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Thermodynamic constraints on temperature distribution in a stationary system with heat engine or refrigerator

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Abstract
In this paper we consider a stationary thermodynamic system that includes a transformer of mechanical energy into heat energy or heat into mechanical energy. We derive conditions that determine temperature distribution (temperature field) inside such a system permitted by thermodynamics. We obtain conditions that divide feasible temperature fields into two classes—one where mechanical energy has to be spent and another where it is extracted. Closed-form expressions for the minimal supplied/maximal extracted power are derived. It is shown that for a linear heat transfer law and heat engine operating at maximal power the ratio of engine working body's temperatures during contact with reservoirs is equal to the square root of the ratio of reservoirs' temperatures irrespective of the system's structure and whether the engine is internally irreversible or not. Therefore, an engine's efficiency at maximal power does not depend on its internal structure. The problem of maintaining given temperatures in a subset of inter-connected chambers is considered. The conditions that determine optimal temperatures in the chambers where temperatures are not fixed which minimize energy are derived.

1. Introduction

Finite-time thermodynamics began with studies of the maximal power problem for a heat engine (e.g. [1–6, 9, 14]). It then extended to studies of direct and inverse cycles, maximal efficiency for given power [10, 11], etc.

We extend the Novikov–Curzon–Ahlborn maximal power problem for a heat engine with two reservoirs by considering a general-type system, which includes a transformer of mechanical into heat and heat into mechanical energy (heat engine or refrigerator), heat reservoirs and finite capacity subsystems with different temperatures that contact reservoirs and each other (see an example of such a system in figure 1). We assume that each subsystem (reservoir with constant temperature, finite capacity subsystem with a temperature that depends on its extensive variables, the heat/mechanical energy transformer) is in internal equilibrium and all irreversibility arises at the boundaries of subsystems. That is, we consider endoreversible systems. The thermodynamic description for a subsystem is valid only for such a system. We consider open systems, which receive and/or reject energy flows from the environment. We consider stationary non-equilibrium states of such systems. If the law of heat transfer and heat transfer coefficients are given then such a system has a stationary regime. It is described by the temperature distribution between the system's subsystems (that is, by the discrete temperature field).

If there is no transformer in the system then a unique distribution of temperatures will be reached. The temperatures of subsystems here are determined by the energy balance equations. The number of equations is the same as the number of unknowns. If exchange flows are monotone functions with respect to each of its temperatures, then the stationary temperatures of the subsystems are unique. We denote this self-settled temperature field in the system as Θ_0. If the system includes a transformer then different configurations of

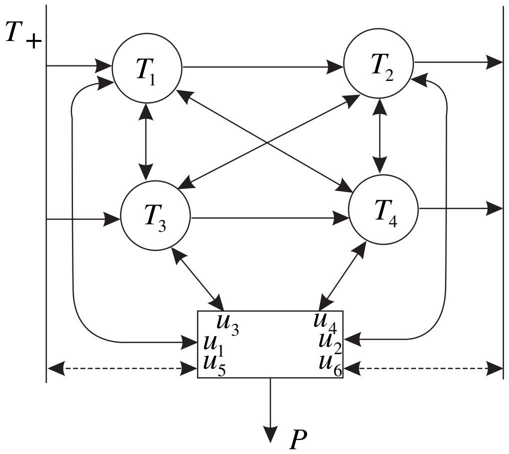

Figure 1. An open thermodynamic system.

temperature fields are possible depending on the heat flows supplied/ removed from each subsystem by the transformer. In its turn, the transformer itself uses or rejects power equal to the net heat flows it receives. For some temperatures the transformer's maximal power P^* will be positive and for others it will be negative. For the latter the absolute value of this maximal power will be equal to the minimal power needed for the heat pump to maintain some stationary field of temperatures which differ from Θ_0.

In the particular case of a system with only one reservoir (the environment) the maximal power of the transformer is always negative. The maximal power problem here becomes the optimal thermostatting problem of maintaining the given field of temperatures using the minimal energy [8, 12]. It was demonstrated in [8,12,13] that if the system consists of multiple chambers and it is required to maintain the given temperature in one target chamber only, and that temperature is significantly different from the environment temperature, then it is optimal to supply/ remove the heat not only into the target chamber but also into other adjoint chambers.

If reservoirs' temperatures are given then the temperatures of the finite-capacity subsystems cannot be arbitrary, no matter how much power is supplied/removed from the system by the transformer. That is why the problem of a transformer's limiting possibilities can be formulated as finding conditions that determine temperature distributions that are permitted by thermodynamics and dividing all thermodynamically feasible temperature distribution into two non-overlapping classes:

- the temperature fields that allow us to obtain power from the system—*power generating*;
- the temperature fields that require us to spend power to maintain them—*power consuming*.

Solutions of these problems immediately follow from the solution of the maximal power problem. Indeed if the dependence of P^* on the temperature field Θ is found then the condition $P^*(\Theta) = 0$ determines the boundary between generating and consuming systems.

The maximal power problem for two reservoirs and linear heat transfer has been studied in detail. The optimal thermostatting problem has been formulated and solved for a system that consists of sequentially connected subsystems only in [12–14]. The problem of determining thermodynamic bounds for a temperature was considered some 20 years ago by Sertorio *et al* (e.g. [7]). To the best of our knowledge the problem of maximal power in a general-type system and the problem of constructing the set of realizable temperature fields and its division into generating and consuming power subsets has not been considered in the literature.

In this paper we consider these problems for a general type system with arbitrary structure. The general solutions are then specified for Newton's laws of heat transfer. In many cases it turns out that the extremal conditions are reduced to the requirements that some function has the same value during every contact between the heat engine (heat pump) and every subsystem. This allows us to construct a control system to maintain maximal power when external conditions change. In the following we will refer to the heat engine and heat pump as transformers.

All these problems can be extended into systems which are non-homogeneous with respect to pressure or to other intensive variables. We limited the scope to temperature non-homogeneous systems to keep the results in a compact form.

