the following structure.

2.1. \mathcal{M}_0 is the cone with zero as vertex over the orbit of the matrix $\text{diag}(1, 1, -2)$. This 2-dimensional orbit resides on a 4-dimensional sphere having center in zero.

2.2. The orbit of each non-scalar matrix in \mathcal{M} is a 2-dimensional algebraic surface residing on a 4-dimensional sphere. It is diffeomorphic to the projective plane $\mathbb{RP}(2)$.

(3) The orbit of each matrix in \mathcal{M} is obtained from the orbit of the matrix $D = \text{diag}(1, 1, -2)$ by some scalar shift and some homothety.

(4) The set of diagonal matrices in \mathcal{M}_0 consists of the three straight lines which intersect in zero and can be parameterized respectively as $\text{diag}(\lambda, \lambda, -2\lambda)$, $\text{diag}(\lambda, -2\lambda, \lambda)$, $\text{diag}(-2\lambda, \lambda, \lambda)$.

Remark. A natural smooth parameterization of \mathcal{M}_0 in some neighbourhood of each form $0 \neq A$ in \mathcal{M}_0 is the fibering over the line $\overline{0A}$ with the fiber being the orbit of a matrix from this line, and with this orbit intersecting orthogonally this line.

All the remaining discourse serves the proof of the theorems DS, DS3.

7.1 Investigating an orbit by orthogonal changes

Lemma (AlgSO) (1) The tangent space to the surface $SO(n)$ in the point E (the Lie algebra $so(n)$) consists of all the antisymmetric matrices.

(2) The tangent space to the orbit of a diagonal matrix D in the point D consists of the commutators of D with all the antisymmetric matrices.

This lemma a) is known, b) is an exercise for the reader, c) is simply derived in Appendix (AlgSO).

Lemma (Orbit2).

(1) The orbit of any non-scalar matrix in $\text{Sym}(n)$ intersects orthogonally the plane of *diagonal* matrices.

(2) Orbit of any matrix in $\text{Sym}(n)$ resides in the hyper-plane orthogonal to the line Scal .

(3) Orbit of any matrix in \mathcal{MH} is a smooth, compact, algebraic, connected surface of dimension $\dim SO(n) - 1$.

Proof.

Proving (1) Due to the symmetry by the group action, and by Theorem DO, it is sufficient to prove this statement for the orbit of any diagonal matrix D . By Lemma AlgSO, the tangent space T to the orbit of D in the point D consists of commutators of D with all antisymmetric matrices. Hence, by Lemma A0, each matrix M in T has zero diagonal. Then, it follows from the definition of the s-product (Section 3) that $\langle_s D_1, M \rangle = 0$ for each diagonal matrix D_1 . *The statement (1) is proved.*

Proving (2) Let $A \in \text{Sym}(n)$. If A is scalar, then its orbit is $\{A\}$, and the statement (2) is satisfied. It remains the case of a non-scalar A . Denote Π^* the hyper-plane orthogonal to Scal and containing A . Let A_0 be the base of this projection of A , that is the intersection point of Π^* with Scal . By Lemma S, each orthogonal change is an isometry in $\text{Sym}(n)$. And each point in Scal is a fixed point with respect to these changes. Hence, as the vector $\overline{A_0 A}$ is orthogonal to Scal , the vector $\overline{A_0 (g^c A)}$ is also orthogonal to Scal for each $g \in SO(n)$. *This proves the statement (2).*

Proving (3) First, let us present a finite polynomial system defining the orbit of a matrix A from \mathcal{M} . Let $c_i(X)$ be the coefficient in the monomial of λ^i in the polynomial $\text{charPol}_X(\lambda)$ for an arbitrary symmetric matrix X (c_i is a polynomial in the variables $x_{i,j}$ — elements of X). The value $c_i(A)$ for each concrete symmetric matrix A (consisting of real numbers) is expressed by the Viète formula via $\text{spec}(A)$:

$$c_i(A) = \text{sgn}(i) \cdot \text{elSym}_i(\text{ev}(A)), \quad 0 \leq i \leq n-1.$$

Here $\text{ev}(A)$ is the number sequence made of $\text{spec}(A)$, with repetition of multiple values,

elSym_i is the elementary symmetric polynomial of degree $n-i$ in the variables y_1, \dots, y_n (in this formula the real eigenvalues are substituted for the variables y_j in elSym_i),

$\text{sgn}(i)$ is the appropriate sign: plus or minus.

