# Irreversibility factor and limiting performance of financial systems (thermodynamic approach)

*Anatoliy M. Tsirlin*
*Program System Institute, Russian Academy of Science,*
*set."Botic",Perejaslavl-Zalesky, Russia 152020. e-mail: tsirlin@sarc.botik.ru*
*and Vladimir Kazakov*
*School of Finance and Economics, University of Technology, PO Box 123,*
*Broadway NSW 2007 Australia, e-mail:Vladimir.Kazakov@uts.edu.au*

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## 1 introduction

Thermodynamic approach to the modelling of economic systems has been developed by Samuelson ([1]), Lichnerovich ([2]), Rozonoer ([3]), Martinesh ([4]) end others ([5],[6]). The majority of these works relied on reversible thermodynamics' analogy.

New branch of thermodynamics - Finite Time Thermodynamics (FTT) - has emerged in last decades. It studies limiting possibilities of thermodynamic systems (heat engines, separation systems etc.) caused by constraints on processes' duration and rates ([7]– [8]). Microeconomic analogies of these problems were studied in ([9]),([10]), ([11])).

Financial systems are similar to microeconomic systems. Their distinguishing characteristic is the use of credit as one of traded assets. In this paper the approach, developed in ([11]), is applied to financial systems.

Many problems considered in this paper can be solved without using thermodynamic approach. But this approach allows us to consider all these problems within unified conceptual and methodological framework. We expect that further development of this approach and its application on micro rather than on macro level will lead to discoveries of many thermodynamic analogies in financial systems including such as Onsager conditions, self-organization processes, etc.

We consider financial systems where financial assets (stocks, bonds, currencies, derivatives, credits, etc.) are traded. Each financial system consists of subsystems. We shall call them financial agents (FAs). FA estimates how valuable is the asset for it by constructing this asset's price estimate $P$. This number represents the minimal price at which FA is willing to sell this asset and the maximal price at which it is willing to buy it. This estimate depends on FA's stock of this asset $V$ and its current net capital $M$. For a credit this estimate $P$ represents the minimal rate at which the FA is willing to provide credit loan to somebody and the maximal rate FA is willing borrow. The will classify all FA as members of one of three classes depending on how they construct these estimates:

1. FA with infinite internal capacity with respect to all or some of the assets. The estimates for these FAs do not depend on their capital and stock of assets. These estimates can change under the influence of external factors but not under the influence of the internal exchange processes.

2. FA with finite internal capacity where the price estimates of assets $P$ and capital $r$ depend on its capital $M$ and its stock of assets $V$

$$P = P(V, M), \quad r = r(V, M). \tag{1}$$

3. Financial intermediary (bank) that sets up prices for buying and selling and credit rates for deposits and loans. It does it in order to achieve its economic objectives.

If there is no exchange of assets/capital between the financial system and the environment then its is closed. If there is such exchange then it is open. In this paper we study optimal behaviour of banks. We use methodologies of FTT and irreversible microeconomics.

In section one we consider systems interaction between a bank and FA in stationary and non-stationary conditions. In section two we introduce the notions of profitability of a closed system, of profitability's dissipation and of irreversibility of exchange processes in financial systems. In section three we study limiting possibilities of intermediaries in open financial systems with and without competition and when assets' estimates are stochastic variables.

When a contact between two financial agents is established, assets/capital are exchanged between FAs. Because FAs establish contact voluntarily, they do that if and only if each participating FA increases it's wealth function $S(V, M)$. Exchange kinetics $m_{ij}$ is described by the dependence of the flow of assets between $i$-th and $j$-th FAs on the vectors of their price estimates. If these two vectors are the same then $m_{ij} = 0$. The estimates' differential is the "driving force" of the exchange processes. The existence of the wealth

function for the scalar case was proven in ([3]) and for the vector case in ([11]). This function is assumed to be continuously differentiable, convex and increasing on $M$ and on each component of vector $V$.
The wealth function relates to the price estimates as

$$P = \frac{\partial S}{\partial V_i}, \qquad r = \frac{\partial S}{\partial M}. \tag{2}$$

From the existence of the wealth function the following equations follow

$$\frac{\partial^2 S}{\partial V_i \partial M} = \frac{\partial P_b^0}{\partial M} = \frac{\partial r}{\partial V_i} = \frac{\partial^2 S}{\partial M \partial V_i}. \tag{3}$$

That is, **the sensitivity of the estimate of $i$-th type of asset to the changes of FA's capital is equal to the sensitivity of the capital estimate to the changes of stock of this asset**. Equations (3) are economic analogies of Maxwell thermodynamic equations.

# 2 Optimal bank strategy in a system with one FA

## 2.1 Buying and selling of assets

Consider a system that includes a finite capacity FA and a bank. We assume that the bank is required to buy the given amount $G$ of assets in the given time $\tau$. It minimizes its expenses by controlling the price it offers to FA

$$\begin{array}{c} \dot{V} = -m(P_b^0, P), \quad V(0) = V_0, \quad V(\tau) = V_0 - G, \\ \dot{M} = Pm(P_b^0, P), \quad M(0) = M_0. \end{array} \tag{4}$$

where $P(t)$ is the price paid for the asset and $P_b^0$ is asset's price setimate by FA. The optimality criterion here is

$$I = \int_0^\tau m(P_b^0, P) P dt \to \max_{P(t)} \tag{5}$$

The conditions of optimality of the problem (4),(5), were obtained in ([12]) and have the following form

$$\frac{d}{dV}\left[\frac{\frac{\partial g}{\partial P}}{g^2(P_b^0, P)}\right] = \frac{\frac{\partial g}{\partial P_b^0}\frac{\partial P_b^0}{\partial M}}{g^2(P_b^0, P)}. \tag{6}$$

In the particular case of estimate $P_b^0$ independent on the FA's $M$ and dependent only on its stock of assets $V$, the price is such that

$$\frac{\frac{\partial g}{\partial P}}{g^2(P_b^0, P)} = const, \tag{7}$$

despite the changes of $P_b^0$ caused by changes of $V$.
In even more particular case when the flow is independent on $M$ and linear on price ($\frac{\partial g}{\partial P} = const$), then from condition (7) follows that the optimal price is such that the flow of purchases $g$ is constant and equal to $\frac{G}{\tau}$. One does not need here to know the dependence of $p_b^0$ on $V$ to find the optimal process.
The problem of selling given amount of assets in a given period of time with maximal profit has the same conditions of optimality.

### 2.2 Optimal lending and borrowing

Consider the same system where FA with finite capacity is a depositor to the bank. The bank wants to receive the given net volume of deposits $G$ in the period of time $\tau$. This deposits are not to be withdrawn until time $T$, $\tau << T$. The dependence of the credit estimate on the time of depositing here can be neglected

$$P_c^0 = P_c^0(M, T)$$

and the problem of minimal cost of borrowing (attracting deposits) takes the following form

$$\int_0^\tau m(P_c^0, P)dt = G, \tag{8}$$

$$\dot{M} = -m(P_c^0, P), \quad M(0) = M_0, \tag{9}$$

$$I = \int_0^\tau m(P_c^0, P)Pdt \to \min_{P(t)}. \tag{10}$$

Here $m$ is the flow of credits that depends on the bank interest rate for deposits $P$ and FA's credit estimate $P_c^0$. In its turn $P_0^c$ depends on the capital $M$ and term/duration of the deposit. We assume that the function $m$ is continuously differentiable on all arguments, positive for $P > P_c^0$ and equal zero for $P = P_c^0$. We can then rewrite this problem in terms on new independent variable $M$. Here

$$dt = -\frac{dM}{m(P_c^0, P)}. \tag{11}$$

The problem (8)-(10) takes the form

$$I = -\int_{M_0}^{M_0-G} P dM \to \min_{P(M)} \tag{12}$$

subject to constraints

$$-\int_{M_0}^{M_0-G} \frac{dM}{m(P_c^0, P)} = \tau. \tag{13}$$

For the non-degenerate solution ($\lambda_0 = 1$) of the problem (12), (13) the Lagrange function takes the form

$$L = P + \frac{\lambda}{m(P_c^0, P)}.$$

Its optimality condition on $P$ is

$$\frac{\partial L}{\partial P} = 0 \to 1 - \frac{\lambda}{m^2(P_c^0, P)} \frac{\partial m}{\partial P} = 0,$$

which yield (similarly to (7)) the conditions of optimality

$$\frac{m^2(P_c^0(M,T), P^*(M))}{\frac{\partial m}{\partial P}} = \lambda = const, \quad \forall M. \tag{14}$$

where $P^*(M)$ is the optimal interest rate for deposit.
For the linear dependence of the flow of deposits on the differential between the offered interest rate and credit estimate

$$m = \alpha(P - P_c^0)$$

the condition (14) takes the form

$$\alpha(P - P_c^0)^2 = const.$$

That is, differential between the rate, offered by the bank and the credit estimate, which evolves in accordance with equation (9) must be constant. From equation (8) it follows that

$$P^*(M) = P_c^0(M,T) + \frac{G}{\alpha\tau}, \tag{15}$$

the net capital of FA depends on time as

$$M(t) = M_0 - G\frac{t}{\tau}$$

and the minimal average credit interest rate is

$$I_{\min} = \frac{G}{\tau}\int_{M_0-G}^{M_0} P^*(M)dM. \quad (16)$$

If the bank wants to "sell" credit volume $G$ in the given period of time $\tau$ at the maximal-possible interest rate then it has to offer the rates that obey the same conditions of optimality. The only difference is that the second term in (15) changes sign and the flow $m$ also changes sign.
The term (duration) of the deposit/loan $T$ is an exogenous parameter of this problem.