2. Transformer's maximal power

We consider a stationary state of a thermodynamic system which consists of $(n - m)$ reservoirs with constant temperatures, m finite-capacity subsystems, whose temperatures are determined by their internal energies and the transformer. We denote the heat exchange flows between subsystems as q_{ij}. These flows are caused by the temperature differences between subsystems. The transformer generates power by contacting the subsystems when it receives heat from them or rejects heat into them. It is required to find such temperatures u_i for the contact between the transformer and each of the subsystems such that the power P is maximal. If the maximal power is negative then it corresponds to the minimum of the external power consumed by the system.

2.1. Problem formulation and conditions of optimality

We denote the temperature of the ith subsystem as T_i, the heat flow from the ith and the jth subsystems as $q_{ij}(T_i, T_j)$, the temperature of the working body when it contacts the ith subsystem as u_i, the heat flow between the ith subsystem and the transformer as $q_i(T_i, u_i)$ and the power of the transformer as P. We define the flow entering each subsystem as positive. When T_i increases, q_{ij} decreases monotonically, and when T_j increases, the flow increases monotonically. If $T_i = T_j$ then $q_{ij} = 0$. If there is no contact between subsystems then $q_{ij} = 0$. We assume that functions $q_{ij}(T_i, T_j)$ are continuously differentiable and that the working body is internally reversible and the entropy production in it is equal to zero.

The maximal power problem takes the form

$$P = \sum_{i=1}^{n} q_i(T_i, u_i) \to \max_{u_i > 0} \qquad (1)$$

subject to

$$\sum_{i=1}^{n} \frac{q_i(T_i, u_i)}{u_i} = 0, \tag{2}$$

$$\sum_{j=1}^{n} q_{ij}(T_j, T_i) = q_i(T_i, u_i), \qquad i = 1, \dots, m. \tag{3}$$

The form of objective (1) follows from the energy balance for a transformer's working body. Condition (2) follows from the working body's entropy balance, and condition (3) follows from the energy balance for the ith subsystem with finite capacity.

We divide m finite-capacity subsystems into two categories—the subsystems with free temperatures T_i ($i = 1, \dots, r$), which can be controlled jointly with u_i to maximize P, and the subsystems with fixed temperatures ($i = r + 1, \dots, m$). We shall call them the subsystems with fixed and free temperatures. The unknown variables in the problem (1)–(3) are the n contact temperatures of the working body u_i and r free temperatures of subsystems T_i ($i = 1, \dots, r$), which are related to each other via heat balance equations.

We assume that problem (1)–(3) is not singular. Then its conditions of optimality can be expressed in terms of its Lagrange function L

$$L = \sum_{i=1}^{m} q_i + \sum_{i=m+1}^{n} q_i - \Lambda \sum_{i=1}^{m} \frac{q_i}{u_i} - \Lambda \sum_{i=m+1}^{n} \frac{q_i}{u_i} + \sum_{i=1}^{m} \lambda_i \left(\sum_{j=1}^{n} q_{ij} - q_i \right), \tag{4}$$

where λ_i and Λ are Lagrange multipliers. $\lambda_0 = 1$.

The stationarity conditions for L for $i > m$ yield the following equations

$$\frac{\partial L}{\partial u_i} = \frac{\partial}{\partial u_i} \left[q_i(T_i, u_i) \left(1 - \frac{\Lambda}{u_i} \right) \right] = 0, \quad i = m + 1, \dots, n. \tag{5}$$

Therefore, the contacts' temperatures obey the condition

$$\frac{u_i^2 \partial q_i / \partial u_i}{u_i \partial q_i / \partial u_i - q_i} = \Lambda, \qquad i = m + 1, \dots, n. \tag{6}$$

We shall call the left-hand side of this equation (which has the dimension of temperature) the *reduced contact temperature*. Thus, the following statement holds: *in order to obtain the maximal power the reduced contact temperatures for contacts with all reservoirs of the transformer must be equal.*

From the conditions $\partial L / \partial u_i = 0$ and $\partial L / \partial T_i = 0$ we obtain

$$\frac{\partial}{\partial u_i} \left[q_i \left(1 - \frac{\Lambda}{u_i} - \lambda_i \right) \right] = 0, \qquad i = 1, \dots, m. \tag{7}$$

$$\frac{\partial}{\partial T_i} \left[q_i \left(1 - \frac{\Lambda}{u_i} - \lambda_i \right) \right] + \sum_{j=1}^{n} (\lambda_j - \lambda_i) \frac{\partial q_{ij}}{\partial T_i} = 0, \quad i = 1, \dots, r. \tag{8}$$

We took into account here that $q_{ij} = -q_{ji}$. The multiplier $\lambda_j = 0$ for $j > m$. From condition (7) it follows that the reduced contact temperature for a contact with a finite-capacity subsystem is

$$\frac{u_i^2 \partial q_i / \partial u_i}{u_i \partial q_i / \partial u_i - q_i} = \frac{\Lambda}{(1 - \lambda_i)}, \qquad i = 1, \dots, m. \tag{9}$$

Condition (9) relates the reduced contact temperature for contact with the ith subsystem with the reduced contact temperature with reservoirs Λ and multipliers λ_i, which are determined by the set of equations (3), (8).

These general conditions can be significantly simplified for particular systems. For example, if the system contains only reservoirs then the optimal contact temperatures are found from (6) and Λ from condition (2); if the temperatures of all subsystems with finite capacities are fixed then the balance equations (3) determine all u_i for $i = 1, \dots, m$ and, therefore, all terms in (2) and (1), except the reservoirs' terms. We will show below that the problem here is reduced to maximization of power obtained from reservoirs subject to the given entropy flow from reservoirs to the transformer.

