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INTELLECTUAL CONTROL SYSTEMS

# Segregated Systems, Models and Control

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**Abstract**—For systems consisting of multiple assemblies interacting with the uniform environment, consideration was given to the mathematical models and optimal control. The optimality conditions were established. A structural approach to calculation of the time distribution densities of assembly sojourn in the system was developed.

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## 1. INTRODUCTION

The macrosystems consisting of many elementary subsystems of which each is not individually controlled and observed represent an important class of the controlled systems. One can act upon them and follow their response only at the macrolevel by measuring their averaged characteristics (see [1, 2]). A special case of the macrosystems is represented by the systems with segregation.

By the segregated systems are meant those consisting of quite a few assemblies interacting through a uniform environment which depends on the averaged state of the assemblies. The segregated systems of physico-chemical nature are exemplified by crystallization and dissolution, drying and granulation, biosynthesis, growing of plants, fishes, and animals, and so on (see [3]).

The segregated systems are adequate to the socio-economic systems where the set of elementary economic agents (cells) generates the common normative-legislational and pricing environment. The state of environment depends on the averaged interaction of cells.

A mathematical specialty of the segregated system models lies in averaging in the right-hand sides of the differential equations describing evolution of the environment, as well as the fact that the control actions can be applied only to the environment and change conditions common to all assemblies. As it is the case with all macrosystems, each assembly in the systems with segregation cannot be controlled, and its state cannot be measured.

## 2. ASSEMBLY-ISOLATED SYSTEMS

### *2.1. System Model*

The vectors of states of the assembly and environment are denoted, respectively, by $x$ and $y$. The evolution of an assembly is defined by its kinetic equation

$$\dot{x}(t,\gamma) = f(x(t,\gamma), y(t)), \quad x(0,\gamma) = x_0(\gamma), \tag{2.1}$$

where $\gamma$ is a random parameter with the probability distribution density $P(\gamma)$ and $t$ is the time of assembly sojourn in the system. Therefore, the state $x(t,\gamma)$ is random for any instant $t$. The initial values of the assembly state vector is one of the components of the vector $\gamma$.

The state of environment at each time instant obeys the equation

$$\dot{y} = \overline{\varphi(y, x(t,\gamma))}^{\gamma} + g(y,t) = \int \varphi(y, x(t,\gamma))P(\gamma)d\gamma + g(y,t), \quad y(0) = y_0, \tag{2.2}$$

where $\overline{\varphi(y, x(t,\gamma))}^{\gamma}$ denotes the $\gamma$-average value of the function $\varphi(y, x(t,\gamma))$.

If the system is controllable, then the control actions $u(t)$ enter only the right-hand sides of Eqs. (2.2) in the form

$$\dot{y} = \overline{\varphi(y, u, x)}^{\gamma} + g(y, u, t), \quad y(0) = y_0, \quad u \in V_u, \tag{2.3}$$

where the set $V_u$ of permissible controls is defined by the restrictions imposed on them at each time instant or over the control interval $[0, \tau]$. The first term in the right-hand side of (2.3) characterizes the kinematics of interaction with the assemblies, and the second, the external actions on the environment.

### *2.2. Problems of Control and Optimality Conditions*

The optimality criterion is given by

$$y_0(T) = \int_0^T \left[\overline{\varphi_0(y, u, x)}^{\gamma} + g_0(y, u, t)\right] dt \to \max, \quad y_0(0) = 0. \tag{2.4}$$

A wide class of the optimality criteria can be rearranged in this form by introducing corresponding variables.

We assume that the set $V_u$ is closed and bounded for any $t \in [0, \tau]$. We determine the necessary optimality conditions using the principle of maximum for the variational problems with the scalar argument in the form [4]. According to the formalism proposed there, in the integrand $R$ of the generalized Lagrange functional a term is assigned to each of the problem's conditions, and the problem variables are decomposed into two groups using a certain rule. For the variables of the first group, the function $R$ reaches maximum on the optimal solution, and for those of the second group, it is stationary.

Under the assumption of nondegenerate solution, for problem (2.1), (2.3), (2.4) the function $R$ (integrand of the generalized Lagrange functional) is given by

$$\begin{aligned} R = \int \{[\varphi_0(y, u, x(t, \gamma)) + \xi\varphi(y, u, x(t, \gamma))] \\ + \psi(\gamma, t) f(x(t, \gamma), y) + \dot{\psi}(\gamma, t) x(t, \gamma)\} P(\gamma) d\gamma + g_0(y, u, t) + \xi g(y, u, t) + \dot{\xi} y, \end{aligned} \tag{2.5}$$

where the integral is taken over the definitional domain of the distribution density $P(\gamma)$. Controls $u(t)$ belong to the variables of the first group, and the variables characterizing the states of environment and assemblies belong to the second group.