Therefore, $\text{Orbit}(A)$ is defined by the following equations for unknown matrix $X \in \text{Sym}(n)$:

$$c_i(X) = \text{sgn}(i) \cdot \text{elSym}_i(\text{ev}(A)), \quad 0 \leq i \leq n-1 \quad (1).$$

Here each lefthand side is the fixed polynomial c_i (in the variables X) defined above, and the righthand side is a real number — the value of the Viète expression at the spectrum of A .

Let us prove that the system (1) defines the set $\text{Orbit}(A)$. If $X \in \text{Orbit}(A)$, then $\text{spec}(X) = \text{spec}(A)$, this defines the characteristic polynomial for X , and X satisfies (1). Reversely, if for $X \in \text{Sym}(n)$ the equations (1) on X are satisfied, then $\text{charPol}(X) = \text{charPol}(A)$, and so, $\text{spec}(X) = \text{spec}(A)$. By the statement 5 of Lemma Orbit1 of the Section 3, the orbit of A consists exactly of the forms in \mathcal{M} which spectrum is $\text{spec}(A)$. Hence, $X \in \text{Orbit}(A)$.

The statement about algebraic equations for the orbit is proved.

Now, prove the remaining part of the statement (3) of the lemma

7.1.1 Further discourse on the orbit

The map $\text{DO}_A : \text{SO}(n) \rightarrow \text{Orbit}(A)$, $\text{DO}_A(g) = g^c A$ is a polynomial map of algebraic surfaces. By the definition of orbit, DO_A is surjective. As $\text{SO}(n)$ is compact and connected, and DO_A is continuous, $\text{Orbit}(A)$ is compact and connected.

Now, irreducibility of $\text{Orbit}(A)$ follows from

- a) that $\text{SO}(n)$ is irreducible ([Bo], head I, Section 1.2, first Proposition),
- b) that DO_A is a surjective map onto $\text{Orbit}(A)$, c) Lemma A2.

So: $\text{Orbit}(A)$ is an irreducible, connected, compact, algebraic surface contained in some sphere in the hyper-plane Π^* . The minimal dimension of embracing plane for $\text{Orbit}(A)$ depends on the multiplicities in $\text{spec}(A)$.

Let us describe a smooth parameterization of the orbit for the form $A \in \mathcal{MH}$. Let $\text{Bas} = \{e_1, \dots, e_n\}$ be an eigenvector basis for $A \in \mathcal{MH}$, and let e_1 and e_2 belong the (unique) multiple eigenvalue for A . By Lemma Stab, the connected component of unity in the stabilizer $\text{St}(A)$ is the group of rotations about 0 of the plane $V_1 = L(e_1, e_2)$ (taking in account that each operator from this component is identical on the orthogonal complement to V_1).

$\text{St}(A)$ is a Lie subgroup of dimension 1, and a homogeneous space $\text{SO}(n)/\text{St}(A)$ of the left congruence classes by $\text{St}(A)$ is a smooth manifold of dimension $\dim \text{SO}(n) - 1$. Respectively, $\text{Orbit}(A)$ is a smooth and closed sub-manifold in $\text{Sym}(n)$ of dimension $\dim \text{SO}(n) - 1$ locally diffeomorphic to $\text{SO}(n)/\text{St}(A)$. The theory for these three latter statements about the stabilizer and the orbit is referenced in the Remark 2 after Theorem DS.

Due to the action of $\text{SO}(n)$, small neighbourhoods of different points in $\text{Orbit}(A)$ are isometrical. More precisely, for each points A and B in this orbit there exists a neighbourhood of A in the orbit which is mapped onto the corresponding neighbourhood of B

in the orbit by the change of some operator from $SO(n)$. This provides a homogeneous smooth atlas parameterizing the orbit.

Lemma Orbit2 is proved.