2.2. *Newton's laws of heat transfer*

In many cases it is assumed that the flows q_i, q_{ij} depend linearly on the temperatures

$$q_i = \alpha_i (T_i - u_i), \qquad q_{ij} = \alpha_{ij} (T_j - T_i), \tag{10}$$

where α_{ij}, α_i are the heat transfer coefficients. This is called Newton's heat transfer law.

For the Newton heat transfer the problem (1)–(3) takes the form

$$P = \sum_{i=1}^{n} \alpha_i (T_i - u_i) \to \max_{u_i > 0}, \tag{11}$$

subject to

$$\sum_{i=1}^{n} \bar{\alpha}_i^u \frac{T_i}{u_i} = 1, \qquad \bar{\alpha}_i^u = \frac{\alpha_i}{\sum_{\nu=1}^{n} \alpha_\nu}, \tag{12}$$

$$\sum_{j=1}^{n} \alpha_{ij} (T_j - T_i) = \alpha_i (T_i - u_i), \qquad i = 1, \dots, m. \tag{13}$$

The stationarity conditions for its Lagrange function become

$$\frac{\partial L}{\partial u_i} = 0 \Rightarrow u_i^2 (1 - \lambda_i) = \Lambda T_i, \qquad i = 1, \dots, n, \tag{14}$$

where $\lambda_i = 0$ for $i > m$.

$$\frac{\partial L}{\partial T_i} = 0 \Rightarrow \alpha_i \left(1 - \frac{\Lambda}{u_i} - \lambda_i \right) + \sum_{j=1}^{n} (\lambda_j - \lambda_i) \alpha_{ij} = 0, \quad i = 1, \dots, r. \tag{15}$$

Substitution of the Newton law of heat transfer into (6) gives the reduced temperatures of the working body during its contact with the reservoirs

$$\frac{u_i^2}{T_i} = \Lambda, \qquad i = m + 1, \dots, n. \tag{16}$$

Thus, the optimal temperatures for contacts with reservoirs are proportional to the square root of the reservoirs' temperatures. If heat transfer is linear and the system has two reservoirs then we get the known results [1, 2] that the heat engine's maximal efficiency for a cycle with maximal power is equal to $\eta_{\mathrm{NCA}} = 1 - \sqrt{T_2/T_1}$. Thus, condition (6) generalizes the Novikov–Curzon–Ahlborn for arbitrary heat transfer and arbitrary number of heat reservoirs.

For the subsystems

$$\frac{u_i^2}{T_i} = \frac{\Lambda}{(1-\lambda_i)}, \qquad i = 1, \dots, m. \tag{17}$$

2.3. System which consists of reservoirs and transformer

We are seeking the maximal power ${}_rP^*$ for a system which consists of n reservoirs ($m = 0$) and the transformer. From Newton's law of heat transfer from (16) it follows that the optimal temperatures for contacts with reservoirs are

$$u_i = \sqrt{\Lambda}\sqrt{T_i} = \frac{\sum_{j=1}^{n} \alpha_j \sqrt{T_j}}{\alpha_\Sigma}\sqrt{T_i}, \qquad \alpha_\Sigma = \sum_{j=1}^{n} \alpha_j, \tag{18}$$

and the power is

$$P_r^* = \sum_{i=1}^{n} \alpha_i \left(T_i - \sqrt{T_i} \frac{\sum_{j=1}^{n} \alpha_j \sqrt{T_j}}{\alpha_\Sigma} \right). \tag{19}$$

Example 1. Consider the system which consists of four reservoirs and a heat engine. Reservoir temperatures and heat transfer coefficients are given

$$T_1 = 300\,\mathrm{K}, \quad T_2 = 700\,\mathrm{K}, \quad T_3 = 500\,\mathrm{K}, \quad T_4 = 350\,\mathrm{K},$$

$$\alpha_1 = 200\frac{\mathrm{W}}{\mathrm{K}}, \quad \alpha_2 = 100\frac{\mathrm{W}}{\mathrm{K}}, \quad \alpha_3 = 250\frac{\mathrm{W}}{\mathrm{K}}, \quad \alpha_4 = 300\frac{\mathrm{W}}{\mathrm{K}}.$$

The optimal contact temperatures found from (18) are

$$u_1 = 352.78\,\mathrm{K}, \quad u_2 = 538.88\,\mathrm{K},$$
$$u_3 = 455.43\,\mathrm{K}, \quad u_4 = 381.04\,\mathrm{K}.$$

The maximal power which can be obtained from this system is $P_r^* = 7.38$ kW. The reduced contact temperatures are the same for all reservoirs. From (16) it follows that they are equal to $\Lambda = 414.84$ K.

2.4. Transformer's maximal power and optimal temperatures of the subsystems

Consider a system where the temperatures of all subsystems are free ($r = m$). Since the subsystems' temperatures are chosen optimally, it is clear that if the number of reservoirs is larger than one where at least two reservoirs have different temperatures the maximal power is positive and not lower than P_r^* given by (19).

We decompose this problem into three subproblems.

- To maximize the power $P_r^*(\sigma_r)$ derived from the contact between transformer and reservoirs for the given entropy flow from the reservoirs to the transformer σ_r

$$P_r(\sigma_r) = \sum_{i=m+1}^{n} q_i \to \max_{q_i}, \tag{20}$$

subject to

$$\sum_{i=m+1}^{n} \frac{q_i}{u_i} = \sigma_r. \tag{21}$$

- To maximize the power $P_s^*(\sigma_s)$ generated from contact between the transformer and finite-capacity subsystems subject to the given flow of entropy from the subsystems to the transformer's working body σ_s

$$P_s(\sigma_s) = \sum_{i=1}^{m} q_i \to \max_{q_i}, \tag{22}$$

subject to

$$\sum_{i=1}^{m} \frac{q_i}{u_i} = \sigma_s \tag{23}$$

and to the balance equations (13). We will show later that σ_s is bounded from above.