If the parameters $a$ that are invariable over the interval $(0, \tau)$ must be selected in the problem, then over this interval the integral $S$ of the function $R$ must be locally nonimprovable on the set $V_a$ of the permissible values of parameters.

In terms of the function $R$, the necessary conditions for optimality of problem (2.1), (2.2), (2.4) are as follows:

$$u^*(t) = \arg\max_{u \in V_u} R(u, y^*(t), x^*(t, \gamma)), \tag{2.6}$$

$$\frac{\partial R}{\partial y} = 0, \quad \frac{\partial R}{\partial x(t, \gamma)} = 0 \quad \forall \gamma. \tag{2.7}$$

To reduce notation, we denote

$$H = \overline{\varphi_0(y, u, x) + \xi\varphi(y, u, x)}^{\gamma} + g_0(y, u, t) + \xi g(y, u, t) \tag{2.8}$$

and with regard for (2.5) rearrange conditions (2.6), (2.7) in

$$u^*(t) = \arg\max_{u \in V_u} H(y, u, x), \tag{2.9}$$

$$\dot{\xi} = -\frac{\partial}{\partial y}\left[H(y, u, x(t,\gamma)) + \overline{\psi(t,\gamma) f(x(t,\gamma), y)}^{\gamma}\right], \tag{2.10}$$

$$\dot{\psi}(\gamma, t) = -\frac{\partial[\varphi_0(y, u, x) + \xi\varphi(y, u, x) + f(x(t,\gamma), y)]}{\partial x(t,\gamma)}, \tag{2.11}$$

$$\xi(\tau) = \psi(\gamma, \tau) = 0 \quad \forall\gamma, \tag{2.12}$$

where

$$\overline{\psi, f(x(t,\gamma), y)}^{\gamma} = \int \psi(\gamma, t) f(x(t,\gamma), y) P(\gamma) d\gamma. \tag{2.13}$$

## 3. ASSEMBLY-OPEN SYSTEMS. STATIONARY MODE

In the open-loop system, exchange with the environment takes place not only by the flows influencing the environmental state, but assembly flows as well. We consider only the stationary mode of such systems where the state of environment and the distributions of random variables affecting the assemblies are independent of the calendar time.

### *3.1. Mathematical Model*

In the static mode, the environment state $y$ is constant and equal to the system output state because the environment is uniform. The state of an assembly varies with its age $\tau_i$, that is, the time from arriving to the system till the current instant, so that

$$\frac{dx}{d\tau_i} = f(x(\gamma, \tau_i), y), \quad x(0) = x_0(\gamma). \tag{3.1}$$

In some cases, Eqs. (3.1) can be solved as

$$x = x(\gamma, \tau_i, y), \tag{3.2}$$

which enables one to simplify essentially solution of the system optimization problem. Solution of (3.2) is called the kinetic curve.

The age of an assembly is a random variable. We assume that it is independent of the vector $\gamma$, and its distribution density is denoted by $P_1(\tau_i)$. The time of assembly sojourn in the system $\tau_f$, is another random parameter called sometimes the assembly life time. The sojourn time is random, its distribution density is denoted by $P_2(\tau_f)$. We demonstrate below that the distribution densities of age and sojourn time are related to one another.

The environmental state is defined by the averaged conditions like

$$\overline{\varphi(y, u, x(\gamma, \tau_i))}^{\gamma, \tau_i} = g(y, u), \tag{3.3}$$

where $u$ is the vector of control actions and the overline stands for averaging in $\tau_i, \gamma$ according to the distribution densities $P_1(\tau_i)$ and $P_3(\gamma)$. In particular, the right-hand side of equality (3.3) can be $\frac{V}{g}(y - y_0)$, where $V$ is the system volume and $g$ is the consumption of the environment which is one of the controls.

The size of the vector function $\varphi$ coincides with that of the vector $y$, the functions $f$ and $\varphi$ are continuous and continuously differentiable in the totality of their arguments, as well as the function defining the optimality criterion

$$\overline{\varphi_0(y, u, x(\tau_f))}^{\gamma, \tau_f} \to \max_{u \in V_u}. \tag{3.4}$$

The parameters defining the form of the age and time distribution functions and the time of assembly sojourn can be among the controls.