So, Theorem DS is proved. The proof parts are:

- (1): irreducibility (Theorem “Irreducibility”),
- (2): the structure of \mathcal{M} of a straight cylinder (Lemma Orbit1, point 8),
- (3) — Lemma Orbit1, point 5,
- (4.1): the orbit of any matrix in \mathcal{MH} orthogonally intersects the space of diagonal matrices (Lemma Orbit2, point 1),
- (4.2) — Lemma Orbit2,
- (5) (surjectiveness of DO) — in the proof of Theorem “Irreducibility”.

7.2 Orbit structure details in the case of $n = 3$

Lemma (ST). In the case of $n = 3$, the following holds for the orbit of any non-scalar matrix A in \mathcal{M} .

- (1) It is a bi-dimensional, algebraic surface residing on a 4-dimensional sphere and diffeomorphic to the projective plane $\mathbb{RP}(2)$.
- (2) It contains the union O_{cir} of a smooth single-parameter family of circumferences of the same radius having a common point A (we guess that $\text{Orbit}(A) = O_{cir}$, but do not prove, so far).
- (3) It is not contained in any four-dimensional plane.

Proof.

The statement of that $\text{Orbit}(A)$ is algebraic is proved in Lemma Orbit2.

By the point 2 of Lemma Orbit2, $\text{Orbit}(A)$ resides in the 5-dimensional plane — the orthogonal complement to the line Scal . The action of $SO(V)$ preserves the s-metric (Lemma S). This adds the equation of a sphere for an arbitrary matrix in $\text{Orbit}(A)$, and it turns out that this orbit resides on a four-dimensional sphere (by the s-metric).

Let us prove the statement about a diffeomorphism from $\mathbb{RP}(2)$ onto the orbit. Due to the action of $SO(3)$, and by the statement (7) of Lemma Orbit1, it is sufficient to prove this for the diagonal matrix $D = \text{diag}(1, 1, -2)$ (of zero trace).

Represent $\mathbb{RP}(2)$ as the smooth variety of axes in the 3-dimensional space V , and define the map

$$s : \mathbb{RP}(2) \rightarrow \text{Orbit}(D), \quad s(l) = \text{Rt}(e_3, l)^c D.$$

Here 1) the axis e_3 belongs the eigenvalue -2 of D ,

2) l is an arbitrary axis in V — an element of the projective plane,

3) $g = \text{Rt}(e_3, l)$ is the operator of rotation of V from the axis of e_3 to the axis l about the normal to the plane $L(e_3, l)$, and we put $g = E$ for the case of $l = e_3$,

4) g^c is the change in $\text{Sym}(3)$ by this rotation.

By this construction, $g \in SO(3)$, so the image of s resides on the orbit of D . And we need to prove that s is a diffeomorphic imbedding from the projective plane into

$\text{Sym}(3)$, onto the orbit.

First, s is smooth due to its formula — the matrix transposition and multiplication. Then, notice that 1) by the point (5) of Lemma Orbit1, each operator A on $\text{Orbit}(D)$ has the spectrum $\text{spe} = \{1, 1, -2\}$,

2) there exists an axis l for this A which belongs the eigenvalue -2 . Such an axis is unique, because otherwise, the value -2 occurs (by Theorem DO) multiple in $\text{spec}(A)$ — which contradicts the invariant value of the multiset spe .

Further, consider the operators $s(l_1), s(l_2)$ for any axes $l_1 \neq l_2$. These two operators have different axes belonging to the eigenvalue -2 . As it is proved earlier, such an axis is unique for each form in $\text{Orbit}(D)$. Hence, $s(l_1) \neq s(l_2)$, which proves *injectiveness* of the map s .

Let us prove surjectiveness of s . Let $A \in \text{Orbit}(D)$. It has the spectrum $\text{spe} = \text{spec}(A) = \{1, 1, -2\}$, and there exists a unique axis l for A belonging the eigenvalue -2 . Put $g = \text{Rt}(e_3, l)$, and let us prove that $s(l) = g^c D = A$ by investigating the following two cases.