- To find the maximal combined power subject to the entropy balance for the working body of the transformer

$$P(q_i) = P_s^*(\sigma_s) + P_r^*(\sigma_r) \to \max / \sigma_r + \sigma_s = 0. \tag{24}$$

The former problem was considered above for $\sigma_r = 0$. The condition for finding optimal temperatures for contacts with reservoirs from (6), (18) remains valid. After substitution of (6) into (21) we obtain

$$\Lambda = \frac{\sum_{i=m+1}^{n} u_\nu(\partial q_\nu/\partial u_\nu)}{\sum_{i=m+1}^{n}(\partial q_\nu/\partial u_\nu) - \sigma_r}. \tag{25}$$

After taking into account (16) we obtain for the Newton law of heat exchange

$$\sqrt{\Lambda} = \frac{\sum_{i=m+1}^{n} \alpha_i \sqrt{T_i}}{\sigma_r + \alpha_\Sigma^r}, \qquad \alpha_\Sigma^r = \sum_{i=m+1}^{n} \alpha_i. \tag{26}$$

The maximal power generated from contacts with reservoirs is

$$P_r^*(\sigma_r) = \sum_{i=m+1}^{n} \alpha_i \sqrt{T_i} \left(\sqrt{T_i} - \frac{\sum_{j=m+1}^{n} \alpha_j \sqrt{T_j}}{\sigma_r + \alpha_\Sigma^r} \right). \tag{27}$$

This power increases monotonically when σ_r increases.

When the second problem is solved in addition to the contact temperatures u_i it is also necessary to find the subsystems' temperatures T_i for $i = 1, \dots, m$. If the heat transfer law has the Newton form (10) then u_i can be expressed in terms of q_i and T_i

$$u_i = T_i - \frac{q_i}{\alpha_i} \tag{28}$$

and the balances (13) can be rewritten as the set of linear equations with respect to the temperatures T_i, $i = 1, \dots, m$,

$$\sum_{j=1}^{m} \alpha_{ij} T_j - T_i \sum_{j=1}^{n} \alpha_{ij} = q_i - \sum_{j=m+1}^{n} \alpha_{ij} T_j, \qquad i = 1, \dots, m. \tag{29}$$

Or in matrix form

$$A(\alpha) \cdot T = C(q),$$

where

$$A(\alpha) = \begin{Vmatrix} \alpha_{11} - \tilde{\alpha}_1 & \alpha_{12} & \cdots & \alpha_{1m} \\ \alpha_{21} & \alpha_{22} - \tilde{\alpha}_2 & \cdots & \alpha_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{1m} & \alpha_{m2} & \cdots & \alpha_{mm} - \tilde{\alpha}_m \end{Vmatrix},$$

$$\tilde{\alpha}_i = \sum_{j=1}^{n} \alpha_{ij},$$

$$C_i(q) = q_i - \sum_{j=m+1}^{n} \alpha_{ij} T_j = q_i - R_i.$$

We denote matrix $A(\alpha)$ components as $a_{ij}(\alpha)$ and the components of its inverse matrix A^{-1} as $b_{ij}(\alpha)$. The subsystems' temperatures can be expressed in terms of the flows q_i and of the fixed reservoir temperatures

$$T = A^{-1} \cdot C(q),$$

$$T_i(q) = b_{i,1}(q_1 - R_1) + b_{i,2}(q_2 - R_2) + \cdots + b_{i,m}(q_m - R_m). \tag{30}$$

Here $(\partial T_i/\partial q_\nu) = b_{i,\nu}$, $i = 1, \ldots, m$, $\nu = 1, \ldots, m$.

Now after the problem (22), (23), (13) with respect to u_i, T_i is solved so we can consider the optimization problem with respect to the heat flows q_i

$$P = \sum_{i=1}^{m} q_i \to \max_{q_i} \qquad \text{subject to} \sum_{i=1}^{m} \frac{\alpha_i q_i}{\alpha_i T_i(q) - q_i} = \sigma_s. \tag{31}$$

The Lagrange function for the problem (31) is

$$L = \sum_{i=1}^{m} q_i \left(1 - \frac{\Lambda_s \alpha_i}{\alpha_i T_i(q) - q_i}\right). \tag{32}$$

Its stationarity condition on q_j is

$$\frac{\partial L}{\partial q_j} = 0 \Rightarrow \sum_{i=1}^{m} \frac{\alpha_i^2 q_i \Lambda_s b_{i,j}}{(\alpha_i T_i(q) - q_i)^2} + 1 - \frac{\alpha_j^2 T_j(q) \Lambda_s^*}{(\alpha_j T_j(q) - q_j)^2} = 0,$$
$$j = 1, \ldots, m. \tag{33}$$

Solving the set of $m + 1$ equations (33), (23) with respect to q_i gives the optimal $q_i^*(\sigma_s)$ and $\Lambda(\sigma_s)$ and then the optimal $T_i^*(\sigma_s)$, $u_i^*(\sigma_s)$ and $P_s^*(\sigma_s)$. It is convenient here to set Λ_s^* rather than σ_s and then to use (23) to calculate the corresponding entropy flow σ_s.

In the third problem it is required to find the maximal net power extractable from the reservoirs and subsystems. The Lagrange function of the problem (24) is

$$L = P_s^*(\sigma_s) + P_r^*(\sigma_r) + \lambda(\sigma_r + \sigma_s). \tag{34}$$

Its stationarity condition on σ_r, σ_s yields

$$\frac{\partial P_s^*}{\partial \sigma_s} = \frac{\partial P_r^*}{\partial \sigma_r}. \tag{35}$$

Since the derivatives $\partial P_s^*/\partial \sigma_s$ and $\partial P_r^*/\partial \sigma_r$ are equal to $\Lambda_s^*(\sigma_s)$ and $\Lambda_r^*(\sigma_r)$ correspondingly [15], the power $P(q_i)$ is maximal when

$$\Lambda_r^*(\sigma) = \Lambda_s^*(-\sigma), \qquad \sigma = \sigma_r = -\sigma_s. \tag{36}$$

Example 2. Consider a system which consists of two reservoirs, four subsystems and a transformer. The structure of the system is shown in figure 1. The matrix of heat transfer coefficients has the form

$$\{\alpha_{ij}\} = \begin{pmatrix} 0 & 800 & 900 & 700 & 400 & 0 \\ 800 & 0 & 500 & 900 & 0 & 100 \\ 900 & 500 & 0 & 300 & 200 & 0 \\ 700 & 900 & 300 & 0 & 0 & 250 \\ 400 & 0 & 200 & 0 & 0 & 0 \\ 0 & 100 & 0 & 250 & 0 & 0 \end{pmatrix}. \tag{37}$$

α_{ij} for $i, j = 1, \ldots, 4$ correspond to the interaction between subsystems and for $i > 4$ or $j > 4$ to the interaction between the subsystems and reservoirs.

