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Irreversibility and Limiting Possibilities of Macrocontrolled Systems: I. Thermodynamics*

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Abstract. In this paper, two types of systems — thermodynamic and economic — are considered, which include a large number of micro subsystems and are controlled on the macro level (macrocontrolled systems). The analogy between the maximal work problem in thermodynamics and the maximal profit problem in a microeconomic system is investigated. The notion of exergy is generalized for the systems which do not contain reservoirs, and the conditions of maximal power of heat engines are generalized for systems with arbitrary structure. The notion of system profitability and the measure of irreversibility of an microeconomic processes are introduced. The extremal principle which determines an equilibrium state of open microeconomic system, is formulated. The conditions of optimality of resource trading and the expression for profitability of resource exchange are formulated for systems which include market with perfect competition, and for systems which do not include it. Economic analogues of the second law of thermodynamics are formulated using introduced concepts. The first part of the paper is devoted to thermodynamic systems and the second to microeconomic systems.

1. Introduction

Many systems — physico-chemical, economic, social, etc. — consist of a large number of subsystems, where neither control nor measurement of the current states of individual subsystems are possible. The control in such systems changes the conditions for all the constititive subsystems. That is, the control here is a macrocontrol. The examples are the changes are the contact between thermodynamic systems, changes of prices, interest rates, taxes, etc. We shall call such systems macro controlled and the processes that occur in them — macrodynamic processes (following L. I. Rozonoer [8]). The important feature of macrocontrolled systems is that when a contact is established between such systems, the exchange processes between subsystems of these systems occur which change the macroscopic states of these systems. In order to separate subsystems and return them into their previous states it is necessary to use a control, which changes the state of the environment. This determines the irreversible character of the processes here. Please verify

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The factor of irreversibility has been studied in great detail in thermodynamics, but it plays no lesser role in other macrocontrolled systems. Meanwhile, there are very few results available regarding quantitative estimates of irreversibility for such systems. Also the methodology of thermodynamics has been rarely applied to solve problems here.

The maximal work is a typical problem related to irreversibility of thermodynamic systems. The concept of exergy (availability) is widely used for the analysis of thermodynamic systems [3, 4], etc. The exergy analysis of a system, which includes mechanical and heat processes, uses the exergy Ec of the amount of heat Q_+ derived from the reservoir with the temperature T_+ in the system, which includes the reservoir with the temperature T_-. It is defined as the maximal work that can be derived using Q_+. Since the limiting work corresponds to the reversible Carnot cycle of a heat engine, the exergy depends on the temperature of the cold reservoir T_- and is equal to the product of the amount of heat on the Carnot efficiency

$$E_0 = Q_+\Big(1 - \frac{T_-}{T_+}\Big). \tag{1}$$

The exergies of other forms of energy are defined similarly, via a reversible process of deriving work in a system with reservoir whose temperature, chemical potential, pressure, etc. are fixed. These fixed parameters are sometimes called the parameters of the "global environment". If the initial parameters of the system have the same values as the parameters of the global environment then the entropy of such system is zero. In the majority of cases some ambiguity of the parameters of the "global environment" is of no importance because calculations use exergy increments in various processes and not their absolute values. Later we consider the maximal work problem in various systems and subject to various conditions. In particular, it is possible that a system does not include reservoir, or that work has to be produced with a given power or in given time, etc. Accordingly, we define exergy as *the limiting work that can be derived in a thermodynamic system subject to given constraints*. When this generalized definition of exergy is used, irreversible processes are included into maximal work processes, which are used in exergy analysis. Moreover, in this case exergy cannot be defined as availability, because the maximal work in the system depends not only on the amount and potential of energy, but also on the structure of the system and the time frame of the process. This generalized definition of exergy is equivalent to the traditional definition for systems with reservoir and without constraints on rates of processes.

The maximal work can be derived from the system in a reversible process of equalization of the intensive parameters of its sub-systems. In such a process the entropy increment is zero, the rates of the exchange fluxes are infinitely small, and the duration of the processes tends to infinity. If the system is adiabatically insulated then its Ec is equal to the difference between its internal energy in the beginning of the process and in the end of it, when intensive variables of subsystems become equal. We shall denote by E_∞ the exergy of a system calculated without taking into account the constraints on the duration of the process.