### *3.2. Optimization of the Static Mode of Systems with Segregation*

For the nondegenerate solution $\lambda_0 = 1$, the generalized Lagrange function of problem (3.1)–(3.4) assumes the form

$$R = \overline{\varphi_0(y, u, x(\gamma, \tau_f))}^{\gamma,\tau_f} + \lambda \left[ \overline{\varphi(y, u, x(\gamma, \tau_i))}^{\gamma,\tau_i} - g(y, u) \right] + \left[ \overline{\frac{d\psi(\gamma, \tau_i)}{d\tau_i} x(\gamma, \tau_i) + \psi(\gamma, \tau_i) f(x(\gamma, \tau_i), y)}^{\gamma,\tau_i} \right], \tag{3.5}$$

where $P_1(\tau_i)$ and $P_2(\tau_f)$ are related to one another by equality (4.3) established in Section 4. The variables of the first group do not exist in this problem.

The necessary optimality conditions [5] are given by

$$\frac{\partial R}{\partial x} = 0, \quad \frac{\partial R}{\partial u} \delta u \leqslant 0, \quad \frac{\partial R}{\partial y} = 0, \tag{3.6}$$

where $\delta u$ is a permissible variation of controls with regard for the imposed constraints $u \in V_u$.

For the function $R$ of form (3.5), conditions (3.6) are given by

$$\frac{d\psi}{d\tau_i} = -\frac{\partial}{\partial x} \left[ \psi(\gamma, \tau_i) f(x, y) + \lambda \varphi(y, u, x) \right], \qquad \psi(\gamma, \tau_f) = \frac{\partial}{\partial x(\gamma, \tau_f)} \varphi_0(y, u, x), \tag{3.7}$$

$$\delta u \int_0^\infty \frac{\partial}{\partial u} \left[ \overline{\varphi_0(y, u, x(\gamma, \tau)) P_2(\tau) + \lambda \varphi(y, u, x(\gamma, \tau)) P_1(\tau)}^{\gamma} \right] d\tau \leqslant 0, \tag{3.8}$$

$$\int_0^\infty \frac{\partial}{\partial y} \left[ \overline{\varphi_0(y, u, x(\gamma, \tau)) P_2(\tau) + \lambda \varphi(y, u, x(\gamma, \tau))} + \overline{\psi(\gamma, \tau) f(x(\tau), y) P_1(\tau)}^{\gamma} \right] d\tau = 0. \tag{3.9}$$

Together with Eqs. (3.1) and averaged conditions (3.3), these conditions define the vectors $u, y, \lambda$ and the functions $x(\gamma, \tau)$ and $\psi(\gamma, \tau)$.

The optimality conditions become much simpler if one manages to determine the kinetic curve $x(\gamma, \tau_i, y)$. In this case, the problem comes to

$$I = \overline{\varphi_0(x(\gamma, \tau, y), u, y)}^{\gamma,\tau_f} \to \max, \tag{3.10}$$

$$J = \overline{\varphi(x(\gamma, \tau, y), u, y)}^{\gamma,\tau_i} - g(y, u) = 0. \tag{3.11}$$

Averaging here is done in $\tau$, but with the distribution $P_2$ of the time of assembly sojourn for the functional $I$ and the distribution $P_1$ of the assembly age for the function $\varphi$.

The necessary optimality conditions for the functions $\varphi_0$ and $\varphi$ that are continuously differentiable in $y$ and $u$ come to existence of nonzero vector $\lambda = (\lambda_0, \lambda_1, \ldots)$ such that on the optimal solution the Lagrange functional $S = \lambda_0 I + \lambda J$ is stationary in $y$ and locally nonimprovable in $u \in V_u$:

$$\frac{\partial}{\partial u} \left[ \overline{\varphi_0(x(\gamma, \tau, y), u, y)}^{\gamma,\tau_f} + \lambda (\overline{\varphi(x(\gamma, \tau, y), u, y)}^{\gamma,\tau_i} - g(y, u)) \right] \delta u \leqslant 0, \tag{3.12}$$

$$\left( \frac{\partial}{\partial x} \frac{\partial x}{\partial y} + \frac{\partial}{\partial y} \right) \left[ \overline{\varphi_0(x(\gamma, \tau, y), u, y)}^{\gamma,\tau_f} + \lambda (\overline{\varphi(x(\gamma, \tau, y), u, y)}^{\gamma,\tau_i} - g(y, u)) \right] = 0. \tag{3.13}$$

If the distribution functions depend on the parameters to be selected, this must be allowed for in the optimality conditions.