Reservoir temperatures are $T_+ = 700$ K, $T_- = 300$ K. The heat transfer coefficients for transformer–subsystems and transformer–reservoirs interactions are

$$\alpha_1 = 1000, \quad \alpha_2 = 1300, \quad \alpha_3 = 900,$$
$$\alpha_4 = 800, \quad \alpha_5 = 100, \quad \alpha_6 = 50.$$

Here α_i for $i = 1, \ldots, 4$ correspond to the interaction with subsystems and for $i > 4$ to the interaction with reservoirs. The dimension of these coefficients is W/K.

First we calculated $\Lambda_r(\sigma_r)$ using (26), then we calculated $P_r^*(\sigma_r)$ from (27) for σ_r, within the interval [0, 0.5]. Then we solved the set of m equations (33) for fixed Λ_s on the interval [400 K, 600 K] and σ_s was calculated using (23).

Then (36) was used to find the optimal σ and Λ, (30) to find subsystem temperatures T_i and (28) to find temperatures for contacts with the transformer u_i.

The calculations yield the following optimal temperatures for the subsystems

$$T_1 = 564.8\,\text{K}, \quad T_2 = 540.2\,\text{K},$$
$$T_3 = 563.7\,\text{K}, \quad T_4 = 527\,\text{K}$$

and the following optimal contact temperatures for the transformer

$$u_1 = 557.8\,\text{K}, \quad u_2 = 543.6\,\text{K}, \quad u_3 = 557.7\,\text{K},$$
$$u_4 = 536.3\,\text{K}, \quad u_5 = 620.2\,\text{K}, \quad u_6 = 406\,\text{K}.$$

The power obtained here is

$$P^* = 3.22\,\text{kW}.$$

2.5. Taking into account internal irreversibility of the transformer

Until this point we assume that the transformer is internally reversible. It is interesting that some of the above derived results hold even when the transformer operates irreversibly if heat transfer is linear. Suppose the transformer's operations are internally irreversible which yields entropy production $\sigma > 0$. The entropy balance for the working body and Newton heat transfer then takes the form

$$\sum_i \frac{\alpha_i (T_{i0} - u_i)}{u_i} + \sigma = 0. \tag{38}$$

In the maximal power problem u_i are found by maximizing

$$P = \sum_i \alpha_i (T_{i0} - u_i) \rightarrow \max \tag{39}$$

subject to (38).

The optimality conditions for the problem (38), (39) were obtained above for an internally reversible engine from stationarity of its Lagrangian function. They have the following form

$$u_i = \sqrt{\Lambda T_{i0}}, \qquad i = 1, \ldots, r. \tag{40}$$

It is clear that the Lagrange function of the problem with the internally irreversible engine is the same as of the problem with the reversible one. Thus, if σ does not depend on u_i then conditions (40) holds. In particular, for the system with two reservoirs and $T_{10} \approx T_+$ and $T_{20} \approx T_-$, the efficiency of a heat engine for maximal power is equal to the Novikov–Curzon–Ahlborn efficiency

$$\eta = 1 - \frac{u_2}{u_1} = 1 - \sqrt{\frac{T_-}{T_+}} \tag{41}$$

and does not depend on σ. Thus, if the entropy production σ for an internally irreversible engine operating at maximal power increases then both power P and the heat flow from the hot reservoir q_+ decrease in the same proportion.

Let us find such a value of σ that the maximal power P of a transformer operating in a two reservoir system is zero. After taking into account (41) we express $\sqrt{\Lambda}$ in terms of σ from the condition (38)

$$\frac{\alpha_1 (T_+ - \sqrt{\Lambda T_+})}{\sqrt{\Lambda T_+}} + \frac{\alpha_2 (T_- - \sqrt{\Lambda T_-})}{\sqrt{\Lambda T_-}} + \sigma = 0$$

or

$$\sqrt{\Lambda} = \frac{\alpha_1 T_+ + \alpha_2 T_-}{\alpha_1 + \alpha_2 - \sigma}. \tag{42}$$

Since the left-hand side of this equality and the numerator of its right-hand side are non-negative, the transformer's internal entropy production must obey the constraint

$$\sigma \leqslant \alpha_1 + \alpha_2. \tag{43}$$

After taking into account (40), (42) we get

$$\begin{aligned} P &= \alpha_1 (T_+ - \sqrt{\Lambda}\sqrt{T_+}) + \alpha_2 (T_- - \sqrt{\Lambda}\sqrt{T_-}) \\ &= \alpha_1 T_+ + \alpha_2 T_- - \sqrt{\Lambda}(\alpha_1 \sqrt{T_+} + \alpha_2 \sqrt{T_-}) \\ &= \frac{\alpha_1 \alpha_2 (\sqrt{T_+} - \sqrt{T_-})^2 - \sigma(\alpha_1 T_+ + \alpha_2 T_-)}{\alpha_1 + \alpha_2 - \sigma}. \end{aligned} \tag{44}$$

The power $P > 0$ if

$$\sigma < \frac{\alpha_1 \alpha_2 (\sqrt{T_+} - \sqrt{T_-})^2}{\alpha_1 T_+ + \alpha_2 T_-}. \tag{45}$$