1) $l = e_3$. Then, $g = E$, $Ae_3 = -2e_3$; the 2-dimensional eigenspace $V_1 = L(e_1, e_2)$ of the eigenvalue 1 is orthogonal to e_3 ; the operator A is identical on the space V_1 . Hence $A = D = g^c D = s(l)$.

2) $l \neq e_3$. $g e_3 = l$, and as g is orthogonal, $g L(e_1, e_2)$ is the orthogonal complement l^\perp to l . The axis l belongs to the eigenvalue -2 for A . Hence, by Theorem DO, l^\perp is the eigenspace of 1 for A . Hence the operator A has the matrix $\text{diag}(1, 1, -2) = D$ in the basis $(g e_1, g e_2, g e_3)$. This means $g^c D = A$, and this proves surjectiveness of s .

Remark.

1. The idea of parameterizing this orbit by the axis of the eigenvalue -2 has been communicated to me by S. Yu. Orevkov.

2. Even though the projective plane is not a sub-manifold in V , the above orbit — the image of the map s — is a smooth algebraic variety in $\text{Sym}(3)$.

The orbit also has certain remarkable metrical properties. Thus, let us show the orbit construction by a circumference family.

Proving (2) Consider first the 1-orbit of D by the axis e_1 . This is a circumference (residing in its bi-dimensional plane), which we denote Cir_1 . This is visible from the explicit formulae for a change in D by a rotation about e_1 . The general matrix of this 1-orbit is

$$\text{C1G} = e_1^\phi D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c^2 - 2b^2 & -3cb \\ 0 & -3cb & -2c^2 + b^2 \end{pmatrix},$$

where $c = \cos \phi$, $b = \sin \phi$. The only diagonal matrices in this circumference are D and $D_2 = \text{diag}(1, -2, 1)$. When the angle ϕ changes from zero to $\pi/2$, the matrix C1G passes (without repetition) the arch from D to D_2 . At this stage, the matrix element $\text{C1G}(2, 3)$ first changes monotonously from zero to $-3/2$, and then changes monotonously back to zero. When the angle ϕ changes from $\pi/2$ to π , the matrix C1G passes (without

repetition) the arch from D_2 to D ; this is the “lower” half of the 1-orbit, it differs from the upper half only in the sign of the element $C1G(2, 3)$.

The subspace L_1 of the 1-orbit Cir_1 is linearly generated by the vector $\overline{D D_2}$ and the tangent vector to Cir_1 in the point D . After norming by the factor -3 , these two vectors produce the basis of the following three matrices for the space L_1 :

$$M_1 = \text{diag}(0, 1, -1), \quad M_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Now, let l be the axis in the space $L(e_1, e_2)$ obtained from e_1 by a rotation $h = \text{RtA}(e_3, \psi)$. Then the 1-orbit $\text{Cir}(\psi)$ for D by l is a circumference containing D and isometric to Cir_1 ; this isometry is the operator h^c (in the space $\text{Sym}(3)$), and it also maps the plane L_1 to the plane $L(\psi)$ of the 1-orbit $\text{Cir}(\psi)$. This property holds due to that the operator h^c conjugates the subgroups $G(e_1)$ and $G(l)$. Therefore, for these two 1-orbits it holds the equation

$$\text{Orbit}(l)D = h^c \text{Orbit}(e_1)(h^{-c}D),$$

where $\text{Orbit}(l)A$ denotes the orbit of A by the changes by rotations about the axis l . But in this case, h is a rotation about e_3 , and $D = \text{diag}(1, 1, -2)$. Hence h commutes with the operator D , and the relation between these two 1-orbits is simplified:

$$\text{Orbit}(l)D = h^c \text{Orbit}(e_1)D.$$

That is these 1-orbits are conjugated by h .

So, it appears that the orbit for D contains the union of a smooth single-parameter family of circumferences having a common point D and obtained from Cir_1 by various rotations about D .

The point (2) is proved.

Let us also note (informally, for intuition) that the plane $\Pi(\psi)$ of 1-orbit in this family turns dependently on ψ in various directions in a five-dimensional space.

Proving (3) We need to prove that the orbit of the above matrix D is not contained in any four-dimensional plane. The simplest way to see this is to consider any embracing plane for the orbit of $D = \text{diag}(0, 0, 1)$. Because by the statement (7) of Lemma Orbit1, the latter orbit differs metrically from the former only by some scalar shift and homothety by some non-zero factor.