The finite-time thermodynamics, which emerged in 1980-ties [1, 2], considers thermodynamic problems with fixed average rates. One of the ways to fix average rate is by imposing a constraint on the duration of the process τ. Then the availability of the heat fluxes in an open system is sought subject to additional constraint on the power N is the heat engine (which is fixed or maximal-possible).

In particular, if the system includes a number of reservoirs with the temperatures T_i and it is required that the rate of work production, that is, the power N of a hypothetical heat engine is given, then the exergy of the heat Q_+, derived from the reservoir with the temperature T_+, will be different from the expression (1). We shall denote this Ec as E_N.

If the system is adiabatically insulated and includes a number of subsystems with finite capacities whose temperatures change when their internal energies change, then the exergy can be defined as the maximal work which can be produced by temperature equalization in such system. Here the duration of the process can be unlimited or fixed and equal τ. Accordingly, we get the values of E_∞ and E_τ.

Maximal work problems for many thermodynamic systems have been solved. We will list their solutions. Other maximal work problems will be considered in this paper. It is clear, that the exergies E_N and E_τ (the exergy of thermodynamic system in finite time) are lower than E_0 and E_∞ correspondingly, because non-zero rates of the exchange processes lead to dissipative losses and the increase of system entropy. E_N and E_τ depend not only on the initial state of the system, capacities of the sub-systems, etc., but also on the coefficients and laws, which determine the kinetics of heat and mass exchange.

For example, in a simplest system, which includes reservoirs with the temperatures T_+ and T_-, and linear dependencies of the heat fluxes on the temperatures of the reservoirs and the working body T,

$$q_+ \;=\; \alpha_+(T_+ - T)\,, \qquad q_- \;=\; \alpha_-(T - T_-)\,,$$

the following solution was found [9]

$$E_N \;=\; \frac{2NQ_+}{N + \frac{\alpha}{4}(T_+ - T_-) - \sqrt{N^2 - \frac{\alpha}{2}(T_+ - T_-)N + \frac{\alpha^2}{16}(T_+ - T_-)^2}}\,, \tag{2}$$

where

$$\alpha \;=\; \frac{4\alpha_+\alpha_-}{(\sqrt{\alpha_+} + \sqrt{\alpha_-})^2}\,, \tag{3}$$

if the working body contacts reservoirs in turn, and

$$\alpha \;=\; \frac{4\alpha_+\alpha_-}{\alpha_+ + \alpha_-}$$

if there is a permanent contact with reservoirs.

As a rule, calculation of E_τ is reduced to the realization of minimal dissipation process with fixed average rate. This fact determines the importance of the study of minimal dissipation processes if finite-time [10].

In the first part of the paper, the new solutions of maximal work in finite time E_τ are derived for a number of thermodynamic systems of various configurations.

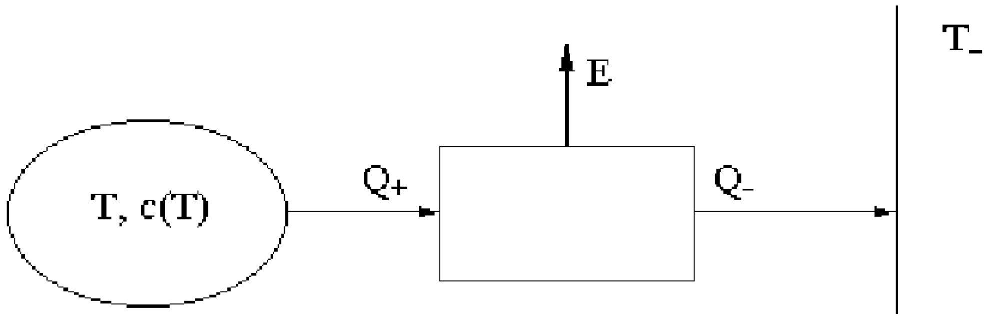

Fig. 1: A system which includes subsystems with finite capacities, working body and reservoir.

The second part of this paper is devoted to the introduction of similar notions and solution of similar problems in microeconomic systems.