Suppose the internal irreversibility of the transformer σ is proportional to P

$$\sigma = K_n P. \tag{46}$$

In this case the entropy balance (38) takes the form

$$\sum_i \alpha_i \frac{(T_{i0} - u_i)}{u_i} + K_n P = 0. \tag{47}$$

The Lagrange function for the problem (39), (47) is

$$L = (1 - \Lambda K_n) \sum_i \alpha_i (T_{i0} - u_i) - \Lambda \sum_i \alpha_i \left(\frac{T_{i0}}{u_i} - 1 \right).$$

Its stationarity conditions on u_i yield equations similar to (40)

$$u_i = \sqrt{\frac{\Lambda}{1 - \Lambda K_n}} \sqrt{T_{i0}}. \tag{48}$$

Thus, the efficiencies of internally reversible and internally irreversible heat engines in a two-reservoir system have the same form (41).

Similar derivations show that the condition (41) for the efficiency of a heat engine in a two reservoir system holds if σ is an arbitrary smooth and monotone function of power.

3. Temperature distributions permitted by thermodynamics and their classification

Suppose that subsystem temperatures as well as the reservoir temperatures are fixed. The system consists of m finite capacity systems and $(n - m)$ reservoirs. Let us find the conditions that hold only for a system with such a distribution of temperature between its subsystems that the maximal power generated by the transformer is positive.

From energy balances (3) we get

$$P_s = \sum_{i=1}^{m} \sum_{j=1}^{n} q_{ij}(T_j, T_i). \tag{49}$$

The flow of entropy from a subsystem to the transformer is

$$\sigma_s = \sum_{i=1}^{m} \frac{\sum_{j=1}^{n} q_{ij}(T_j, T_i)}{u_i}. \tag{50}$$

The contact temperatures u_i are uniquely determined by the heat balances

$$q_i(T_i, u_i) = \sum_{j=1}^{n} q_{ij}(T_j, T_i) \qquad i = 1, \ldots, m. \tag{51}$$

Since the temperatures u_i, $i = 1, \ldots, m$, are non-negative, the balance equations (51) constrain the minimal temperatures of the subsystems that are feasible in a system with transformer. In the following we will assume that these constraints hold.

Therefore, if the temperature field inside the system is fixed then P_s and σ_s are also fixed. The optimal temperatures for contacts with the reservoirs and the corresponding power are found similarly as was done above for the problem (20) for $\sigma_r = -\sigma_s$. The optimal reduced temperature during contact with each reservoir Λ is

$$\Lambda = \frac{\sum_{\nu=m+1}^{n} (\partial q_\nu / \partial u_\nu) u_\nu}{\sigma_s + \sum_{\nu=m+1}^{n} (\partial q_\nu / \partial u_\nu)}. \tag{52}$$

Since the reservoir temperatures are positive Λ should be positive too. The numerator in (52) and the second term in the denominator are always negative. Therefore, temperatures of the contacts with reservoirs as well as the reduced temperature are positive if

$$\sigma_s < \left| \sum_{\nu=m+1}^{n} \frac{\partial q_\nu}{\partial u_\nu} \right|. \tag{53}$$

This condition further constrains the space of feasible temperatures of the subsystems.

Substitution of (52) into (6) gives

$$q_i(T_i, u_i) = u_i \frac{\partial q_i}{\partial u_i} \left[1 - u_i \left(\frac{\sigma_s + \sum_{\nu=m+1}^{n} (\partial q_\nu / \partial u_\nu)}{\sum_{\nu=m+1}^{n} (\partial q_\nu / \partial u_\nu) u_\nu} \right) \right],$$
$$i = m+1, \ldots, n. \tag{54}$$

Its solution gives $(n-m)$ optimal temperatures u_i^* $(i = m+1, \ldots, n)$ for contacts between the transformer and reservoirs.

The maximal total power that can be generated by the system with the given heat transfer coefficients and given temperatures of the subsystems is

$$P^* = P_s + \sum_{i=m+1}^{n} q_i(T_i, u_i^*). \tag{55}$$

Suppose the heat flow dependence on the contacting temperatures has the form (10). The temperatures for contacts between the working body and the subsystems are uniquely determined by equation (3) as

$$u_i = T_i - \frac{\sum_{j=1}^{n} q_{ij}}{\alpha_i} = T_i - \frac{\sum_{j=1}^{n} \alpha_{ij}(T_j - T_i)}{\alpha_i},$$
$$i = 1, \ldots, m. \tag{56}$$

Here the power produced during contacts with subsystems is

$$P_s = \sum_{i=1}^{m} q_i(T_i, u_i) = \sum_{i=1}^{m} \sum_{j=m+1}^{n} \alpha_{ij}(T_j - T_i). \tag{57}$$

In this expression we omitted the heat flows between subsystems because each such flow enters the double sum twice with opposite signs.

The entropy production during contact between the working body and subsystems is

$$\sigma_s(\alpha, T) = \sum_{i=1}^{m} \frac{q_i(T_i, u_i)}{u_i} = \sum_{i=1}^{m} \frac{\alpha_i \sum_{j=1}^{n} \alpha_{ij}(T_j - T_i)}{\alpha_i T_i - \sum_{j=1}^{n} \alpha_{ij}(T_j - T_i)}.$$

P_s and σ_s are determined by the problems' conditions. The optimal reduced temperature for contact with reservoirs is

$$\Lambda = \frac{\sum_{i=m+1}^{n} \alpha_i u_i}{\sum_{i=m+1}^{n} \alpha_i - \sigma_s}. \tag{58}$$