### 2.3 Maximal profit from buying/selling assets when price estimates change

Consider FA with infinite capacity and price estimate $P_b^0$ that changes deterministically or stochastically under the influence of external parameters. In this situation the bank can generate a profit by buying assets from FA when their price estimates are low and accumulating some stock in a process and then by selling this stock when these price estimates are high. Let us find out the maximal profit and the corresponding optimal strategy here.
Suppose that the estimate $P_b^0(t)$ is known and we seek such $P(t)$ that corresponds to maximal increase of capital in the given period of time (when the assets are sold the bank's capital increases, when it is bought it decreases) The optimality criterion here is

$$I(\tau) = -\int_0^{\tau} m(P_b^0(t), P(t))P(t)dt \to \max_{P(t)}. \quad (17)$$

The change of the bank's capital is governed by the equation

$$\dot{M} = -M(P_b^0, P)P, \quad M(0) = M_0 \quad (18)$$

and the change of the stock of asset is governed by the equation

$$\dot{V} = m(P_b^0, P), \quad V(0) = V_0. \quad (19)$$

Assume that the bank does not accumulate assets but only resells them, $V(0) = V(\tau)$ or

$$\int_0^\tau m(P_b^0, P)dt = 0. \tag{20}$$

If bank has no capital it does not buy assets and if it does not hold assets it does not sell them, therefore

$$g(P_b^0(t), P(t)) \le 0, \quad \text{if} \quad M = 0, \quad g(P_b^0(T), P(t)) \ge 0, \quad \text{if} \quad V = 0. \tag{21}$$

The problem is reduced to maximization of the capital increase subject to constraints (18)-(21). The price offered by the bank $P(t)$ is the control variable and the stocks of assets and caital are state variables.

*Estimate of the maximal profit.*

First, we relax the problems constraints by assuming that the initial stock $M_0$ and capital $V$ are so large that the conditions (21) always hold. That is, that the bank has constant access to unlimited source of interest-free funds. Because the constraints (18), (19) are Lyapunov equations (that is, their r.h.s. do not depend on $M$ and $V$) they always hold if (21) holds. The maximal capital increase $I^*(\tau)$ in this problem gives an upper bound for the initial problem.

The Lagrange function for the problem (17), (21) takes the form

$$L(P_b^0, P) = -g(P_b^0, P)P + \lambda g(P_b^0, P) = g(P_b^0, P)(\lambda - P).$$

Its stationarity condition on $P$ is

$$\frac{\partial g(P_b^0, P)}{\partial P}(\lambda - P) - g(P_b^0, P) = 0,$$

that can be rewritten as

$$\frac{g(P_b^0, P)}{\frac{\partial g(P_b^0, P)}{\partial P}} + P = \lambda. \tag{22}$$

$\lambda$ from (20) is

$$\lambda = \frac{\int_0^\tau \frac{\partial g}{\partial P} P dt}{\int_0^\tau \frac{\partial g}{\partial P} dt}. \tag{23}$$

Its substitution into (22) yields the link between the optimal price $P^*(t)$ and $P_b^0(t)$

$$g(P_b^0, P^*) = \frac{\partial g}{\partial P} \frac{\int_0^\tau \frac{\partial g}{\partial P} P^* dt}{\int_0^\tau \frac{\partial g}{\partial p} dt - P^*}. \tag{24}$$

Consider linear flow of buying/selling with respect to the price differential

$$m(P_b^0, P) = \alpha(P - P_b^0), \tag{25}$$

where $\alpha$ is some positive constant. The condition (24) then takes the form

$$M = \frac{\alpha}{\tau}\int_0^\tau P dt - P(t).$$

We denote the average price over the interval $\tau$ as $\overline{P} = \frac{1}{\tau}\int_0^\tau P dt$. From (20) follows that $\overline{P} = \overline{P}_b^0$ and the optimal flow is

$$m^* = \alpha(P(t) - P_b^0(t)) = \alpha(\overline{P_0} - P(t)),$$

$$P^*(t) = \frac{1}{2}(\overline{P}_b^0 + P_b^0(t)) \tag{26}$$

and the optimal flow of asset is

$$g^*(P_b^0, P^*) = \alpha(P^*(t) - P_0(t)) = \frac{\alpha}{2}(\overline{P}_b^0 - P_0(t)) \tag{27}$$

Thus, the optimal flow of asset is proportional to the deviation of its price from the average price over the period $\tau$.
The upper bound on the rate of profit from trading of this asset is

$$J^* = \frac{I^*(\tau)}{\tau} = \frac{1}{\tau}\int_0^\tau -m^*(P_b^0(t), P^*(t))P^*(t)\, dt =$$

$$= \frac{1}{\tau}\int_0^\tau \frac{\alpha}{2}(P_0 - \overline{P}_0)\frac{1}{2}(P_0 + \overline{P}_0)\, dt = \frac{\alpha}{4\tau}\int_0^\tau (P_0^2 - \overline{P}_0^2)\, dt =$$

$$= \frac{\alpha}{4\tau}\int_0^\tau (P_0 - \overline{P}_0)^2\, dt = \frac{\alpha}{4} D_{P_0}. \tag{28}$$

The obtained results also hold if the estimate $P_b^0(t)$ is the stationary random process with expectation $\overline{P}_0$, variance $D_{P_b^0}$ and density $\mu(P_b^0)$ on the feasible set $D$ of $P_b^0$. After replacing time averaging with the set averaging, the expression for the optimal flow of assets (24) takes the form

$$m(P_b^0, P^*) = \left(\frac{\partial m(P_b^0, P)}{\partial P}\right)_{p^*} \left(\frac{\left[\frac{\partial m}{\partial P} P\right]}{\left[\frac{\partial m}{\partial P}\right]} - P^*\right). \tag{29}$$

The overline here denotes expectation of the corresponding expression on $P_b^0$. Thus,

$$\left[\frac{\partial m}{\partial P}\, P\right] = \int\limits_D \mu(P_b^0) \frac{\partial m(P_b^0, P)}{\partial P}\, P\, dP_b^0.$$

We see that (29) is an integral equation that relates $P^*$ and $P_b^0$ for each $t$. This relationship depends on the distribution of $P_0$. For the linear exchange kinetics (24) the equation (29) becomes much simpler. Here the optimal price for buying (selling) of assets depends on $\overline{P}_0$ only for any distribution of $P_b^0$. From (28) it follows that the maximal rate of profit here is

$$J^*(\tau) = \frac{\alpha}{4} D_{p_0}. \tag{30}$$

**Estimate of stock exhaustion**

Let us estimate how inaccurate this estimate (28) is. Assume that random process $P_b^0$ is a gaussian stationary process with the following the following correlation function

$$R_{P_b^0}(\tau) = \sigma^2 \exp[-\alpha|\tau|].$$

It is easy to show that for the linear dependence (25) the flow of buying for $P = P^*(P_b^0)$ is also Gaussian random process with the correlation function

$$R_{m^*}(\tau) = \frac{\alpha^2}{4} R_{P_b^0}(\tau) = \frac{\alpha^2\sigma^2}{4} \exp(-\alpha|\tau|).$$

This correlation function corresponds to the following spectral density

$$S_{m^*}(w) = \frac{2\alpha D_{P_b^0}}{\alpha^2 + w^2} = \frac{2\alpha \frac{\alpha^2\sigma^2}{4}}{\alpha^2 + w^2} = \frac{\alpha^3\sigma^2}{2(\alpha^2 + w^2)}.$$

The spectral density of the process $V(t)$, which is governed by equation (21) has the form

$$S_V(w) = \frac{\tilde{V_0} - S_m(w)}{w^2}.$$

The constant $V_0$ is the expectation of the random process $V(t)$ and is to be found from the condition of bounded variance. The variance of $V(t)$ is

$$D_A = \int\limits_0^\infty S_V(w) dw.$$

It is bounded for $\tilde{V_0} = \frac{\alpha\sigma^2}{2}$, and the corresponding value of $D_V$ is

$$D_V = \frac{\sigma^2}{4}. \tag{31}$$

The probability of stocks' exhaustion is the same as the probability that the negative deviation of $V(t)$ from $V_0$ is larger than the initial stock of asset

$$P(V(t) \le 0) = \int_{-\infty}^{+\infty} f_V dV = \frac{1}{\sqrt{2\pi}\sqrt{D_V}} \int_{-\infty}^{+\infty} \exp\left[-\frac{(V-\tilde{V_0})^2}{2D_V}\right] dV =$$

$$= \frac{2}{\sqrt{2\pi}\sigma} \int_{-\infty}^{+\infty} \exp\left[-\frac{2(V-\frac{\alpha\sigma^2}{2})^2}{\sigma^2}\right] dV.$$

On of (many possible) near-optimal strategies here is when $P(t)$ is chosen from the condition (24) for all momets of time when $V(t) > 0$; and from the condition $m(t) = 0$ for all momets of time when $V(t) = 0$. After taking into account $m(t) = 0$, $\dot{M} = 0$, the expression for the average rate of profit takes the form

$$\check{J}(V_0) = \check{J}^*\Big(1 - P(V_0)\Big).$$

It gives a lower bound on the rate of bank's profit. When the initial stock tends to infinity these two estimates tend to each other.
Similar analysis can be performed for the probability that the second constraint (18) holds (capital is exhausted). Unlike $V(t)$, capital $M(t)$ increases and the probability that $M(t) = 0$ decreases when time increases.
Obtained estimates show which characteristics of market prices should be forecasted in order to choose optimal flows and prices for trading. This approach can be generalized for the multi-asset models where the optimal selection of assets for buying and selling and their prices are sought.