2. Exergy of a Closed Thermodynamic System

If a system includes two or more reservoirs with different values of intensive variables, then it is possible to derive infinite amount of work from such a system. Thus the standard definition of exergy cannot be applied here. But it is possible to consider exergy of one given reservoir in the system. A large number of publications is devoted to the study of limiting possibilities of heat engines with fixed and with free power. Formulae, similar to (2), can be obtained not only for reservoirs whose temperatures differ, but also for reservoirs with different pressures, chemical potentials, etc. Here we will consider only thermal-mechanical systems.

2.1. System with reservoir

2.1.1. The duration of the process is not limited

Assume the system includes a reservoir with temperature T_-, k subsystems with finite heat capacities $c_i (i = 1, \dots, k)$ and initial temperatures T_{i0} and the working body. In [9] the maximal work problem was solved under the assumption that the temperatures of subsystems are controlled by changing their volumes. Here we assume that the volumes of the subsystems are constant and the work can be obtained by changing the volume, and consequently the temperature of the working body during its contact with subsystems.

Let us find E_∞ for one subsystem. First, we write the energy and entropy balances for the reversible heat engine which operates by exchanging energy between the subsystem and the reservoir (Fig. 1)

$$dQ_+ - dQ_- - dE = 0 , \quad \frac{dQ_+}{T} - \frac{dQ_-}{T_-} = 0 \quad \Rightarrow \quad dQ_- = dQ_+ \frac{T_-}{T} . \qquad (4)$$

The second equation states that there is no increase in the working body entropy.

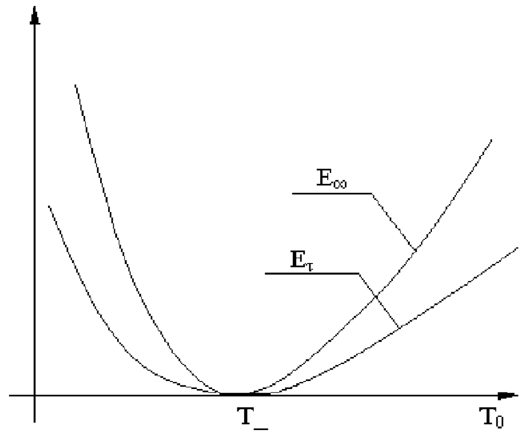

Fig. 2: Availability of the system shown in Fig. 1 for finite and for unlimited duration of the process.

From (4) it follows that

$$dE \;=\; dQ_+\Big(1 - \frac{T_-}{T}\Big) \;= -c\;(T)\Big(1 - \frac{T_-}{T}\Big)dT\,,$$

so

$$E_\infty \;=\; \int_{T_-}^{T_0} c(T)\Big(1 - \frac{T_-}{T}\Big)dT\,. \tag{5}$$

For the constant heat capacity

$$E_\infty \;=\; c\Big[T_0 - T_-\Big(1 + \ln\frac{T_0}{T_-}\Big)\Big]\,.$$

This function is non-negative and is equal zero if $T_0 = T_-$ (Fig. 2).

For k sub-systems $E_\infty = \sum_{i=1}^n E_{i\infty}$. For $c_i = \text{const}$

$$E_\infty \;=\; \sum_{i=1}^{n} c_i\Big[T_{0i} - T_-\Big(1 + \ln\frac{T_{i0}}{T_-}\Big)\Big]\,. \tag{6}$$

2.1.2. Fixed duration of the process

In this case for each subsystem (we omit subscript i here) we consider the problem of choosing such temperatures of the working body $T_1(t)$ and $T_2(t)$, that the derived work is maximal

$$E_\tau \;=\; \int_0^\tau (q_+(T,T_1) - q_-(T_2,T_-))\,dt \quad\to\quad \max_{T_1,T_2} \tag{7}$$

subject to the constancy of the working body entropy

$$\int_0^\tau \Big(\frac{q_+(T,T_1)}{T_1} - \frac{q_-(T_2,T_-)}{T_2}\Big)\, dt \;=\; 0\,, \tag{8}$$

and subject to the subsystem temperature changing, when the heat is removed from it, according to the equation

$$\frac{dT}{dt} \;=\; -\frac{q_+(T,T_1)}{c(T)}\,, \qquad T(0) \;=\; T_0\,. \tag{9}$$

Here again the condition (7) follows from the energy balance and the condition (8) follows from the entropy balance. q_+ and q_- denote the heat fluxes from the subsystem to the engine and from the engine to reservoir.