It is easy to compute that the tangent space to the orbit in the point D is generated by the vectors

$$T_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

These are the results of changes in D by infinitesimal rotation about e_1 and about e_2 respectively, they are obtained by commuting the matrix D with two (anti-symmetric) matrices from a basis of the Lie algebra $\text{so}(3)$.

Also the orbit contains the matrices $D_2 = \text{diag}(0, 1, 0)$ and $D_3 = \text{diag}(1, 0, 0)$. Respectively, the tangent space in D_3 contains the vector

$$B = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Denote $D_2' = D_2 - D$, $D_3' = D_3 - D$.

If a plane L contains the orbit for D , then L contains the point D and also the directions T_1 , T_2 , B , $D_2' = \text{diag}(0, 1, -1)$, $D_3' = \text{diag}(1, 0, -1)$. The vectors T_1 , T_2 and B are mutually orthogonal (recall the s-metric). The vector D_i' is orthogonal to T_1 , T_2 and B for $i = 2, 3$, and the vectors D_2' and D_3' constitute a staircase matrix of rank 2. So: the above five directions are linearly independent and belong to each plane containing $\text{Orbit}(D)$.

The lemma is proved.

7.3 Proof for the structure of \mathcal{M} for $n = 3$

By statement (8) of Lemma Orbit1, \mathcal{M} is the straight cylinder over the surface restriction \mathcal{M}_0 . It remains to describe the geometric structure of \mathcal{M}_0 . Each matrix in \mathcal{M}_0 has spectrum of the kind $\{\lambda, \lambda, -2\lambda\}$, because the three eigenvalues satisfy the conditions of zero sum and existence of a multiple value. Therefore, by Theorem DO, the diagonal matrices in \mathcal{M}_0 are all the matrices of the kind $D(\lambda) = \text{diag}(\lambda, \lambda, -2\lambda)$ or $D_2(\lambda) = \text{diag}(\lambda, -2\lambda, \lambda)$ or $D_3(\lambda) = \text{diag}(-2\lambda, \lambda, \lambda)$.

The statement (4) of Theorem DS3 is proved.

Remark. 1) For each λ , the matrices $D(\lambda)$, $D_2(\lambda)$ and $D_3(\lambda)$ belong to the same orbit (by the statement 5 of Lemma Orbit1).

2) The points $D(1)$ and $D(-1)$ are central-opposite to each other on the sphere S_4 (to which $D(1)$ belongs), but they belong to different orbits.

The conic set \mathcal{M}_0 is union of the “positive” half-cone — union of orbits of the family $D(\lambda)$ for $\lambda \geq 0$, and the “negative” half-cone — union of orbits of the family $D(\lambda)$ for $\lambda \leq 0$.

3) Intersection of \mathcal{M}_0 with the sphere S_4 (to which $D(1)$ belongs), is $\text{Orbit}(D(1)) \cup \text{Orbit}(D(-1))$.

For any form $A \in \mathcal{M}_0$ $\text{spec}(A)$ is proportional to the multiset $\{1, 1, -2\}$ with a factor λ depending on A . Therefore, $\text{Orbit}(A) = \text{Orbit}(\lambda(A)D) = \lambda(A) \cdot \text{Orbit}(D)$. Here $D = \text{diag}(1, 1, -2)$, $\lambda(A)$ are all real numbers, and the last equality is true due to that $g^c \lambda D = \lambda g^c D$ for all invertible $g \in \mathcal{L}(n)$.

Therefore, \mathcal{M}_0 is the cone with its vertex in zero and its base being $\text{Orbit}(D)$. And by Lemma ST of Section 7.2, $\text{Orbit}(D)$ is a diffeomorphic image of the projective plane, and it resides on the sphere S_4 with center in zero.

Theorem. Zero matrix is the only singular point on the surface \mathcal{M}_0 .

Proof.