# 3 Profitability of financial system and capital dissipation

Consider a closed system that includes $n$ FA including not more than one agent with infinite capacity (market). Each FA has different price estimate. In this system bank can profit by buying asset from FA with lower estimate and selling it to FAs with higher estimate. **We shall call the maximal profit which can be extracted by a bank subject to process' constraint the system's profitability**.

One of such constraints is the fixed duration of contact with FAs. We shall denote the profitability of the system with unlimited contact time as $E_0$ and with fixed contact time as $E_\tau$. It is clear that $E_\tau \leq E_0$. If all FAs in the system have equal price estimates (the system is uniform), then an intermediary (bank) is unable to extract profit $E_0 = E_\tau = 0$. The wealth functions $S_i$, $i = 1, \ldots, n$ are convex on stocks of asset, and the price estimate of the asset/credit are equal to the partial derivatives of wealth function

$$P_{bi}^0 = \frac{\partial S_i}{\partial V_i}, \quad P_{ci}^0 = \frac{\partial S_i}{\partial M_i}. \tag{32}$$

Therefore when contact between FAs is voluntarily established, the asset is transferred from the contacting FA with a lower to the contacting FA with the higher estimate in exchange for the capital or other assets. As a result, the contacting FAs' wealth functions increase and their estimates of this assets's price tend to each other. This reduces the profitability of the system, that is, the potential for an intermediary to extract profit by reselling the goods. Direct exchange process(barter) is irreversible, because one needs to buy goods from FA with higher estimate and then to resell it to FA with the lower one, that is, to spent capital, in order to return the system back to its initial state. The degree of irreversibility can be estimated as the loss of potentially extractable capital $\Delta E$ (**capital dissipation**).

If exchange is performed via an intermediary and its duration is not bounded then this intermediary extracts profit $\Delta E$. It can return the system into its initial state, by re-investing this profit back into the system (when there is no constraints on process duration). Therefore, the exchange via intermediary is reversible.

Let us emphasize the thermodynamic analogy of this reasoning. When a direct contact is established between two bodies with equal temperatures their combined entropy increases. This process is irreversible because one needs to spend energy in order to return the system in its initial state. Entropy production determines energy dissipation here. If two bodies exchange heat via an ideal heat engine this engine produces enough work to return the system back in its initial state, that is, the process is reversible.

Therefore, a bank in a financial system plays a role similar to the role of a heat engine in thermodynamics. It reduces dissipation of capital and makes exchange processes reversible.

Let us describe this qualitative reasoning quantitatively. We again denote the FA's stock of assets (bonds, stocks, etc) as $V$ and its capital as $M$. In ([11]) it was proven that the wealth function in this problem exists and has the following form

$$S = rM + \sum_i P_i N_i. \tag{33}$$

The estimates $r$ and $P_i$ are constant for infinite capacity FAs (markets). These estimates can depend on $M$ and $V$ for final-capacity FAs.
Suppose that FA can establish contact with the environment. We denote environment's estimates as $\overline{r}$ and $\overline{P}$. If FA is transferred from initial state to the final state, which is close to the initial one, then the amount of capital extracted here is

$$\delta E^0 = -\delta M_\Sigma = -(\delta M + \delta\overline{M}) = [\frac{\delta S}{r} + \frac{\delta\overline{M}}{\overline{r}} - \sum_i (P_i \delta V_i - \overline{P}_i \delta\overline{V}_i)]$$

In a reversible process $(\delta S + \delta\overline{S}) = 0$, $\delta\overline{V}_i = -\delta V_i$ so that

$$\delta E^0 = \delta S(\frac{1}{\overline{r}} - \frac{1}{r}) + \sum_i \delta V_i (P_i - \overline{P}_i) \tag{34}$$

The increase of wealth function in the irreversible process will be larger because the flow of assets is positive and is accompanied by the positive price differential

$$\delta S = -\delta\overline{S} + \sigma, \quad \sigma > 0.$$

From (34) follows that that the capital extracted in irreversible process is

$$dE = dE^0 - \frac{\sigma}{\overline{r}}.$$

Its integration along the system's trajectory yields the net amount of extracted capital

$$E = E^0 - \frac{\Delta S}{\overline{r}}. \tag{35}$$

This formula links profitability in reversible and irreversible processes with the increment of the combined wealth function and the price estimate of capital by the market. The equality (35) is analog of the Sotole formula in thermodynamics, which relates the reversible and irreversible works with the environment temperature. It holds for a system with a number of FAs.

The given definition of profitability depends heavily on the type of constraints imposed on the process. Various types of constraints can be imposed. For example,
1. The initial as well as final states of all or some FAs can be fixed.
2. The FA's price estimates $\{P_i, r\}$ and environment (market) prices estimates can be required to be equal.
3. The duration of the process can be fixed.
These constraints reduce profitability of the system. The given definition of system's profitability is applicable if there is no environment with constant estimates $\overline{P}$, $\overline{r}$.
Note that profitability in a system with one market for given initial state and no constraints on process duration is the analog of exergy, which is widely used in engineering. Consider an economic system that includes $k$ FAs with stocks of asset $V_i$ $(i = 1, \ldots, k)$ and capitals $M_i$. Their price estimates depend on $V_i$ and $M_i$ One of the subsystems can be a market with fixed estimate.
Assume that the system is closed. When contact is established between $i$-th and $j$-th FAs the flow of assets $n_{ij}$ and flow of capital $m_{ij}$ occur. The former is directed from FA with lower price estimate to the one with the lower estimate and the capital flows in opposite direction.
Consider a system that includes an intermediary (bank). The bank's objective is to organize exchange in such a way that it can extracted capital from the system. We will assume that a direct exchange between FAs is not possible, bank offers to buy asset from FAs at one price and offers to sell it to them at the other price. Bank maximize its profit by controlling these prices. The exchange flows depend on price estimate of asset $P_i$ by the $i$-th FA and the price offered by the bank $c_i$ as follows

$$n_i = n_i(P_i, c_i), \quad n_i = 0, \quad \text{if} \quad P_i = c_i, \quad sign(n_i) = sign(c_i - p_i).$$

We assume that the flow of assets is positive if it is directed to the bank. It is clear that

$$m_i(P_i, c_i) = -c_i n_i(P_i, c_i).$$

The stocks of assets and capital of the $i$-th FA changes in accordanve with the equations

$$\dot{V}_i = -n_i(p_i, c_i), \quad V_i(0) = V_{i0},$$
$$\dot{M}_i = c_i n_i(p_i, c_i), \quad M_i(0) = M_{i0}.$$

We assume that the estimates $P_i(V_i, M_i)$ are monotonically decreasing functions of $V_i$ for given $M_i$. As a rule, these functions are also increasing on $M_i$

for fixed $V_i$. It is also possible that the price estimate does not on the FA capital. Let us find out what amount of capital can be extracted by a bank from the system.

**Unlimited duration of the process**

*System with the market.* We denote the market price of the asset $P^0_{b-}$. If $t \to \infty$ the estimates of asset price for all FA tend to $P^0_{b-}$. From the condition of equilibrium for $t \to \infty$ we get

$$P^0_{bi}(\overline{V_i}, \overline{M_i}) = P^0_{b-}, \qquad i = 1, \ldots, k. \tag{36}$$

$\overline{V_i}$, $\overline{M_i}$ denote equilibrium stocks of asset and capital.

If the duration of the process is not limited then the intermediary buys asset at the price that is infinitely close to the $P^0_{bi}$, thus

$$\frac{dM_i}{dV_i} = -P^0_{bi}(V_i, M_i), \qquad M_i(V_{i0}) = M_{i0}. \tag{37}$$

Profitability of economic system (limiting amount of extracted capital) here is

$$E_\infty = \sum_{i=1}^{k}(M_{i0} - \overline{M_i}) = \sum_{i=1}^{k} \int_{V_{i0}}^{\overline{V_i}} P^0_{bi}(V_i, M_i(V_i))dV_i. \tag{38}$$

Conditions (35), (37) determine $2k$ unknowns $\overline{V_i}$ $\overline{M_i}$, therefore $E_\infty$ is also determined.

*Example.* Consider a system that includes two FAs and a market. The initial stocks of asset and capital for EA are $V_{i0}$, $M_{i0}$, $i = 1, 2$, and the market price of asset is $P^0_{b-}$.