We denote the entropy increment of working body when it receives heat as ΔS. Then the problem (7)–(9) can be decomposed into two subproblems. The first problem is

$$Q_+ \;=\; \int_0^\tau q_+(T,T_1)\, dt \quad\to\quad \max_{T_1(t)} \tag{10}$$

subject to the constraint (9) and the condition

$$\int_0^\tau \frac{q_+(T,T_1)}{T_1}\, dt \;=\; \Delta S\,. \tag{11}$$

From the constancy of T_-, and therefore the constancy of T_2, it follows that the second subproblem takes the form

$$Q_- \;=\; q_-(T_2,T_-)\tau \quad\to\quad \min_{T_2} \tag{12}$$

subject to the constraint

$$\frac{q_-(T_2,T_-)}{T_2} \;=\; \frac{\Delta S}{\tau}\,. \tag{13}$$

Since the condition (13) links T_2 and ΔS, Q_- depends on ΔS, and the maximum of Q_+ depends on ΔS. ΔS should be chosen in such a way that

$$E_\tau(\Delta S) \;=\; [Q_+^*(\Delta S) - Q_-^*(\Delta S)] \quad\to\quad \max_{\Delta S>0}\,. \tag{14}$$

The problem (9), (10), (11) was studied in [5]. Using (9) and taking into account the monotone dependence of $T(t)$, one can substitute t with the temperature of the working T as a free variable of the problem. This yields the equivalent problem

$$Q_+ \;=\; \int_{\overline{T}}^{T_0} c(T)\, dT \quad\to\quad \max_{\overline{T},T_1(T)} \tag{15}$$

subject to the constraints

$$\int_{\overline{T}}^{T_0} \frac{c(T)}{T_1(T)}\, dT \;=\; \Delta S\,, \tag{16}$$

$$\int_{\overline{T}}^{T_0} \frac{c(T)}{q_+(T,T_1)}\, dT \;=\; \tau\,. \tag{17}$$

The conditions of optimality of this problem have the following form [5]

$$\frac{\partial q_+}{\partial T_1}\left(\frac{T_1^*(T)}{q_+(T,T_1^*(T))}\right)^2 \;=\; \text{const}\,, \quad \forall\, T\,. \tag{18}$$

They determine $T_1(T)$ up to the constant k.

The value of this constant k is to be found by substituting the dependency $T_1^*(T,k)$, found from (18), into (16), (17). Its dependence on ΔS leads to the dependence of Q_+^* on ΔS or, which is equivalent, to its dependence on k.

For Newton laws of heat exchange

$$q_- \;=\; \alpha_-(T_2 - T_-)\,, \qquad q_+ \;=\; \alpha_+(T - T_1)\,, \tag{19}$$

and constant heat capacity c, the equation (18) takes the form

$$\frac{T_1^2}{\alpha_+(T-T_1)^2} \;=\; \text{const} \quad\Longrightarrow\quad T_1 \;=\; kT\,,$$

where k is some constant $(0 < k < 1)$.

The substitution of $T_1(T,k)$ into the conditions (16), (17) yields equations, which link $k, \Delta S$ and $\overline{T}$

$$\Delta S \;=\; \frac{c}{k}\ln\frac{T_0}{\overline{T}}\,, \qquad \tau \;=\; \frac{c}{\alpha_+(1-k)}\ln\frac{T_0}{\overline{T}}\,, \tag{20}$$

or

$$\overline{T} \;=\; T_0 \exp\Big(-\frac{\tau\alpha_+(1-k)}{c}\Big)\,. \tag{21}$$

The substitution of $\overline{T}$ into the condition (20) yields

$$\Delta S \;=\; \frac{\tau\alpha_+(1-k)}{k}\,, \tag{22}$$

and the substitution of $T_1 = kT$ and $\overline{T}$ into (15) gives

$$Q_+^*(k) \;=\; T_0 c\Big[1 - \exp\Big(-\frac{\tau\alpha_+(1-k)}{c}\Big)\Big]\,. \tag{23}$$