*Example 1.* Optimal choice of the mean time of assembly sojourn in the system.

Let the system be uniform in terms of both environment and assemblies. The kinetic curve

$$x(\tau, y) = \tau^2 e^{-y\tau}, \tag{3.14}$$

the distribution densities

$$P_1(\tau, \Theta) = P_2(\tau, \Theta) = \frac{1}{\Theta} e^{-\tau/\Theta}. \tag{3.15}$$

The optimality criterion

$$I = \frac{1}{\Theta} \int_0^\infty P_2(\tau, \Theta) x(\tau, y) d\tau \to \max, \tag{3.16}$$

$$J = \int_0^\infty P_1(\tau, \Theta) \left[y - x(\tau, y)\right] d\tau = 0, \quad \Theta \geqslant 0. \tag{3.17}$$

The time $\Theta = \frac{V}{g}$ of assembly sojourn in the system is to be selected.

After obvious calculations, the optimality conditions (3.12), (3.13) assume the form of equations

$$\frac{\partial S}{\partial \Theta} = 0 \to \int_0^\infty \tau^2 e^{-\tau} \left(y + \frac{1}{\Theta}\right) \left[\left(\frac{\tau}{\Theta} - 2\right) \frac{1}{\Theta^2} + \lambda(y - \tau^2 e^{-y\tau})\right] d\tau = 0, \tag{3.18}$$

$$\frac{\partial S}{\partial y} = 0 \to \int_0^\infty e^{-\tau/\Theta} \left[\frac{1}{\Theta}(1 + \tau^3 e^{-y\tau}) + \lambda(1 + \tau^2 e^{-y\tau})\right] d\tau = 0 \tag{3.19}$$

defining $\lambda, y$ and $\Theta$ together with condition (3.17).

## 4. DENSITIES OF DISTRIBUTION OF THE TIME OF ASSEMBLY SOJOURN

The time of assembly sojourn is one of the most important parameters common to all segregated systems. Let us consider the means of calculating the distribution densities of the sojourn time.

### *4.1. Relation between the Distributions of the Sojourn Time and the Assembly Age*

We assume that the initial state $x_0$ is fixed and the times of assembly sojourn in the system (age) $\tau_i$ and assembly sojourn at the system output $\tau_f$ are random. These random variables are interrelated and, consequently, their distribution densities $P_1(\tau_i)$ and $P_2(\tau_f)$ are interrelated as well.

It is important to establish this relation because the distribution of model time $P_2(\tau_f)$ defines the characteristics of the output flow, and the distribution of age $P_1(\tau_i)$, the kinetics of interaction of assemblies and environment within the system. Additionally, in many cases the distribution $P_2(\tau_f)$ can be determined experimentally using tracers [6] where a portion of assemblies is fed in a single step to the system input and the part $P_2(\tau_f)$ of assemblies discharging the system is measured. Needed is to compute the age distribution $P_1(\tau_i)$ from $P_2(\tau_f)$.

Let $P_2$ be given. Then, the portion of assemblies having ages from $\tau_i$ to $\tau_i + d\tau_i$ at the instant $t$ is proportional to the product of flow $g$ by $d\tau_i$, except for the portion of the assemblies that discharged the system at the time from $(t-\tau_i)$ to $t$. In the stationary case, the portion of assemblies discharging the system is given by

$$F(\tau_i) = \int\limits_0^{\tau_i} P_2(\tau_f) d\tau_f.$$

The age distribution density is equal to the portion of the remaining assemblies having the age from $\tau_i$ to $\tau_i + d\tau_i$. With regard for normalization, we obtain

$$P_1(\tau_i) = \frac{1 - \int_0^{\tau_i} P_2(\tau_f) d\tau_f}{\int_0^{\infty} \left(1 - \int_0^{\tau_i} P_2(\tau_f) d\tau_f\right) d\tau_i}. \tag{4.1}$$

We demonstrate that the denominator of this expression is equal to the mean time of assembly sojourn in the system

$$\Theta = \int\limits_0^{\infty} \tau_f P_2(\tau_f) d\tau_f. \tag{4.2}$$

Indeed, the integral in the denominator of (4.1) is equal to the limit for $s \to 0$ of the Laplace image of the integrand

$$\lim_{s\to 0} L\left[1 - \int\limits_0^{\tau_i} P_2(\tau_f) d\tau_f\right] = \lim_{s\to 0} \frac{1}{s}\left(1 - P_2(s)\right).$$