Assume that FAs' estimates have the form

$$P^0_{bi} = \alpha_i \frac{M_i}{V_i}, \qquad i = 1, 2.$$

The system (37) can be rewritten as

$$\frac{dM_i}{dV_i} = -\alpha_i \frac{M_i}{V_i}, \qquad M_i(V_{i0}) = M_{i0};$$

which yields $M_i(V_i)$

$$M_i = \frac{M_{i0} \cdot V_{i0}^{\alpha_i}}{V_i^{\alpha_i}}, \qquad i = 1, 2.$$

Let us find the equilibrium stocks of asset $\overline{V_1}$ and $\overline{V_2}$ from the condition of equilibrium

$$\alpha_i \frac{M_{i0} \cdot V_{i0}^{\alpha_i}}{\overline{V_i}^{\alpha_i+1}} = P_{b-}^0, \qquad i = 1, 2.$$

We get

$$\overline{V_i} = \left( \frac{\alpha_i}{P_{b-}^0} M_{i0} \cdot V_{i0}^{\alpha_i} \right)^{1/(\alpha_i+1)}, \qquad i = 1, 2,$$

and the corresponding equilibrium capitals are

$$\overline{M_i} = \frac{P_{b-}^0}{\alpha_i} \overline{V_i}, \qquad i = 1, 2. \tag{39}$$

This expression yields (38) for system's profitability.
*System does not include market.* If $t \to \infty$ here then the asset price estimates are the same and equal to some $\overline{P}_b^0$ for all FA. Instead of condition that the market prices were equal to estimates, we have

$$P_{bi}^0(\overline{V_i}, \overline{M_i}) = \overline{P}_b^0, \qquad i = 1, \ldots, k. \tag{40}$$

$\overline{P}_b^0$ is to be found from the condition that the rate of change of bank's stock was equal zero (the bank resells everything it buys)

$$\sum_{i=1}^{k} (\overline{V_i} - V_{i0}) = 0. \tag{41}$$

Equalities (40), (41) jointly with (37) determine $\overline{V}$, $\overline{M}$ and $\overline{P}_b^0$ . They determine $E_\infty$ here.
*Example.* Consider the system from the previous example. The initial stocks of assets and capitals are $V_{10}$, $V_{20}$, $M_{10}$, $M_{20}$. The dependencies of the price estimates on $M_i$ and $V_i$ $(i = 1, 2)$ have the form

$$P_{b1}^0 = \alpha \frac{M_1}{V_1}, \qquad P_{b2}^0 = \beta \frac{M_2}{V_2}.$$

After taking them into account the equations (37) take the form

$$\frac{dM_1}{dV_1} = -\alpha \frac{M_1}{V_1}, \quad M_1(V_{10}) = M_{10},$$

$$\frac{dM_2}{dV_2} = -\beta \frac{M_2}{V_2}, \quad M_2(V_{20}) = M_{20}.$$

The solutions of these equations $M_1(V_1)$ and $M_2(V_2)$ are:

$$M_1 = \frac{M_{10} \cdot V_{10}^{\alpha}}{V_1^{\alpha}}, \quad M_2 = \frac{M_{20} \cdot V_{20}^{\beta}}{V_2^{\beta}}.$$

Conditions (40) and (41), yield the stock of asset $\overline{V}_i$ for each FA after exchange. They can be rewritten as

$$V_{10} + V_{20} = \overline{V}_1 + \overline{V}_2,$$

$$\alpha \frac{M_{10} V_{10}^{\alpha}}{\overline{V}_1^{\alpha+1}} = \beta \frac{M_{20} V_{20}^{\beta}}{\overline{V}_2^{\beta+1}}.$$

For the particular case $\alpha = \beta = \gamma$, we get

$$\overline{V}_1 = \frac{(V_{20} + V_{10}) \cdot (M_{10} V_{10}^{\gamma})^{1/(1+\gamma)}}{(M_{10} V_{10}^{\gamma})^{1/(1+\gamma)} + (M_{20} V_{20}^{\gamma})^{1/(1+\gamma)}},$$

$$\overline{V}_2 = \frac{(V_{20} + V_{10}) \cdot (M_{20} V_{20}^{\gamma})^{1/(1+\gamma)}}{(M_{10} V_{10}^{\gamma})^{1/(1+\gamma)} + (M_{20} V_{20}^{\gamma})^{1/(1+\gamma)}}.$$

$\overline{V}_1$ and $\overline{V}_2$ determine equilibrium stocks of capital

$$\overline{M}_1 = (M_{10} \cdot V_{10}^{\gamma})^{1/(1+\gamma)} \cdot W,$$

$$\overline{M}_2 = (M_{20} \cdot V_{20}^{\gamma})^{1/(1+\gamma)} \cdot W,$$

where

$$W = \left( \frac{(M_{10} V_{10}^{\gamma})^{1/(1+\gamma)} + (M_{20} V_{20}^{\gamma})^{1/(1+\gamma)}}{V_{20} + V_{10}} \right)^{\gamma}.$$

Substitution of these expressions into (38) yields profitability of the system. During a direct (without intermediary) exchange of asset the combined stock of asset does not change. The combined stock of capital does not change also. The unique single estimate of asset price $\overline{P_b^0}$ is then established in the system. Equations (37) take the form

$$\frac{dM_i}{dV_i} = -\overline{P_b^0}, \qquad i = 1, ..., k,$$

Thus

$$\overline{M_i} = M_{i0} - \overline{P_b^0}(\overline{V} - V_{i0}), \quad i = 1, \ldots, k.$$

Substitution of these expressions into (40), (41) determines $\overline{V_i}$ and $\overline{P_b^0}$.
**Limited duration of asset exchange**. Assume that the duration of exchange $\tau$ is given. The intermediary here has to increase the prices it offers to buyers and decrease its prices for sellers from the equilibrium estimate $P_{bi}^0$. This leads to irreversible losses and reduces the amount of capital extracted from the system. The maximal possible value $E_\tau$ will then be lower than $E_\infty$. The difference

$$\Delta S = (E_\infty - E_\tau) > 0$$

describes irreversibility of asset exchange.

## 4 Minimal capital dissipation processes

We have obtained the conditions of optimal choice of prices for buying (selling) the given amount of asset $\Delta V$ from FA in a given period of time $\tau$. It minimizes (maximizes) the amount of capital to be extracted from the system in a process. The wealth function increment for the FA is minimal here. Therefore it corresponds to minimization of capital dissipation. These processes are similar to minimal dissipation processes in thermodynamics ([13]).
The optimal selling price of the asset for selling to FA with estimate $p_b^0(V)$ is

$$c_\tau^*(V, \overline{V}) = P_b^0(V) - \frac{\overline{V} - V_0}{\alpha\tau}, \tag{42}$$

and capital obtained from this sale is

$$E_\tau(\overline{V}) = E_\infty(\overline{V}) - \frac{(\overline{V} - V_0)^2}{\alpha\tau}, \tag{43}$$

where $E_\infty$ is the capital that can be obtained by the bank for $\tau \to \infty$, by selling at equilibrium prices $c(V) = P_b^0(V)$. The function $E(\tau)$ is shown on Figure 1.

Fig. 1. The dependence of profitability on duration of exchange

Here $V_0$ and $\overline{V}$ are stocks of asset in the beginning and at the end of the process, the flow of assets depends on price differential as

$$m(P, c) = \alpha(P - c)$$

and $\alpha$ is constant coefficient. If $\tau < \tau_0 = \dfrac{\Delta V^2}{\alpha E_\infty}$ then the bank is forced pay extra premium to the buyer. If $\tau^* = 2\tau_0$ then the average rate of profit $e(\tau) = \dfrac{E(\tau)}{\tau}$ is maximal and equal to

$$e^* = \frac{\alpha}{4}\left[\frac{E_\infty(\overline{V} - V_0)}{\overline{V} - V_0}\right]^2 .$$

The difference between capital loss in this exchange and the capital loss in equilibrium process measures the loss of profitability (capital dissipation)

$$\sigma = n(P_b^0, c)(P_b^0 - c). \tag{44}$$

The amount of dissipative losses is defined as ($r = 1$)

$$\Delta S(\tau) = \int_0^\tau \sigma(t)\, dt = \int_0^\tau n(P_b^0, c)(P_b^0 - c)\, dt. \tag{45}$$

For the last example these losses are

$$\Delta S(\tau, \overline{V}) = \int_0^\tau \alpha(P_b^0(V) - c(V))^2\, dt = \frac{(\overline{V} - V_0)^2}{\alpha\tau},$$

therefore (compare with equality (34) for $\overline{P}_k^0 = 1$)

$$E(\tau) = E_\infty(\overline{V}) - \Delta S(\tau, \overline{V}) = E_\infty(\overline{V}) - \int_0^\tau n(P_b^0, c)(P_b^0 - c)\, dt. \tag{46}$$

The expression (46) is valid for any dependence $n(P_b^0, c)$. Indeed, after substitution of $dt$ with $dV$ the integral in (45) can be rewritten as

$$\Delta S(\overline{V}) = S(\overline{V}) - S(V_0) = \int_{V_0}^{\overline{V}} (P_b^0(V) - c_\tau(V, \overline{V}))\, dV.$$

In its turn the extracted capital is

$$E(\tau,\overline{V}) = \int\limits_{V_0}^{\overline{V}} c_\tau(V,\overline{V})\,dV, \quad E_\infty(\overline{V}) = \int\limits_{V_0}^{\overline{V}} P_b^0(V)\,dV. \tag{47}$$

From the comparison of these equations (46) follows. Thus, the optimal buying (selling) processes are the minimal dissipation processes. These processes are singled out by the conditions (24).

**Maximal profit in a system of FAs**. This problem is reduced to selling (buying) asset from each of FAs. Because the trading must be executed optimally from the viewpoint of capital extraction the price $c$ and asset price estimate $P_b^0$ must obey the conditions (24) at each moment of time. The volumes of sales/purchases $\Delta V_i$ for each of $m$ FAs must be chosen optimally and obey the condition

$$\sum_{i=1}^{m} \overline{V_i} = \sum_{i=1}^{m} V_{i0}. \tag{48}$$

We consider market as one of FAs with asset price estimate $P_{b-}^0$ that is independent of stock of asset and capital. Thus for any dependence $n(c, P_{b-}^0)$ the optimal price of buying and selling on such market $c$ must not depend on time.