From the condition (13)

$$\alpha_-(T_2 - T_-) \;=\; \frac{\Delta S T_2}{\tau} \;=\; \frac{\alpha_+(1-k)}{k}T_2$$

it follows that

$$T_2 \;=\; T_- \frac{\alpha_- k}{\alpha_- k - \alpha_+ (1-k)}\,.$$

Because $T_2 > 0$, $\alpha_- k - \alpha_+(1-k) > 0$, and therefore $k > \alpha_+/(\alpha_- + \alpha_+)$.

$$Q^*_-(k) \;=\; \frac{\tau T_- \alpha_+ \alpha_- (1-k)}{\alpha_- k - \alpha_+(1-k)}\,. \tag{24}$$

The condition (14) yields the equation for k

$$\frac{\partial Q^*_+}{\partial k} \;=\; \frac{\partial Q_-}{\partial k} \quad\Rightarrow\quad T_0 e^{-\frac{\alpha_+ \tau}{c}(1-k)} \;=\; \frac{T_- \alpha_-^2}{(\alpha_- k - \alpha_+(1-k))^2}\,. \tag{25}$$

The left hand side of this equation is a monotonically increasing function of k, when k increases from zero to one. Its right hand side has discontinuity at $k^0 = \alpha_+/(\alpha_- + \alpha_+)$. If $k < k^0$ then the right hand side of this equation is non-negative. If $k > k^0$ then this right hand side monotonically decreases from infinity to T_-. The solution of the equation (25) exists, is unique and for $T_0 > T_-$, it obeys the inequality

$$\frac{\alpha_+}{\alpha_+ + \alpha_-} \;<\; k \;\le\; 1\,.$$

After the value of k_i, which maximizes $E_{i\tau}(\Delta S_i(k_i))$, is found for each of the subsystems the availability is determined as

$$E_\tau \;=\; \sum_i E^*_{i\tau}\,.$$

2.2. Systems without reservoirs

2.2.1. Unlimited duration of the process

If there are no restriction on the duration, then the process which extracts maximal work from the subsystem (Fig. 3) is a reversible process in which the temperatures of all subsystems equalize and approach the same value Θ.

Here the entropy of the system is not increasing, because the working body receives and returns heat at the temperature which is infinitely close to the temperature of the subsystem. The exergy of the system E_∞ is equal to the decrease of the internal energy of the system

$$E_\infty \;=\; \sum_{i=1}^{k} \int_{\Theta}^{T_{i0}} c_i(T)\, dT\,.$$

The value of Θ is to be found from the condition of constancy of the entropy of the working body

$$\sum_{i=1}^{k} \Delta S_i \;=\; \sum_{i=1}^{k} \int_{\Theta}^{T_{i0}} \frac{c_i(T)}{T}\, dT \;=\; 0\,. \tag{26}$$

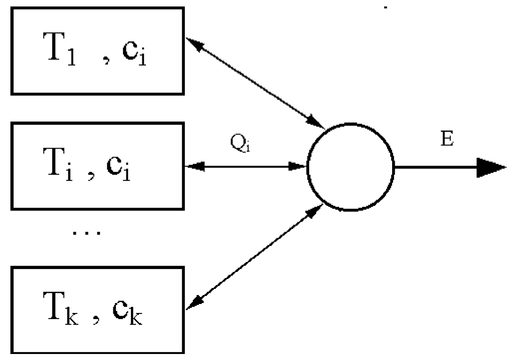

Fig. 3: A system without reservoir.

In particular for the constant heat capacities from the equality (26) follows that the finite temperature Θ is

$$\Theta \;=\; \prod_{i=1}^{k} T_{i0}^{\gamma_i}\,, \qquad \gamma_i = \frac{c_i}{\sum\limits_{\nu=1}^{k} c_\nu}\,,$$

so that

$$E_\infty \;=\; \sum_{i=1}^{k} c_i(T_{i0}-\Theta) \;=\; (\overline{T}-\Theta)\sum_{i=1}^{k} c_i\,. \tag{27}$$

2.2.2. Relationship between exergy losses and change of system's entropy

If the traditional definition of exergy as potential availability is used for the system with a reservoir, then the loss of exergy in irreversible equilibrium process is proportional to the increase of the system entropy. So for the heat engine which includes a reservoir with temperature T_- and subsystems with heat capacities c_i and initial temperatures T_{i0} the loss of exergy during irreversible temperature equalization process is

$$E_\infty \;=\; T_- \Delta S\,,$$

where the entropy increment is

$$\Delta S \;=\; \sum_i c_i\Big(\frac{T_{i0}}{T_-} - \ln\frac{T_{i0}}{T_-} - 1\Big)\,.$$