Therefore, we can express the maximal power in terms of subsystems' given temperatures and heat transfer coefficients. We obtain the condition for the set of generating temperature fields

$$P^*(T, \alpha) = \sum_{i=m+1}^{m} \sum_{j=m+1}^{n} \alpha_{ij}(T_j - T_i)$$
$$+ \sum_{i=m+1}^{n} \alpha_i \left(T_i - \sqrt{T_i} \frac{\sum_{j=m+1}^{n} \alpha_j \sqrt{T_j}}{(\sum_{j=m+1}^{n} \alpha_j) - \sigma_s(\alpha, T)} \right) \geqslant 0. \tag{59}$$

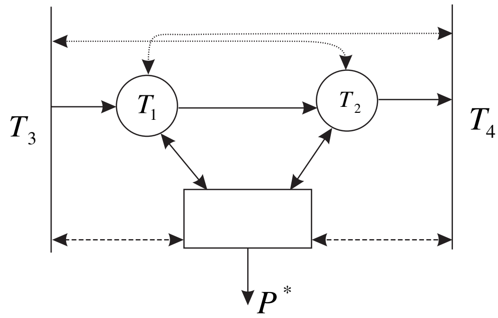

Figure 2. The structure of a thermodynamic system.

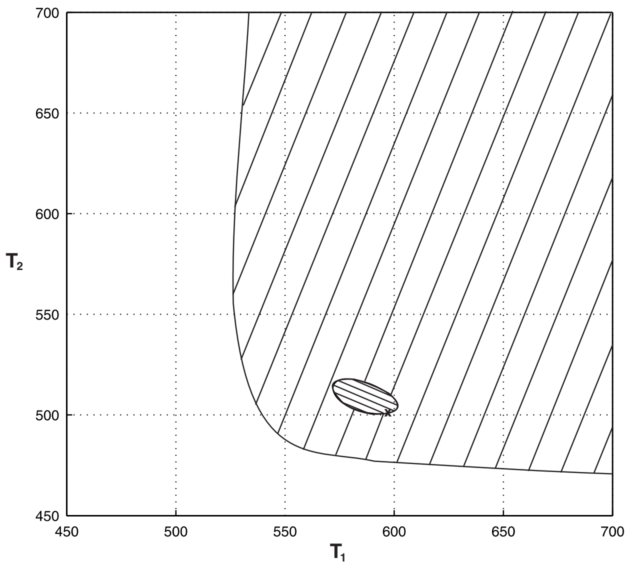

Figure 3. The area of thermodynamically feasible temperature fields in example 3. The smaller internal subset of the area above the boundary corresponds to the power generating temperature fields.

If inequality (59) does not hold then it is necessary to use the power not lower than $-P^*(T, \alpha)$ in order to maintain the required field of temperatures in the system. It is necessary here that the constraint on the entropy production similarly to (43) holds irrespectively of the sign of P.

Example 3. We consider the system that includes two subsystems with the temperatures T_1 and T_2, two reservoirs with the temperatures $T_3 = 700$ and $T_4 = 300$ and the transformer (figure 2). The transformer contacts subsystems and reservoirs. The heat transfer coefficients are given

$$\alpha_1 = 100, \quad \alpha_2 = 200, \quad \alpha_3 = 25, \quad \alpha_4 = 10$$

$$\alpha_{12} = \alpha_{21} = 50, \quad \alpha_{13} = 300, \quad \alpha_{14} = 100,$$
$$\alpha_{23} = 400, \quad \alpha_{24} = 400.$$

The dependence of the maximal power on subsystem temperatures $P^*(T_1, T_2)$ is determined by equation (59), and the boundary separating two subspaces of generating and of discrete temperature fields obeys equation $P^*(T_1, T_2) = 0$.

The area of thermodynamically feasible temperatures is shown in figure 3. The subspace of generating temperatures is located inside this area. The cross shows the point which

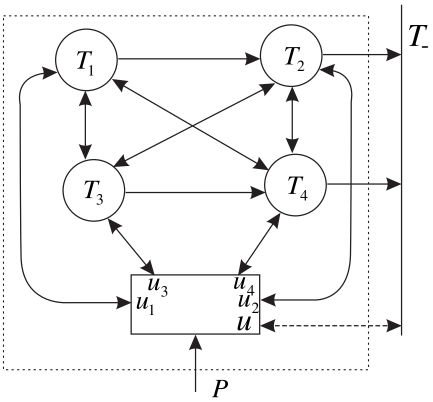

Figure 4. The structure of the thermostatting system.

corresponds to the system without a transformer. Zero power transformer provides control of the heat transfer from reservoirs to subsystems and between different subsystems. It allows us to achieve any temperature field located at the boundary between the generating and consuming temperature fields. The feasible temperatures are singled out by constraint (43) and by non-negativity of the working body's temperatures u_i.

4. The optimal thermostatting problem

The optimal thermostatting problem is a particular case of the maximal power problem for a system with one reservoir. Some subsystems here have fixed temperatures T_i $(i = r+1, \ldots, n-1)$; the temperatures of the others T_i $(i = 1, \ldots, r)$ are free. The unknowns here are the contact temperatures of the transformer u_i $(i = 1, \ldots, n)$ and the subsystems' free temperatures.

For example, such a problem arises when an air-conditioning system for a building, where at times only some of the rooms are occupied, is designed. Here the temperatures of unoccupied rooms are to be selected optimally and the reservoir is the environment. $-P^*$ gives the lower bound on energy consumption necessary to air-condition this building. The important particular case is the problem of active insulation when only the temperature in the central 'chamber' is fixed [13, 14].

Let us demonstrate that if at least one of the fixed temperatures differs from the reservoir's temperature then the optimal power P^* is negative. The structure of the system which consists of the reservoir with temperature T_- and the subsystems with finite capacity with temperature T_i $(i = 1, \ldots, m)$ is shown in figure 4. The flows between the reservoir and the ith subsystem are denoted as $q_{ri}(T_-, T_i)$. The transformer consumes the power P and receives from or rejects to subsystems the heat flows q_i $(i = 1, \ldots, m)$. The power of the transformer is

$$P = \sum_{i=1}^{m} q_i, \tag{60}$$

and at least for one i $T_i \neq T_-$. Let us demonstrate that the maximal power of the transformer here is negative, that is, it is required to supply energy to the system to maintain the given distribution of temperatures inside the system.