The problem of maximization of extracted capital in a limited time in a closed financial system is reduced to solution of $m$ problems (17)–(20) of optimal selling/buying for each of FAs with given initial and final stocks of asset ($V_{i0}$ and $\overline{V_i}$). The optimal $\overline{V_i}$ are then found from the condition

$$\sum_{i=1}^{m} E_i(\tau,\overline{V_i}) \to \max_{\overline{V_i}} \tag{49}$$

subject to constraint (48). The conditions of optimality of the problem (48), (49) have the following form

$$\frac{\partial E_i(\tau,\overline{V_i})}{\partial \overline{V_i}} = \Lambda, \quad i = 1,\dots,m.$$

$\Lambda$ is to be found from (48).

After taking into account (47) we get

$$\frac{\partial E_i(\tau,\overline{V_i})}{\partial \overline{V_i}} = c_{i\tau}(\overline{V_i},\overline{V_i}) + \int\limits_{V_{i0}}^{\overline{V_i}} \frac{\partial c_{i\tau}(V_i,\overline{V_i})}{\partial \overline{V_i}} dV_i = \overline{c_{i\tau}}(\overline{V_i}). \tag{50}$$

The first term in the r.h.s. is the prediction of optimal price at $t = \tau$ and the second term correctes is. This correction is determined by the averaged sensitivity of the optimal price with respect to the stock of sold assets. We shall call the expression (50) the corrected price. The condition of optimal choice of volumes of sold (bought) assets is reduced to the cndition that for all FAs the optimal prices are the same

$$\overline{c_{i\tau}}(\overline{V_i}) = \Lambda, \quad i = 1, ..., m. \tag{51}$$

*Example.* Consider the system where for each FA

$$P_{bi}^0 = \frac{h_i}{V_i}, \quad i = 1, \ldots, m, \tag{52}$$

$$n_i(c, p_b^0) = \alpha_i(c_i - P_{bi}^0), \quad i = 1, \ldots, m. \tag{53}$$

Let us find out what amount of capital can be extracted from $i$-th FA in arbitrary long time. After taking into account (52) we get from the expression (47)

$$E_{i\infty}(\overline{V_i}) = h_i \int\limits_{V_i 0}^{\overline{V_i}} \frac{dV_i}{V_i} = h_i \ln \frac{\overline{V_i}}{V_{i0}}, \quad i = 1, ..., m.$$

According to (47)

$$E_i(\tau, \overline{V_i}) = h_i \ln \frac{\overline{V_i}}{V_{i0}} - \frac{(\overline{V_i} - V_{i0})^2}{\alpha_i \tau}. \tag{54}$$

The optimal choice of $\overline{V_i}$ (51) is reduced to solution of equation

$$\overline{c_{i\tau}}(\overline{V_i}) = \left[ P_{bi}^0(\overline{V_i}) - \frac{\overline{V_i} - V_{i0}}{\alpha \tau} \right] - \frac{\overline{V_i} - V_{i0}}{\alpha \tau} = \Lambda. \tag{55}$$

The problem becomes much simpler if all FAs have constant estimates $P_b^0 =$ const. The condition of optimality (55) then leads to the equations

$$P_{bi}^0 - \frac{2}{\alpha_i \tau}(\overline{V_i} - V_{i0}) = \Lambda \rightarrow \Delta V_i = \frac{\alpha_i \tau}{2}(P_{bi}^0 - \Lambda). \tag{56}$$

From (48) it follows that constant $\Lambda$ is equal to the averaged estimate of asset price

$$\Lambda = \frac{\sum\limits_{i=1}^{m} \alpha_i P_{bi}^0}{\sum\limits_{i=1}^{m} \alpha_i},$$

$$\overline{V_i^*} = \frac{\tau \alpha_i}{2} \left( P_{bi}^0 - \frac{\sum\limits_{\nu=1}^{m} \alpha_\nu P_{b\nu}^0}{\sum\limits_{\nu=1}^{m} \alpha_\nu} \right) + V_{i0}. \qquad (57)$$

Substitution of $\overline{V_i^*}$ into (56) yields maximal possible capital $E_i(\tau, \overline{V_i^*})$ extractable from each FA during the period $\tau$

$$E_\tau^* = \sum_{i=1}^{m} \left[ P_{bi}^0 (\overline{V_i^*} - V_{i0}) - \frac{(\overline{V_i}^* - V_{i0})^2}{\alpha_i \tau} \right].$$

For the estimates that are independent on asset's stock the profitability (after taking into account (57)) is

$$E_\tau^* = \frac{\tau}{4} ((P_{bi}^0)^2 - \Lambda^2).$$

In most cases these estimates $p_i$ decrease when stock of asset increase and the profitability is monotonic convex function of $\tau$.

**Reselling of assets**.

Consider a system where assets are simultaneously sold and bought at different prices. We denote as $\hat{P}$ and $\check{P}$ the limiting maximal and minimal prices of the customers correspondingly, $P_+(t)$ is the buying price and $P_-(t)$ is the selling price. Because intermediary cannot sell asset at a higher price than the maximal price of the customers and because it also cannot buy asset at lower price than the minimal price of customers the following inequalities must hold

$$P_-(t) \le \hat{P}_0, \quad P_+(t) \le \check{P}_0.$$

Then the problem of maximal rate of profit can be written as

$$J = \frac{1}{\tau} \int\limits_0^\tau \left[ m_-(\hat{P}_0, P_-) P_- - m_+(\check{P}_0, P_+) P_+ \right] dt \to \max_{P_+, P_-}. \qquad (58)$$

Here $m_-(\hat{P}_0, P_-)$ and $m_+(\check{P}_0, P_+)$ are the flows of buying and selling of asset.
We again assume that intermediary does not accumulate asset. The condition of non-accumulation can be written as

$$\Delta V = \int_0^\tau \Big[ m_+(\check{P}_0, P_+) - m_-(\hat{P}_0, P_-) \Big] dt = 0. \tag{59}$$

If capital is exhausted then the flows or buying and selling are linked

$$\begin{aligned} m_+(\check{P}_0, P_+)P_+ \le m_-(\hat{P}_0, P_-)P_-, \quad M = 0, \\ m_+(\check{P}_0, P_+) \ge m_-(\hat{P}_0, P_-), \quad V = 0. \end{aligned} \tag{60}$$

The evolution of stocks of assets and capital are governed by the differential equations

$$\begin{aligned} \dot{V}(t) = m_+(\check{P}_0, P_+) - m_-(\hat{P}_0, P_-), \quad V(0) = V_0, \\ \dot{M}(t) = m_-(\hat{P}_0, P_-)P_- - m_+(\check{P}_0, P_+)P_+, \quad M(0) = M_0. \end{aligned} \tag{61}$$

If $V_0$ and $M_0$ are large then the probability of capital exhaustion can be neglected and the equations (60) can be deleted from the problem. The r.h.s. of equations (61) do not depend on $V$ and $M$, and can also be deleted from the problem of choosing the optimal prices $P_+^*$ and $P_-^*$ are chosen.
Consider the problem (58), (59). Its Lagrange function is

$$L = m_-(\hat{P}_0, P_-)(P_- - \lambda) - m_+(\check{P}_0, P_+)(P_+ - \lambda).$$

Its conditions of stationarity on $P_+$ and $P_-$ yield

$$\begin{aligned} \frac{\partial m_+\Big(\check{P}_0(t), P_+(t)\Big)}{\partial P_+} \left[P_+(t) - \lambda\right] + m_+\Big(\check{P}_0(t), P_+(t)\Big) = 0, \\ \frac{\partial m_-\Big(\hat{P}_0(t), P_-(t)\Big)}{\partial P_-} \left[P_-(t) - \lambda\right] + m_-\Big(\hat{P}_0(t), P_-(t)\Big) = 0. \end{aligned} \tag{62}$$

If these two equations have unique solution and corresponds to maximum of $L$, then we get from (62)

$$\begin{aligned} m_+(\check{P}_0(t), P_+(t)) = -\frac{\partial m_+(\check{P}_0(t), P_+(t))}{\partial P_+} \left(P_+(t) - \lambda\right), \\ m_-(\hat{P}_0(t), P_-(t)) = -\frac{\partial m_-(\hat{P}_0(t), P_-(t))}{\partial P_-} \left(P_-(t) - \lambda\right). \end{aligned}$$

Their substitution into (59) gives:

$$\Delta V = \int_0^\tau \Big[m_+(\check{P}_0, P_+) - m_-(\hat{P}_0, P_-)\Big] dt =$$

$$= \int_0^\tau \left[\frac{\partial m_-(\hat{P}_0(t), P_-(t))}{\partial P_-} P_-(t) - \frac{\partial m_+(\check{P}_0(t), P_+(t))}{\partial P_+} P_+(t)\right] dt +$$

$$+ \lambda \int_0^\tau \left[\frac{\partial m_-(\hat{P}_0(t), P_-(t))}{\partial P_-} - \frac{\partial m_+(\check{P}_0(t), P_+(t))}{\partial P_+}\right] dt = 0,$$

$\lambda$ can be expressed as

$$\lambda = \frac{\int_0^\tau \left[\frac{\partial m_-(\hat{P}_0(t), P_-(t))}{\partial P_-} P_-(t) - \frac{\partial m_+(\check{P}_0(t), P_+(t))}{\partial P_+} P_+(t)\right] dt}{\int_0^\tau \left[\frac{\partial m_-(\hat{P}_0(t), P_-(t)}{\partial P_-} - \frac{\partial m_+(\check{P}_0(t), P_+(t))}{\partial P_+}\right] dt}. \tag{63}$$