Any non-zero form $A \in \mathcal{M}_0$ belongs to \mathcal{MH} . By [Ar1] (see Section 2 of this paper), such A is a regular point on \mathcal{M} . As \mathcal{M} is a straight cylinder over \mathcal{M}_0 , A is also regular on \mathcal{M}_0 .

Why zero is singular? In our case, the cone base is a 2-dimensional surface injected in a complex way into a 4-dimensional sphere. So, we cannot see singularity of the vertex by simply referring to the picture of a straight circular cone. Let us prove singularity by finding the enough number of linearly independent tangent vectors.

The vertex (0) is singular because 1) \mathcal{M}_0 is a conic set with vertex in zero, 2) we present certain four forms A_i on the orbit of $D = \text{diag}(1, 1, -2)$ such that their axes $\overline{0A_i}$ are linearly independent (and reside in \mathcal{M}_0).

The two of these forms are $D = \text{diag}(1, 1, -2)$ and $D_2 = \text{diag}(1, -2, 1)$. The orbit of D contains the 1-orbit of D by the axis e_1 . This is a circumference containing the matrices D and D_2 , so that D corresponds to the rotation angle 0, and D_2 corresponds the angle $\pi/2$. We also need a couple of non-diagonal matrices on the orbit. Thus, $e_1^{\pi/4}D$ is the matrix

$$M_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & b & -a-1 \end{pmatrix}, \quad a = -1/2, \quad b = -3/2$$

(recall that e_i^ϕ is the change by the rotation operator $\text{RtA}(e_i, \phi)$).

Similarly, the 1-orbit for D by e_2 contains the matrix

$$M_2 = \begin{pmatrix} a & 0 & b \\ 0 & 1 & 0 \\ b & 0 & -a-1 \end{pmatrix}$$

— with the same a and b . To verify linear independence of the symmetric matrices D , D_2 , M_1 , M_2 , represent each of them in the row-vector form: $[X(i, j) | 1 \leq i \leq j \leq 3]$. The sub-diagonal part of each matrix is skipped in this representation, because it equals the super-diagonal part. This produces one matrix of the four rows:

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & -2 \\ 1 & 0 & 0 & 0 & -2 & 1 \\ 1 & 0 & 0 & a & b & -a-1 \\ a & 0 & b & 1 & 0 & -a-1 \end{pmatrix}.$$

The first loop of clearing of the first column by the staircase form algorithm produces the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & -1 & -2 & 3 \\ 0 & 0 & 0 & a-1 & b & -a+1 \\ 0 & 0 & b & 1-a & 0 & a-1 \end{pmatrix}.$$

Further, the fourth row is moved to the second place, and to the third row it is added the second row multiplied by $a-1$. This produces the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & b & 1-a & 0 & a-1 \\ 0 & 0 & 0 & -1 & -2 & 3 \\ 0 & 0 & 0 & 0 & b_2 & a_2 \end{pmatrix}.$$

Here $b \neq 0$, $b_2 = b - 2(a - 1) = -3/2 - 2(-3/2) \neq 0$. Hence this is a staircase matrix of rank 4, and hence the matrices D , D_2 , M_1 , M_2 are linearly independent, and their corresponding axes are linearly independent. So, we present the four linearly independent axes on the surface \mathcal{M}_0 which intersect in zero. Therefore, zero is a singular point in \mathcal{M}_0 . Let us explain — why. It is proved earlier that all other points of this surface are regular. So, the tangent space to each non-zero point on \mathcal{M}_0 has dimension 3. To prove by contradiction, suppose that zero is regular on \mathcal{M}_0 . Then, by the definition of a regular point, some neighbourhood of zero in \mathcal{M}_0 is diffeomorphic to a ball in \mathbb{R}^3 . Due to its dimension, such a neighbourhood cannot contain the above four linearly independent axes coming out from zero. This contradiction shows that zero is a singular point in \mathcal{M}_0 .

The theorem is proved.

So, 0 is the only singular point on \mathcal{M}_0 . By Theorem DS, \mathcal{M} is the straight cylinder over \mathcal{M}_0 with the line Scal as element. This proves the last remaining statement in Theorem DS3:

singular points on \mathcal{M} are the scalar matrices and only them.

Theorem DS3 is proved.