By removing uncertainty and using the l'Hospital rule we find that this limit is equal to the limit of $-\frac{dP_2(s)}{ds}$ for $s \to 0$, which in turn is equal to the integral in (4.2) so that

$$P_1(\tau_i) = \frac{1}{\Theta}\left(1 - \int\limits_0^{\tau_i} P_2(\tau_f) d\tau_f\right). \tag{4.3}$$

This expression enables one to determine for any segregated system the distribution density of the time assembly sojourn in the volume in terms of the distribution density of the sojourn time of the assemblies discharging the system in the stationary mode.

Using the Laplace transform, we rearrange equality (4.3) in

$$P_1(s) = \frac{1}{\Theta s}\left(1 - P_2(s)\right). \tag{4.4}$$

At that,

$$\Theta = \lim_{s\to 0} \frac{dP_2(s)}{ds}.$$

The distributions $P_2(\tau_f)$ and $P_1(\tau_i)$ are identical if

$$P_2(\tau_f) = \frac{1}{\Theta} e^{-\frac{\tau_f}{\Theta}}.$$

Indeed, in this case

$$P_2(s) = \frac{1}{\Theta s + 1}.$$

According to (4.4), we have

$$P_1(s) = \frac{1}{\Theta s}\left(1 - \frac{1}{\Theta s + 1}\right) = \frac{1}{\Theta s + 1}.$$

The Laplace transform enables us to determine the distribution densities of the assembly sojourn time in arbitrary-structure systems.

*4.2. Structural Analysis of Distributions of the Assembly Sojourn Time and Age*

The segregated system can consist of several subsystems exchanging flows of assemblies. In each subsystem the state of environment depends on the control actions and averaged-in-age state of the assemblies. Relation between the distribution densities for different structures allows one to use the experimental data acquired at any point of the system to calculate other subsystems.

In what follows, we discuss mostly the distributions $P_2(\tau_f)$ of the sojourn time $\tau_f$. Its Laplace transform is denoted by $P_2(s)$. The assembly age distribution $P_1(\tau_i)$ and its transformation $P_1(s)$ can be calculated from (4.3), (4.4) for a single system and from the formulas established below, for a complex system.

4.2.1. Simplest Models and Elementary Operations. Under ordered movement of the assemblies from the system input to the output (hydrodynamic displacement mode, queue), the time of sojourn $\tau_f^0$ of all assemblies is the same

$$P_2(\tau_f) = \delta(\tau_f - \tau_f^0), \quad P_2(s) = e^{-s\tau_f^0}. \tag{4.5}$$

It can be readily demonstrated that under ordered distribution of the assemblies within the system volume (hydrodynamics of ideal mixing)

$$P_2(\tau_f) = \frac{1}{\Theta} e^{-\tau_f/\Theta}, \quad P_2(s) = \frac{1}{\Theta s + 1}, \tag{4.6}$$

where $\Theta$ is the mean time of assembly sojourn in the system equal to the ratio of the number of assemblies to their consumption. Since the portion of assemblies in the volume and output flow is the same, $\Theta$ is the ratio of the system volume to consumption. Flows in system can merge and branch. At that, the sojourn time distribution densities vary.

**Merge of flows.** Let the distribution $P_{2i}(\tau_f)$ be known for each $i$th flow of $n$ flows with the consumption $g_i$. We denote by

$$\gamma_i = \frac{g_i}{\sum\limits_{j=1}^{n} g_j}, \quad \gamma_i \geqslant 0, \quad \sum_{i=1}^{n} \gamma_i = 1.$$

The portion of assemblies in the $i$th flow sojourning in the system during time from $\tau_f$ to $\tau_f + d\tau_f$ is $g_i P_{2i}(\tau_f) d\tau_f$. The same portion in the merge flow,

$$g P_2(\tau_f) d\tau_f = \sum_{i=1}^{n} g_i P_{2i}(\tau_f) d\tau_f,$$

hence

$$P_2(\tau_f) = \sum_{i=1}^{n} \gamma_i P_{2i}(\tau_f). \tag{4.7}$$

At flow branching,

$$P_{2i}(\tau_f) = P_2(\tau_f) \quad \forall i. \tag{4.8}$$

4.2.2. Concatenation of Subsystems. Let two subsystems be connected serially. For the first subsystem, the distribution of the time of sojourn in it $P_{21}(\tau_f)$ and the distribution of age $P_{11}(\tau_i)$ are related by (4.3). For the second subsystem, we separate the age $\tau_i$ of assemblies in system and their age $\tau_2$ only in the second subsystem. Additionally, we discriminate the time of sojourn $\tau_f$ at the system output and the times of sojourn $\tau_{fi}$ in each subsystem.