Elimination of $\lambda$ from equations (62) and (63), yields the set of equations for optimal $P_+^*(t)$ and $P_-^*(t)$.
Consider linear dependence of flow of sold/bought assets on the price differential

$$\begin{aligned} m_+(\check{P}_0(t), P_+(t)) &= \alpha_+(P_+(t) - \check{P}_0(t)), \\ m_-(\hat{P}_0(t), P_-(t)) &= \alpha_-(P_-(t) - \hat{P}_0(t)), \end{aligned} \tag{64}$$

where $\alpha_+$, $\alpha_-$ are constants.
Equations (62) take the form

$$\begin{aligned} \alpha_+(P_+ - \lambda) + \alpha_+(P_+ - \check{P}_0) &= 0, \\ \alpha_-(P_- - \lambda) + \alpha_-(P_- - \hat{P}_0) &= 0. \end{aligned}$$

Substitution of $\lambda$ into these equations yields the optimal solution

$$P_+^*(t) = \frac{1}{2}(\lambda + \check{P}_0) = \frac{1}{2}\left[\frac{\alpha_+\overline{P}_+ + \alpha_-\overline{P}_-}{\alpha_+ + \alpha_-} + \check{P}_0(t)\right],$$

$$P_-^*(t) = \frac{1}{2}(\lambda + \hat{P}_0) = \frac{1}{2}\left[\frac{\alpha_+\overline{P}_+ + \alpha_-\overline{P}_-}{\alpha_+ + \alpha_-} + \hat{P}_0(t)\right].$$

The expressions for $P_+^*$ and $P_-^*$ contain their average values $\overline{P}_+$ and $\overline{P}_-$ over the interval $[0, \tau]$. We can eliminate $\overline{P}_+$ and $\overline{P}_-$, by averaging left and right hand sides of these equations. We get the set of equations with two unknowns $\overline{P}_+$ and $\overline{P}_-$

$$\overline{P}_+ = \frac{\alpha_+ \overline{P}_+ + \alpha_- \overline{P}_-}{2(\alpha_+ + \alpha_-)} + \frac{\overline{\check{P}}_0}{2}, \quad \overline{P}_- = \frac{\alpha_+ \overline{P}_+ + \alpha_- \overline{P}_-}{(\alpha_+ + 2\alpha_-)} + \frac{\overline{\hat{P}}_0}{2},$$

and

$$\overline{P}_+ = \frac{(2\alpha_+ + \alpha_-)\overline{\check{P}}_0 + \alpha_- \overline{\hat{P}}_0}{2(\alpha_+ + \alpha_-)}, \quad \overline{P}_- = \frac{\alpha_+ \overline{\check{P}}_0 + (\alpha_+ + 2\alpha_-)\overline{\hat{P}}_0}{2(\alpha_+ + \alpha_-)}.$$

Therefore if $\overline{P}_+$ and $\overline{P}_-$ are known, then we can express the optimal $P_+^*$ and $P_-^*$ in terms of $\check{P}_0(t)$ and $\hat{P}_0(t)$ and their averaged values only

$$P_+^*(t) = \frac{1}{2}\,\check{p}_0(t) + \frac{\alpha_+ \overline{\check{P}}_0 + \alpha_- \overline{\hat{P}}_0}{2(\alpha_+ + \alpha_-)}, \quad P_-^*(t) = \frac{1}{2}\,\hat{p}_0(t) + \frac{\alpha_+ \overline{\check{P}}_0 + \alpha_- \overline{\hat{P}}_0}{2(\alpha_+ + \alpha_-)}. \quad (65)$$

The upper bound on the profit rate is

$$J = \frac{1}{\tau}\int_0^\tau \Big[m_-(\hat{P}_0, P_-)P_- - m_+(\check{P}_0, P_+)P_+\Big]dt =$$

$$= \frac{1}{\tau}\int_0^\tau \Big[-\alpha_-\Big(P_-^*(t) - \hat{P}_0(t)\Big)P_-^*(t) - \alpha_+\Big(P_+^*(t) - \check{P}_0(t)\Big)P_+^*(t)\Big]\,dt. \quad (66)$$

If prices $\hat{P}_0$ and $\check{P}_0$ are stationary stochastic processes then the expressions obtained hold. They can be generalized for exchange with one market as it was done in Section 2. For exchange kinetic (64) the expressions (65) hold if $\overline{\hat{P}}_0$ and $\overline{\check{P}}_0$ are understood as expectations of these processes. The maximal rate of profit (66) then can be rewritten as

$$J^* = \frac{1}{4}(\alpha_+ D_{\check{P}_0} + \alpha_- D_{\hat{P}_0}) + \frac{\alpha_+\alpha_-}{4(\alpha_+ + \alpha_-)}(\overline{\check{P}}_0 - \overline{\hat{P}}_0)^2,$$

where

$$D_{\check{P}_0} = \overline{\check{P}_0^2} - (\overline{\check{P}}_0)^2, \quad D_{\hat{P}_0} = \overline{\hat{P}_0^2} - (\overline{\hat{P}}_0)^2$$

are the variances of the stochastic variables $\check{P}_0$ and $\hat{P}_0$.

## 5 Optimization of interest rates for loans and deposits by a commercial bank

Bank is an intermediary that operates between two markets, the market of depositors and the market of borrowers, which are unable to establish direct contact. The borrowers are willing to obtain credit at a higher rate than the bank pays to its depositors. Bank's profit is produced by exploiting this differential. Bank maximizes its profit by controlling the rates for deposits and loans. These rates can also depend on the duration and volume of credit. If the latter case the bank should collect information about the dependences on the parameters, which describe how interested market participants are in credits, on the durations and volumes of loans and change its strategy in accordance with these data.
We denote as $\gamma(\tau)$ the interest rate on the loan taken out for $\tau$. It is equal to the fraction of capital that is payed to the bank together with the principle. As a rule $\gamma$, is non-negative non-decreasing function of $\tau$. The situation when the depositor uses bank a safe storage for its capital or where bank gives a loan for charitable purposes are the exceptions.
The choice of rates $\gamma_1(\tau_1)$ and $\gamma_2(\tau_2)$ for deposits and loans effects the volumes of deposited and borrowed funds. These rates should be chosen by the bank from the conditions of maximal average profit. If the bank controls only the yearly rates $\gamma_1^0 = \gamma_1(1)$ and $\gamma_2^0 = \gamma_2(1)$, then the rates for the term of loan are calculated according to some rule. In particular, for continuous compounding this rule is

$$\gamma_i(\tau_i) = (1 + \gamma_i^0)^{\tau_i} - 1, \tag{67}$$

$$\gamma_i(\tau_i) = \gamma_i^0 \tau_i, \quad i = 1, 2. \tag{68}$$

We describe how interested bank's depositors and borrowers are in bank's credit by the minimal rate $r_1$, at which the depositors are willing to deposit their funds into bank and by the maximal rate $r_2$ at which the borrowers are willing to borrow from the bank. We shall call $r_1$ and $r_2 > r_1$ the credit estimates for depositors and borrower's correspondingly. They can depend on the volume and length of borrowing.
We first assume that the parameters which describe how interested the depositors and borrowers are in credit are known. Let us find out the optimal rates of credit and corresponding maximal profit $\Pi^*(\tau_1, \tau_2)$ as the function of $\tau_1$ and $\tau_2$ here. If bank can control the length of loan then it choses $\tau_1$

and $\tau_2$ from the conditions of maximal $\Pi^*$.
We then consider the problem with random lengths of loans and deposits where the rates are to be chosen from the condition of maximal average profit.

**Choosing rates after taking into account the dependence of estimates on the length of loan**

Assume that the terms for the deposits $\tau_1$ and for loans $\tau_2$ and the corresponding credit estimates $r_1$ and $r_2$ are known and such rates for deposits $\gamma_1$ and credits $\gamma_2$ are sought that the bank's profit is maximal. We need to know the dependence of the estimates $r_i$, the bank-set interest rates $\gamma_i (i = 1, 2)$ and the specific volumes of deposits and loans per unit of time. The flows $m_1(\gamma_1, r_1)$ and $m_2(r_2, \gamma_2)$ for bank-monopolist are the functions of demand and supply

$$m_1(\gamma_1, r_1) = \begin{cases} 0 & \text{if} \quad \gamma_1 \leq r_1, \\ > 0 & \text{if} \quad \gamma_1 > r_1, \end{cases}$$
$$m_2(\gamma_2, r_2) = \begin{cases} 0 & \text{if} \quad \gamma_2 \geq r_2, \\ > 0 & \text{if} \quad \gamma_2 < r_2. \end{cases} \tag{69}$$

The profit from the loan $m_2$ given for time $\tau_2$ is

$$\Pi_2 = m_2\Big(r_2(\tau_2), \gamma_2(\tau_2)\Big)\gamma_2(\tau_2),$$

and the average yearly rate of this profit is

$$\overline{\Pi_2} = m_2(r_2(\tau_2), \gamma_2(\tau_2))\frac{\gamma_2(\tau_2)}{\tau_2}. \tag{70}$$

Depositors also receive payments for their deposits that bank hold over the period $\tau_1$

$$\overline{\Pi_1} = m_1(\gamma_1(\tau_1), r_1(\tau_1))\frac{\gamma_1(\tau_1)}{\tau_1}. \tag{71}$$