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8 Appendix

The proofs in Appendix do not use the statements being proved in the main part of the paper.

All the statements presented in Appendix are evident or known, their proofs are given for a formal reason.

Lemma (Stab). Let $M \in \mathcal{MH}$.

1) The set V_1 of the vectors belonging to the multiple eigenvalue (which is unique) for M is a subspace of dimension 2.

2) The connected component G_e of unity in the stabilizer $\text{St}(M)$ in $\text{SO}(n)$ is a single-parameter subgroup of the operators which restriction on V_1 are rotations of this 2-dimensional space about 0, and which restriction to the orthogonal complement to V_1 is identity.

Proof.

Let λ be the (unique) multiple eigenvalue (of the multiplicity 2) of the operator M . By Theorem DO, there exists a basis B in the space V consisting of the eigenvectors of M , and there are exactly two vectors in B belonging to the eigenvalue λ . Denote these vectors u and v ; denote $B_2 = B \setminus \{u, v\}$, $V_1 = L(u, v)$, $V_2 = L(B_2)$.

The condition $g \in \text{St}(M)$ means that g commutes with M : $gM = Mg$. Therefore, if a vector v belongs to any eigenvalue μ for M , then it holds $Mgv = gMv = g\mu v = \mu gv$. Hence, each eigenspace for M is mapped by g on itself.

Let g_1 and g_2 be the restriction of g to V_1 and to V_2 respectively.

Each vector in B_2 belongs to a 1-dimensional eigenspace for M , hence such a vector is an eigenvector for g . Further, an operator from $\text{SO}(n)$ can have only the eigenvalue 1 or -1. Therefore, in the basis V_2 the operator g_2 has a diagonal matrix with each diagonal value being 1 or -1.

Further, g maps the subspace V_1 onto itself, and it is an orthogonal operator (g_1) on V_1 . The operator M is scalar on V_1 , hence its restriction to V_1 commutes with each linear operator on V_1 .

$\det(g) = 1$. Hence, if the multiplicity m of -1 in $\text{spec}(g_2)$ is even, then g_1 is oriented positively, and g_1 is a rotation about 0 at some angle. If m is odd, then g_1 is oriented negatively, and g_1 is a rotation about 0 composed with some reflection mapping.

$\text{St}(M)$ is an algebraic Lie group, as it is widely known (our reference for the Russian language is [VinO], head I, Theorem 1, and the following exercise). It consists of several connected components. Let H be the set of operators from $\text{St}(M)$ which are identical on V_2 . Then $H \subseteq G_e$, because it is proved that for each operator from H its restriction on V_1 is a rotation of a 2-dimensional plane about 0, and the set of such operators contains unity, is connected and smoothly parameterized by the angle of rotation.

It is proved above that each operator g in $\text{St}(M)$ has on V_2 in the basis B_2 a diagonal matrix $D(g)$ with each diagonal value being 1 or -1. The set of such matrices $D(g)$ is finite and discrete. Therefore, for each operator g in the connected component G_e the

operator g_2 has the only eigenvalue 1, hence g_2 is identical. Therefore $H = G_e$ — the connected component of unity.

The lemma is proved.

Lemma (AlgSO) (known).

(1) The tangent space T to the surface $\text{SO}(n)$ in the point E (the Lie algebra $\text{so}(n)$) consists of all the antisymmetric matrices.

(2) The tangent space T to the orbit of a diagonal matrix D by the changes of $\text{SO}(n)$ in the point D consists of the commutators of D with all the antisymmetric matrices.

Proof. Denote $G = \text{SO}(n)$.

Proving (1) By definition, $\text{so}(n)$ consists of the derivatives $g'(0)$ at zero of all smooth curves $g(t)$ on the surface G , such that $g(0) = E$. By the property of the transposed orthogonal matrix, it holds $g(t)g(t)^* = gg^{-1} = E$. Differentiating this equation at $t = 0$, taking in account that transposing a matrix is a linear map, and by the rule of derivative of a product of linear operators depending on t , we obtain:

$$0 = (g(t)g(t)^*)'(0) = g'(0)g(0)^* + g(0)g'(0)^* = g'(0) + g'(0)^* = 0.$$

This proves the statement (1) of that an arbitrary matrix $(g'(0))$ in $\text{so}(n)$ is antisymmetric.