First, we formulate thermodynamic balances for the system shown in figure 4. The energy balance gives

$$\sum_{i=1}^{m} (q_{ri}(T_-, T_i) + q_i) = 0. \tag{61}$$

The entropy balance for the system and for the transformer's working body gives

$$\sum_{i=1}^{m} \left(\frac{q_i + q_{ri}}{T_i} - \frac{q_i}{u_i} \right) + \sigma = 0, \tag{62}$$

$\sigma > 0$ here is the entropy production in the system due to the exchange flows q_{ij} caused by the difference in subsystem temperatures.

From (60), (61) we get

$$P = \sum_{i=1}^{m} q_i = \sum_{i=1}^{m} q_{ri}(T_-, T_i). \tag{63}$$

After denoting

$$\sum_{i=1}^{m} q_i(u_i, T_i) \left(\frac{1}{T_i} - \frac{1}{u_i} \right) = \sigma_r > 0, \tag{64}$$

we obtain from (62)

$$\sum_{i=1}^{m} \frac{q_{ri}(T_-, T_i)}{T_i} = -(\sigma_r + \sigma) < 0. \tag{65}$$

The flows

$$q_{ri}(T_-, T_i) = \begin{cases} > 0 & \text{for } T_i < T_-, \\ < 0 & \text{for } T_i > T_-. \end{cases} \tag{66}$$

If we substitute T_i with T_- in the denominator of each term in (65) then the positive terms will be reduced and the absolute values of the negative ones will increase and we get

$$\sum_{i=1}^{m} \frac{q_{ri}}{T_-} < \sum_{i=1}^{m} \frac{q_{ri}}{T_i} < -(\sigma_r + \sigma).$$

Therefore, the transformer's power which is necessary to use in order to maintain the stationary state of the non-homogeneous system with one reservoir is

$$P_s = -P > T_-(\sigma_r + \sigma) > 0. \tag{67}$$

The power spent will be equal to zero if and only if all $T_i = T_-$ and $\sigma_r = \sigma = 0$.

Minimal power required for the system with one reservoir (59) is

$$P_s = -P^*(T, \alpha) = -\sum_{i=1}^{m} \alpha_{i-}(T_- - T_i)$$

$$-\alpha_- T_- \left(1 - \frac{\alpha_-}{\alpha_- - \sigma_s(\alpha, T)} \right), \tag{68}$$

where

$$\sigma_s(\alpha, T) = \sum_{i=1}^{m} \frac{\alpha_i \sum_{j=1}^{n} \alpha_{ij}(T_j - T_i)}{\alpha_i T_i - \sum_{j=1}^{n} \alpha_{ij}(T_j - T_i)}. \tag{69}$$

Here α_{i-}, α_- are heat transfer coefficients for heat exchange between the environment and the transformer.

The free temperatures T_i for $i = 1, \ldots, r$ are to be found from the condition of P_s minimum

$$\frac{\partial P_s}{\partial T_\nu} = 0 \Rightarrow -\alpha_{\nu n} = \frac{\partial \sigma_s}{\partial T_\nu} \frac{\alpha_n^2 T_n}{(\alpha_n - \sigma_s)^2},$$

$$\nu = 1, \ldots, r,$$

$$\frac{\partial \sigma_s}{\partial T_\nu} = \sum_{\substack{i=1 \\ i \neq \nu^m}} \frac{\alpha_i^2 \alpha_{i\nu} T_i}{(\alpha_i T_i - \bar{q}_i - q_{i-})^2}$$

$$-\alpha_\nu^2 \frac{T_\nu \sum_{j=1}^{n} \alpha_{\nu j} + \bar{q}_\nu + q_{\nu-}}{(\alpha_\nu T_\nu \bar{q}_\nu - q_{\nu-})^2}, \tag{70}$$

where $\bar{q}_i = \sum_{j=1}^{m} \alpha_{ij}(T_j - T_i)$ is the net exchange flow between the ith and all other subsystems and $q_{i-} = \alpha_{i-}(T_- - T_i)$ are the heat flows into the environment. Thus, we obtained (r) equations (70) for the optimal temperatures of the free subsystems. After these temperatures are found, we can calculate the flows q_i and contact temperatures u_i.

Example 4. Consider the thermodynamic system shown in figure 4, which consists of four subsystems, one reservoir with temperature $T_- = 300$ K and the transformer. The temperatures of two subsystems are fixed

$$T_3 = 320\,\text{K}, \qquad T_4 = 325\,\text{K}.$$

All the other parameters here are the same as were used in example 2. Solution of the set of two equations (70) yields the following optimal free temperatures for the subsystems

$$T_1 = 316.11\,\text{K}, \qquad T_2 = 321.8\,\text{K}.$$

The corresponding contact temperatures for the transformer are

$$u_1 = 308.26\,\text{K}, \quad u_2 = 323.82\,\text{K}, \quad u_3 = 325.65\,\text{K},$$
$$u_4 = 338.23\,\text{K}, \quad u_5 = 231.57\,\text{K}.$$

The minimal power used by a heat engine is

$$P_s = 3.6\,\text{kW}.$$

5. Conclusion

The limited possibilities of energy transformation in a thermodynamic system with the given structure and given exchange kinetics were studied. The results obtained include maximal power and the formula which determines the boundary between temperature fields in the system can be maintained only if power is generated (maximal power is positive) or if energy is spent (maximal power is negative). The minimal energy required to maintain the given field of temperatures in a multi-chamber system and the corresponding heat flows and temperatures of chambers with free temperatures have been obtained. It was shown that internal irreversibility of a heat engine operating in a two-reservoir system with Newton's heat transfer laws does not effect the efficiency of the maximal power regime. Thermodynamic bounds on an engine's internal irreversibility were derived.

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