The average profit of the bank is

$$\begin{gathered} \overline{\Pi(\tau_1, \tau_2)} = \overline{\Pi_2(\tau_2)} - \overline{\Pi_1(\tau_1)} = \\ = \frac{\gamma_2(\tau_2)}{\tau_2}\, m_2(r_2(\tau_2), \gamma_2(\tau_2)) - \frac{\gamma_1(\tau_1)}{\tau_1}\, m_1(\gamma_1(\tau_1), r_1(\tau_1)). \end{gathered} \tag{72}$$

Assume that the dependence of estimates on terms of deposits/ loans $r_1(\tau_1)$ and $r_2(\tau_2)$ are known and the rates $\gamma_1(\tau_1)$ and $\gamma_2(\tau_2)$ that maximize the average profit subject to condition that all deposits are re-loaned are to be

found. The lengths of deposits/credits assumed to be random values with distributions $P_1(\tau_1)$ and $P_2(\tau_2)$ correspondingly.
The average yearly profit is

$$\overline{\Pi} = \overline{\frac{1}{\tau_2} m_2(r_2(\tau_2), \gamma_2(\tau_2))\gamma_2(\tau_2)} - \overline{\frac{1}{\tau_1} m_1(\gamma_1(\tau_1), r_1(\tau_1))\gamma_1(\tau_1)} \qquad (73)$$

is maximazed subject to

$$\overline{m_1(\gamma_1(\tau_1), r_1(\tau_1))} - \overline{m_2(r_2(\tau_2), \gamma_2(\tau_2))} = 0. \qquad (74)$$

The averaging in the first term in (73) and in the second in (74) is understood over $\tau_2$, and in other terms over $\tau_1$. For example,

$$\overline{m_1} = \int_0^\infty m_1(\gamma_1(\tau_1), r_1(\tau_1))P_1(\tau_1)d\tau_1,$$

$$\overline{\left(\frac{m_2\gamma_2}{\tau_2}\right)} = \int_0^\infty \frac{1}{\tau_2} m_2(r_2(\tau_2), \gamma_2(\tau_2))\gamma_2(\tau_2)P_2(\tau_2)d\tau_2.$$

The Lagrange function of the problem (73), (74) has the form

$$\overline{L} = \left(\overline{\left(\frac{m_2\gamma_2}{\tau_2}\right)} - \lambda\overline{m_2}\right) - \left(\overline{\left(\frac{m_1\gamma_1}{\tau_1}\right)} - \lambda\overline{m_1}\right).$$

averaging in the first term assumed to be done on $\tau_2$, and in the second on $\tau_1$.
The conditions of optimality of the problem (73), (74) on $\gamma_1(\tau_1)$ and $\gamma_2(\tau_2)$ give

$$\frac{m_{i\gamma_i}\gamma_i(\tau_i) + m_i(\gamma_i(\tau_i), r_i(\tau_i))}{m_{i\gamma_i}} = \lambda\tau_i, \quad i = 1, 2, \qquad (75)$$

which determine optimal dependencies $\gamma_1(\tau_1, \lambda)$ and $\gamma_2(\tau_2, \lambda)$. Their substitution into (74) allow us to find $\lambda$, and the optimal solution. $m_{i\gamma_i}$ denotes partial derivative of the corresponding functions.
Consider the case when

$$\begin{aligned} m_1 &= \alpha_1(\gamma_1(\tau_1) - r_1(\tau_1)), \\ m_2 &= \alpha_2(r_2(\tau_2) - \gamma_2(\tau_2)). \end{aligned} \qquad (76)$$

here

$$m_{1\gamma_1} = \alpha_1, \quad m_{2\gamma_2} = -\alpha_2.$$

and the equations (75) for the flow of capital are

$$\begin{aligned} 2\gamma_1(\tau_1) - r_1(\tau_1) &= \lambda\tau_1, \\ 2\gamma_2(\tau_2) - r_2(\tau_2) &= \lambda\tau_2. \end{aligned} \tag{77}$$

Their substitution into (74)yields

$$\alpha_1\left(\frac{\lambda\overline{\tau_1}+\overline{r_1}}{2} - \overline{r_1}\right) = \alpha_2\left(\overline{r_2} - \frac{\lambda\overline{\tau_2}+\overline{r_2}}{2}\right).$$

here $\overline{\tau_1}$ and $\overline{\tau_2}$ are expectations for $\tau_1$ and $\tau_2$.
Last equation determines $\lambda$

$$\lambda = \frac{\alpha_1\overline{r_1}+\alpha_2\overline{r_2}}{\alpha_1\overline{\tau_1}+\alpha_2\overline{\tau_2}}, \tag{78}$$

and the optimal rates here are

$$\begin{aligned} \gamma_1(\tau_1) &= \frac{1}{2}\left(r_1(\tau_1) + \frac{\alpha_1\overline{r_1}+\alpha_2\overline{r_2}}{\alpha_1\overline{\tau_1}+\alpha_2\overline{\tau_2}}\tau_1\right), \\ \gamma_2(\tau_2) &= \frac{1}{2}\left(r_2(\tau_2) - \frac{\alpha_1\overline{r_1}+\alpha_2\overline{r_2}}{\alpha_1\overline{\tau_1}+\alpha_2\overline{\tau_2}}\tau_2\right). \end{aligned} \tag{79}$$

The maximal average bank profit for these rates is

$$\Pi_{\max} = \frac{1}{4}\left[\alpha_1\left(\overline{\frac{r_1^2(\tau_1)}{\tau_1}}\right) + \alpha_2\left(\overline{\frac{r_2^2(\tau_2)}{\tau_2}}\right) - \lambda^2(\alpha_1\overline{\tau_1}+\alpha_2\overline{\tau_2})\right], \tag{80}$$

where $\lambda$ is expressed in terms of averaged estimates $\overline{r_i}$ and $\alpha_i$ in accordance with (78).

**Optimal on average yearly rates.**

Consider the situation where the bank controls only the yearly rates $\gamma_1^0$ and $\gamma_2^0$, and the functions $\gamma_i(\tau_i, \gamma_i^0)$ depend only on these rates and on the way the compound interest is calculated (see (67), (68)). The optimum in the problem (73), (74) is sought with respect to $\gamma_1^0$ and $\gamma_2^0$.
We denote

$$\frac{\partial\gamma_i(\tau_i,\gamma_i^0)}{\partial\gamma_i^0} = \gamma_i'(\tau_i,\gamma_i^0), \quad i = 1, 2,$$

after elimination of $\lambda$ we obtain

$$\overline{\frac{\gamma_1'}{\tau_1}\left(m_1 + \gamma_1\frac{\partial m_1}{\partial\gamma_1}\right)}\;\overline{\frac{\gamma_2'}{\tau_2}\frac{\partial m_2}{\partial\gamma_2}} = \overline{\frac{\gamma_2'}{\tau_2}\left(m_2 + \gamma_2\frac{\partial m_2}{\partial\gamma_2}\right)}\;\overline{\frac{\gamma_1'}{\tau_1}\frac{\partial m_1}{\partial\gamma_1}}. \tag{81}$$

This equation jointly with equation (74) determines optimal parameters $\gamma_1^0, \gamma_2^0$.
For the flows (76) we get

$$\frac{\partial m_1}{\partial \gamma_1} = \alpha_1, \quad \frac{\partial m_2}{\partial \gamma_2} = -\alpha_2,$$

if $\gamma_i(\tau, \gamma_i^0)$ have the form (68) then

$$\frac{\gamma_i'}{\tau_i} = \gamma_i^0,$$

and the system (81), (74) can be easily solved.
Conditions (81) take the form

$$\overline{\tau_1} + (\gamma_1^0 \overline{\tau_1} - \overline{r_1}) = \overline{\tau_2} - (\overline{r_2} - \gamma_2^0 \overline{\tau_2}), \tag{82}$$

equation (74) here can be written as

$$\alpha_1(\gamma_1^0 \overline{\tau_1} - \overline{r_1}) = \alpha_2(\overline{r_2} - \gamma_2^0 \overline{\tau_2}),$$

and the equation (82) takes the form

$$\overline{\tau_1} + \frac{\alpha_2}{\alpha_1}(\overline{r_2} - \gamma_2^0 \overline{\tau_2}) = \overline{\tau_2} - (\overline{r_2} - \gamma_2^0 \overline{\tau_2}),$$

$$\overline{\tau_1} + (\gamma_1^0 \overline{\tau_1} - \overline{r_1}) = \overline{\tau_2} - \frac{\alpha_1}{\alpha_2}(\gamma_1^0 \overline{\tau_1} - \overline{r_1}).$$

Thus the optimal on average yearly rates are

$$\begin{aligned} \gamma_2^0 &= \frac{\alpha_1(\overline{\tau_1} - \overline{\tau_2})}{(\alpha_1 + \alpha_2)\overline{\tau_2}} + \frac{\overline{r_2}}{\overline{\tau_2}}, \\ \gamma_1^0 &= \frac{\alpha_2(\overline{\tau_2} - \overline{\tau_1})}{(\alpha_1 + \alpha_2)\overline{\tau_1}} + \frac{\overline{r_1}}{\overline{\tau_1}}. \end{aligned} \tag{83}$$