Consider now a system, which consists of subsystems with heat capacities c_i and initial temperatures T_{i0}, but does not include a reservoir. The exergy of such a system is determined by (27), where $\overline{T} = \sum_i T_{i0}\gamma_i$ is the average temperature of subsystems and $\gamma_i = c_i/(\sum_\nu c_\nu)$ is the relative heat capacity of the i-th subsystem.

The increment of the system entropy in irreversible temperature equalization from T_{i0} to $\overline{T}$ is

$$\Delta S \;=\; \sum_i \Delta S_i \;=\; \sum_i c_i \ln\frac{\overline{T}}{T_{i0}} \;=\; \ln\frac{\overline{T}}{\Theta}\sum_i c_i\,,$$

thus

$$\frac{\overline{T}}{\Theta} = \exp \frac{\Delta S}{\sum_i c_i} .$$

After substitution into (27) we get the expression for the exergy loss in the following form

$$E_\infty = \sum_i c_i \overline{T} \Big(1 - \frac{\Theta}{\overline{T}} \Big) = \sum_i c_i T_{i0} \Big[1 - \exp \Big(- \frac{\Delta S}{\sum_i c_i} \Big) \Big] . \tag{28}$$

This is a monotone dependence and its slope for $\Delta S = 0$ is

$$\left(\frac{dE_\infty}{d\Delta S} \right)_{\Delta S = 0} = \overline{T} .$$

2.2.3. Fixed duration of the process

Let us consider the same problem with the fixed process duration τ. The difference between this problem and the problem considered in Sec. 2.2.1 is that the temperatures for each of the subsystems at the end of the process $\overline{T_i}$ are different and that the entropy of the system increases. The increments of the internal energy and the entropy of the working body equal zero. The problem can be written as

$$E_\tau = \sum_{i=1}^{k} \int_{\overline{T_i}}^{T_{i0}} c_i(T)\, dT \quad \to \quad \max_{\overline{T_i}, T_{pi}(T)} , \tag{29}$$

subject to the constraints

$$\Delta S_p = \sum_{i=1}^{k} \int_{\overline{T_i}}^{T_{i0}} \frac{c_i(T)}{T_{pi}(T)}\, dT = 0 , \tag{30}$$

$$\int_{\overline{T_i}}^{T_{i0}} \frac{c_i(T)\, dT}{q_i(T, T_p)} = \tau , \qquad i = 1, \ldots, k . \tag{31}$$

Here T_{pi} is the temperature of the working body during contact with the i-th subsystem. The condition (29) corresponds to the maximal decrease of the internal energy of the system, the condition (30) corresponds to the zero increase of the entropy of the working body and (31) corresponds to the constraint on the duration of the process.

The problem (29)–(31) is separable and can be decomposed into k subproblems about optimal contact between the working body and each of the subsystems. Initially we assume that the entropy increment of the working body during its contact with i-th subsystem ΔS_i is fixed.

The optimal contact problem takes the form (we omit subscript i)

$$E_\tau = \int_{\overline{T}}^{T_0} c(T)\, dT \quad \to \quad \max_{\overline{T}, T_p(T)} , \tag{32}$$

subject to constraints

$$\int_{\overline{T}}^{T_0} \frac{c(T)}{T_p(T)}\, dT \;=\; \Delta S\,, \tag{33}$$

$$\int_{\overline{T}}^{T_0} \frac{c(T)\, dT}{q(T,T_p)} \;=\; \tau\,. \tag{34}$$

Note that the problem (32)–(34) is identical to the problem (15)–(17). Therefore the temperature of the working body during its contact with the i-th subsystem obeys the condition (18), which determines $T_{pi}(T_i,k_i)$. The substitution of this dependence into (33),(34) yields the system of two equations with three unknowns $\overline{T_i}$, ΔS_i, k_i.