Proving (2) By definition, T consists of the derivatives at $t = 0$ of all the curves of the kind $\gamma(t) = g(t)^c D$, where $g(t)$ is a smooth curve on G such that $g(0) = E$. By the rule of derivative of a product of linear operators depending on t , we have

$$(d/dt \gamma)(0) = d/dt (g(t) D g(t)^*)(0) = g'(0) D g(0)^* + g(0) D g'(0)^* = g'(0)D + D g'(0)^* \quad (1).$$

Differentiation at zero of the true equation $g(t)g(t)^* = E$ (for an orthogonal g) gives the equation $g'(0)^* = -g'(0)$. Therefore, the equation (1) is equivalent to $(d/dt \gamma)(0) = [g'(0), D]$ — a commutator of two matrices. In our case, $g'(0)$ is an arbitrary element of the Lie algebra $\text{so}(n)$. By the statement (1), this algebra consists of all the antisymmetric matrices.

The lemma is proved.

Lemma A0 (probably, known).

For a diagonal quadratic matrix D and an anti-symmetric matrix A of the same size, the commutator $[D, A]$ is a symmetric matrix having zero diagonal.

Proof.

Symmetry of the result matrix is proved by the following equations:

$$(DA - AD)^* = (DA)^* - (AD)^* = A^* D^* - D^* A^* = (-A)D - D(-A) = DA - AD.$$

Further, it is sufficient to prove that (a) DA has zero diagonal and (b) AD has zero diagonal. Also (a) implies (b), because $AD = -(DA)^*$, and the maps of transposition and negation $-M$ for any matrix M preserve the property of having zero diagonal. By the matrix product definition, for each position i on the diagonal it holds

$$(DA)_{i,i} = \sum_{j=1}^n D_{i,j}A_{j,i} = D_{i,i}A_{i,i} = 0.$$

This is because $D_{i,j} = 0$ for $i \neq j$, and $A_{i,i} = 0$.

The lemma is proved.

Illustration: for

$$D = \text{diag}(a, b), \quad A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

it holds $[A, D] =$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \times \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} - \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \times \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & b \\ -a & 0 \end{pmatrix} - \begin{pmatrix} 0 & a \\ -b & 0 \end{pmatrix} = \begin{pmatrix} 0 & b-a \\ b-a & 0 \end{pmatrix}.$$

Lemma A1 (known).

For any polynomial map $F : R^n \rightarrow R^m$, the counter-image \mathcal{M}' of an algebraic set \mathcal{M} is an algebraic set.

Proof. Let \mathcal{M} be the set of zeroes for a polynomial system $\{p_1, \dots, p_k\}$. Then \mathcal{M}' is the set of zeroes for the set of compositions $\{p_1(F(X)), \dots, p_k(F(X))\}$. Indeed, if A belongs to \mathcal{M}' , then $F(A)$ belongs to \mathcal{M} ; then $p_i(F(A)) = 0$ for all i .

Conversely: if $p_i(F(A)) = 0$ for all i , then $F(A)$ belongs to \mathcal{M} , hence A belongs to \mathcal{M}' (by the definition of counter-image of a set).

Lemma A2 (known).

For a surjective polynomial map from an algebraic set \mathcal{M} onto an algebraic set \mathcal{M}' , if \mathcal{M} is irreducible, then \mathcal{M}' is irreducible.

Proof.

To prove by contradiction, suppose that $\mathcal{M}' = M_1 \cup M_2$ for algebraic sets M_1 and M_2 , each different from \mathcal{M}' . By Lemma A1, the counter-images C_1 and C_2 of M_1 and of M_2 respectively are algebraic sets. Also it holds $\mathcal{M} = C_1 \cup C_2$. Further, $C_1 \neq \mathcal{M}$, because $M_1 \neq \mathcal{M}'$. For a similar reason, $C_2 \neq \mathcal{M}$. So, this representation $\mathcal{M} = C_1 \cup C_2$ contradicts irreducibility of \mathcal{M} .

The lemma is proved.