They depend only on the estimates of borrowing rate averaged over the term of the loan $\overline{r_i(\tau_i)}$ and $\overline{\tau_i}$.
**Rates optimization by controlling volume of deposits and loans**
Assume that the credit estimates $r_i$ depend not only on the terms of the loans but also on the volumes of deposit/loans $V_i$. Assume that these dependencies are known. Assume that $\tau_i$ and $V_i$ are random variables distributed with the density $P_i(\tau_i, V_i)$ $(i = 1, 2)$. The subscript $i = 1$ again denotes depositors and

$i = 2$ denotes borrowers. We denote as $m_i[r_i(\tau_i, V_i),\ \gamma_i(\tau_i, V_i)]$ the number of depositors/ borrowers that contact bank per unit of time. The problem of maximal average profit then takes the form

$$\overline{\Pi} = \sum_{i=1}^{2} \overline{\frac{V_i}{\tau_i} m_i[r_i(\tau_i, V_i), \gamma_i(\tau_i, V_i)]} \to \max_{\gamma_i} \tag{84}$$

subject to

$$\sum_{i=1}^{2} \overline{V_i m_i[r_i(\tau_i, V_i), \gamma_i(\tau_i, V_i)]} = 0. \tag{85}$$

Here

$$\text{Sign}(m_i) = \text{Sign}(r_i - \gamma_i), \quad i = 1, 2;$$

and the averaging in the $i$-th term is done on $\tau_i, V_i$.
The conditions of optimality of the problem (84), (85) turn out to be identical with the optimality conditions (75), if $m_i$ are replaced with $m_i V_i$.
If the flows of depositors/borrowers is proportional to the differential between the bank rate and its customer's estimates

$$m_i = \alpha_i (r_i - \gamma_i), \tag{86}$$

then

$$m_i = \frac{\alpha_i}{V_i}(r_i - \gamma_i), \quad i = 1, 2.$$

After elimination of $\lambda$ we obtain the conditions of optimality

$$2\left[\frac{\gamma_2(\tau_2, V_2)V_2}{\tau_2} - \frac{\gamma_1(\tau_1, V_1)V_1}{\tau_1}\right] = \frac{r_2(\tau_2)V_2}{\tau_2} - \frac{r_1(\tau_1)V_1}{\tau_1}. \tag{87}$$

Derivations, similar to the derivations used to obtain (79), yield

$$\gamma_i(\tau_i, V_i) = \frac{1}{2} r_i(\tau_i, V_i) + \lambda \tau_i, \quad i = 1, 2, \tag{88}$$

$$\lambda = \frac{\alpha_1 \overline{r_{1v}} + \alpha_2 \overline{r_{2v}}}{\alpha_1 \overline{\tau_{1v}} + \alpha_2 \overline{\tau_{2v}}}.$$

Here

$$\overline{r_{iv}} = \overline{\left(\frac{r_i(\tau_i, V_i)}{V_i}\right)}, \quad \overline{\tau_{iv}} = \overline{\left(\frac{\tau_i}{V_i}\right)}, \quad i = 1, 2.$$

The averaging is assumed on $V_i, \tau_i$ and is carried out using their densities.

**Effects of taxation and inflation on optimal bank behaviour**

The considered simplest model of bank operations does not take into account a number of factors, including risk of default by a bank as well as by a borrower, the market risk from commercial operations bank undertakes using depositor's funds, and finally the effect of inflation and taxes. One can assume that depositors account for bank's reliability by varying depositor's estimates $r_1(\tau_1)$ among different banks and the banks similarly discriminates between borrowers with different risk prfiles. That is, the riskier are the operations, funded by the borrower using bank loans, the higher is the rate it agrees to accept.

Consider predicted inflation with rate $\mu$. It results in losses for market participants that obtain income with time lag, that is, to a borrower and a depositor. If the rate of inflation is known then the depositor increases its estimate

$$r_{1i}(\tau_1) = r_1(\tau_1)\exp(\mu\tau_1);$$

The borrower also has to repay a smaller principle so he will correct its estimate

$$r_{2i}(\tau_2) = r_2(\tau_2)\exp(-\mu\tau_2)$$

The substitution of these estimates in the above derived expressions allows us to find optimal rates that take into account inflation.

Consider now the effect of taxation. We denote the fraction of bank's deposits taken as a tax on the depositor as $\delta_1$, and the fraction of the difference between the deposit and the amount paid by the bank to the depositor, that is taken as a tax on a bank as $\delta$. After accounting for inflation and tax the depositor's income is

$$d_1 = m_1(\gamma_1(\tau_1), r_{1i}(\tau_1))(1-\delta_1)\frac{\gamma_1(\tau_1)}{\tau_1},$$

and the banks income is

$$\Pi_n = \Pi(\gamma_1, \gamma_2, r_{1i}, r_{2i}) - \delta m_2(y, r_{2i})(y-1),$$

where $\Pi$ has the form (9).

Minimization of this expression yields optimality conditions for deposit/loan interest rates.

The dependencies obtained in this section allows us to optimise the credit rates and to estimate the maximal profit of a commercial bank for given demand functions for depositors and borrowers that determine the rates at which they contact bank. One also needs to know customer's distribution on volumes' of credit and terms' of loans.

# 6 Competition and optimal bank rates

Consider a situation where $n$ banks compete with each other. Each bank controls its flows of buying/selling $m_j$, $j = 1, \dots, n$ to maximizes its profit. The demand functions $P_i(m, P_-)$ and supply functions $P_i(m, P_+)$ relate the buying $P_1$ and selling $P_2$ prices to the combined flow $m = \sum_{j=1}^{n} m_j$ and to the market estimates $P_-$ and $P_+$. We denote internal bank's expenses (salaries, lease of premises, service of networks etc.) as $z_j^0(m_j)$. Each bank solves its extremal problem

$$I_j = m_j[P_2(m, P_+) - P_1(m, P_-)] - z_j^0(m_j) \to \max_{m_j \geq 0}, \quad m = \sum_{j=1}^{n} m_j \qquad (89)$$

We assume that the functions $P_1$, $P_2$ and $z_j^0$ are continous and continuously differentiable on all its variables; $P_2$ is monotonically decreasing function of $m$ and increasing on $P_+$; $P_1$ increases on $P_-$ and $m$ and $z_j^0$ is convex and has discontinuity of the first kind at $m_j = 0$. Under these assumptions the profit $I_j$ is a convex function on $m_j$ and the conditions of profit's maximum yield the system of $n$ equations

$$\frac{\partial I_j}{\partial m_j} = 0 \Rightarrow (P_2(m, P_+) - P_1(m, P_-)) + (\frac{\partial P_2}{\partial m} - \frac{\partial P_1}{\partial m})m_j = \frac{dz_j^0}{dm_j}, \quad j = 1, \dots, n. \qquad (90)$$

*Example.* (Duopoly).
Assume $n = 2$,

$$P_1(m, P_-) = P_- + \alpha_1 m,$$

$$P_2(m, P_+) = P_+ - \alpha_2 m,$$

$$z_j(m_j) = z_{j0} + \beta_j m_j.$$

The equations (90) take the form

$$P_+ - P_- - (\alpha_1 + \alpha_2)m - m_j(\alpha_1 + \alpha_2) = \beta_j, \quad j = 1, 2.$$

We get

$$m_1^* = \frac{2(0.5(P_+ - P_-) - \beta_1 + 0.5\beta_2)}{3(\alpha_1 + \alpha_2)} \qquad (91)$$

$$m_2 * = \frac{2(0.5(P_+ - P_-) - \beta_2 + 0.5\beta_1)}{3(\alpha_1 + \alpha_2)} \qquad (92)$$

and the maximal profits are

$$I_1^* = m_1^*(\frac{P_+ - P_-}{3} + 0.5(\beta_2 - \beta_1)) - z_{10}, \tag{93}$$

$$I_2^* = m_2^*(\frac{P_+ - P_-}{3} + 0.5(\beta_1 - \beta_2)) - z_{20}, \tag{94}$$

The optimal norm of profit $\eta_j$ and the optimal average rate of credit $\mu$ obey the inequality

$$\eta_j = \frac{I_j^*}{m_j^* P_1((m_1^* + m_2^*), P_-)} > \mu. \tag{95}$$

This inequality is the necessary condition for a bank to be able to compete on the market.

If market estimates $P_+$ and $P_-$ change randomly then the equations (90) determine optimal flows of buying/selling for each moment of time. The expected profit is to be found by averaging $P_1^*$ and $I_2^*$ over all feasible estimates that takes into account joint density $f(P_+, P_-)$.

The considered model corresponds to Curno competition, when each bank maximizes its profit and the other banks do the same. If there is a collusion between banks then they maximize their combine profit which is then redistributed according to some rules. The problem here is maximize the combined profit

$$I = m[P_2(m, P_+) - P_1(m, P_-)] - z^*(m) \to \max_m, \tag{96}$$

function $z^*(m)$ here is

$$z^*(m) = \min_{m_1, m_2} \sum_j z_j^0(m_j), \quad \sum_j m_j = m, \quad m_j \geq 0. \tag{97}$$

Comparison of the problems (89) and (96) solutions allows one to perform a test of whether or not the banks collude with each other.

# 7 Conclusion

All models considered in this paper assume that the estimate of prices by FAs for traded assets (stocks, bonds, credit, derivatives, etc.) are known. It is also assumed that the demand/supply functions that relate the flows of assets with bank prices are known.

In practice the demand function can only be identified from the market data.

If the system is non-stationary then this identification should be done in real time or some prediction/correction algorithm should be used to adapts demand function model to variation of the process. This is especially important for derivative trading where demand function is constructed as a result of prediction by FA.

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