During the second stage of the problem solution these unknowns are chosen in such a way that

$$E_\tau \;=\; \sum_{i=0}^{m} E_{\tau i}(\Delta S_i, k_i, \overline{T_i}) \quad\to\quad \max_{\Delta S_i, \overline{T_i}, k_i} \tag{35}$$

subject to the constraints

$$\sum_{i=1}^{m} \Delta S_i(k_i, \overline{T_i}) \;=\; \Delta S_p \;=\; 0\,, \tag{36}$$

$$\varphi_i(\Delta S_i, k_i, \overline{T_i}) \;=\; \tau\,, \qquad i=1,\ldots,m\,. \tag{37}$$

Here we denote the function, which is obtained by integrating (34) after substituting there the dependencies $T_{pi}(T,k)$ found from (18), as φ_i.

In particular, for the Newton laws of heat exchange

$$q_i \;=\; \alpha_i(T_i - T_{pi})\,, \qquad i=1,\ldots,m\,, \tag{38}$$

as it was shown above, the conditions (18) lead to the dependence

$$T_{pi}(T_i) = k_i T_i, \qquad i=1,\ldots,m. \tag{39}$$

For the constant heat capacities the substitution of this expression into (33) and (34) gives the dependencies $\overline{T_i}(k_i)$ and $\Delta S_i(k_i)$ in form (21), (22).

The problem (36)–(37) is reduced to finding such k_i, ΔS_i, $\overline{T_i}$, that the maximum is attained

$$E_\tau \;=\; \sum_{i=1}^{m} c_i(T_{i0} - \overline{T_i}) \quad\to\quad \max_{\overline{T_i}, \Delta S_i, k_i} \tag{40}$$

subject to the constraints (36), (21), (22). After elimination of ΔS_i these constraints can be reduced to the following form

$$\overline{T_i}(k_i) \;=\; T_{i0} \exp\Big(-\frac{\tau\alpha_i(1-k_i)}{c_i}\Big)\,, \qquad i=1,\ldots,m\,, \tag{41}$$

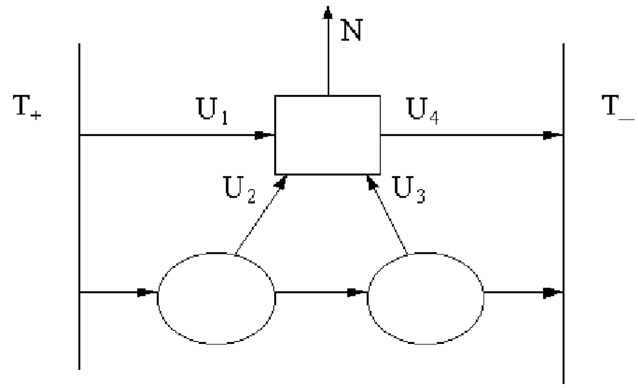

Fig. 4: The structure of an open system with heat engine.

$$\sum_{i=1}^{m} \frac{\alpha_i(1-k_i)}{k_i} = 0\,. \tag{42}$$

Substitution of $\overline{T_i}(k_i)$ into (40) yields a separable problem (40), (42) with m unknowns k_i. The stationarity conditions for the i-th summand the Lagrange function for this problem R_i

Please verify

$$R_i = c_i(T_{i0} - \overline{T_i}(k_i)) - \lambda \frac{\alpha_i(1-k_i)}{k_i}$$

on k_i lead to the system of equations

$$k_i^2 \overline{T_i}(k_i) = \text{const} = \frac{\lambda}{\tau}, \qquad i = 1, \ldots, m\,, \tag{43}$$

which determine $k_i(\lambda)$. The value of λ is to be found after substitution of these dependencies into (42).

3. Maximal Power in an Open Thermodynamic System

In this section, we consider an open thermodynamic system which consists of a number of reservoirs with constant temperatures, and subsystems whose temperatures are determined by their internal energies. When the heat engine contacts with thermodynamic subsystems it receives and rejects fluxes of heat and produces work. It is required to find such contact temperatures u_i for contacts of the heat engine with each of subsystems that the power of the heat engine N is maximal. The total number of thermodynamic subsystems here is n, and at least two of these subsystems are reservoirs (Fig. 4).

The problem of maximal power for a system with two reservoirs with the temperatures T_+ and T_- was first considered by Novikov [6], and later by Curson and Ahlborn [7] and others (see review [11]). Our formulation generalizes this problem for the systems with arbitrary structure.

We denote the temperature of the i-th subsystem as T_i, the heat flux between the i-th and j subsystems as $q_{ji}(T_j, T_i)$ and the heat flux between the i-th subsystem, and the working body as $q_i(T_i, u_i)$. We assume that the heat engine is internally reversible so the entropy production in it is equal to zero. The maximal power problem then is written as

$$N = \sum_{i=m}^{n} q_i(T_i, u_i) \quad \rightarrow \quad \max_{u_i} \tag{44}$$

subject to the constraints

$$\sum_{i=m}^{n} \frac{q_i(T_i, u_i)}{u_i} = 0\,, \tag{45}$$

$$\sum_{j=1}^{n} q_{ji}(T_j, T_i) = q_i(T_i, u_i)\,, \qquad i = 1, \ldots, m\,. \tag{46}$$

The conditions (44), (45) follow from the energy and entropy balances for the working body, and (46) follows from the energy balance for the i-th subsystem whose number m is $m \leq n - 2$. The temperatures of the reservoirs T_i ($i = m + 1, \ldots, n$) are given and constant.

The conditions which determine u_i and T_i for $i \leq m$, follow from the conditions of stationarity of the Lagrange function for the problem (44)–(46)

$$L = \sum_{i=1}^{n} \left[q_i(T_i, u_i)\left(1 + \frac{\Lambda}{u_i} - \alpha_i\right) + \lambda_i \sum_{j=1}^{n} q_{ji}(T_j, T_i) \right],$$

for u_i, T_i. Note that $\lambda_i = 0$ for $i > m$.

$$\frac{\partial L}{\partial u_i} = 0 \;\Rightarrow\; \frac{\partial q_i}{\partial u_i}\left(1 + \frac{\Lambda}{u_i} - \lambda_i\right) = \Lambda \frac{q_i(T_i, u_i)}{u_i^2}\,, \qquad i = 1, \ldots, n\,, \tag{47}$$

$$\frac{\partial L}{\partial T_i} = 0 \;\Rightarrow\; \frac{\partial q_i}{\partial T_i}\left(1 + \frac{\Lambda}{u_i} - \lambda_i\right) + \lambda_i \sum_{j=1}^{n} \frac{\partial q_{ji}}{\partial T_i} = 0\,, \qquad i = 1, \ldots, m\,. \tag{48}$$

Equations (45)–(48) allow us to find $(n + m)$ variables u_i and T_i and $(m + 1)$ Lagrange multipliers.

For Newton heat exchange $q_i = \alpha_i(T_i - u_i)$, $q_{ji} = \alpha_{ji}(T_j - T_i)$ the equations (45)–(48) can be rewritten as

$$\sum_{i=1}^{n} \widetilde{\alpha}_i \frac{T_i}{u_i} = 1\,, \quad \text{where} \quad \widetilde{\alpha}_i = \frac{\alpha_i}{\sum_{i=1}^{n} \alpha_i}\,, \tag{49}$$

$$\sum_{j=1}^{n} \alpha_{ji}(T_j - T_i) = \alpha_i(T_i - u_i)\,, \qquad i = 1, \ldots, m\,, \tag{50}$$

$$u_i^2(1-\lambda_i) \;=\; \Lambda T_i\,, \qquad i=1,\dots,n\,, \tag{51}$$

$$\alpha_i\Big(1+\frac{\Lambda}{u_i}-\lambda_i\Big) \;=\; \lambda_i\sum_{j=1}^{n}\alpha_{ji}\,, \qquad i=1,\dots,m\,. \tag{52}$$

In the particular case when $n=2$, $m=0$, $T_1=T_+$, $T_2=T_-$ from these conditions follow the known results [6], [7] for the maximal power of heat engine. Indeed in this case $\lambda_i=0$, $u_1^*=\sqrt{\Lambda T_+}$, $u_2^*=\sqrt{\Lambda T_-}$, and the efficiency of the heat engine η is

$$\eta \;=\; 1-\frac{u_2^*}{u_1^*} \;=\; 1-\sqrt{\frac{T_-}{T_+}}\,,$$

and its maximal power is

$$N_{\max} \;=\; \frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2}(\sqrt{T_+}-\sqrt{T_-})^2.$$

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