Since

$$\tau_f = \tau_{f1} + \tau_{f2}$$

and both of these random variables are independent, the distribution density is equal to the convolution of the densities of distribution of the addends:

$$P_2(\tau_f) = P_{21}(\tau_{f1}) * P_{22}(\tau_{f2}).$$

In the domain of Laplace transforms the convolution passes into the product, and, therefore, we get

$$P_2(s) = P_{21}(s)P_{22}(s). \tag{4.9}$$

For the first subsystem, the assembly age distribution in the domain of transforms is given by

$$P_{11}(s) = \frac{1}{\Theta_1 s}(1 - P_{21}(s)). \tag{4.10}$$

For the second subsystem, the age of assemblies in the system $\tau_i = \tau_{f1} + \tau_2$, and its distribution

$$P_1(\tau_i) = P_{21}(\tau_{f1}) * P_{12}(\tau_2).$$

In the domain of transforms,

$$P_1(s) = P_{21}(s) * P_{12}(s). \tag{4.11}$$

This distribution is related by (4.4) with the distribution of the time of sojourn in the system:

$$P_1(s) = \frac{1}{(\Theta_1 + \Theta_2)s}(1 - P_2(s)). \tag{4.12}$$

Whence it follows that the distribution of assembly age in the second subsystem is given by

$$P_{12}(s) = \frac{1}{(\Theta_1 + \Theta_2)s}\left(\frac{1}{P_{21}(s)} - P_{22}(s)\right). \tag{4.13}$$

Many characteristics of the distribution densities of sojourn time and age (mean values, variances, and so on ...) may be calculated through their Laplace transforms without passing to the domain of originals. For example, the mean value of time $\tau_i$ and is variance are given by

$$\Theta_i = \lim_{s\to 0}\left[-\frac{d}{ds}P_i(s)\right], \tag{4.14}$$

$$D_i = \lim_{s\to 0}\left[\frac{d^2P_i(s)}{ds^2} - \left(\frac{dP_i(s)}{ds}\right)^2\right]. \tag{4.15}$$

For an arbitrarily structured system incorporating parallel subsystems, assembly recycle subsystems, and so on, the expressions obtained enable one to determine with the use of the Laplace transform the distribution densities of the sojourn time and age of the assemblies in each subsystem.

*Example 2.* We determine the distribution density of the sojourn time in the recycle system consisting of two subsystems. The flow of assemblies passes through system A with certain distribution of the sojourn time $P_{a2}(\tau_f)$ given by

$$P_{ar}(\tau_f) = \frac{1}{\Theta} e^{-\frac{\tau_f}{\Theta}}, \quad P_{a2}(s) = \frac{1}{\Theta s + 1}. \tag{4.16}$$

Then, the portion $r$ of the assembly output flow returns to the system input through the subsystem B for which

$$P_{b2}(\tau_f) = \delta(\tau_f - \tau_0), \quad P_{b2}(s) = e^{-\tau_0 s}. \tag{4.17}$$

Here, $\Theta$ and $\tau_0$ depend on the consumption and correspond to the consumption $g$ at the system input. Since the consumption through recycle is $rg$ and the consumption through subsystem A is $g(1+r)$,

$$P^r_{b2}(s) = e^{-\frac{\tau_0 s}{r}}, \quad P^r_{a2}(s) = \frac{r+1}{\Theta s + r + 1}. \tag{4.18}$$

We determine $P_2(\tau_f)$ for the entire system and, to solve this problem, use the Laplace transform. After flow merge at the input of subsystem A, with regard for (4.18) we get in the domain of transforms

$$P_{abx}(s) = \frac{1 + rP_2(s)\, e^{-\frac{s\tau_0}{r}}}{1+r}. \tag{4.19}$$

On the other hand,

$$P_2(s) = P_{abx}(s)\, \frac{r+1}{\Theta s + r + 1}. \tag{4.20}$$

By substituting (4.20) in (4.19), we get

$$P_2(s)(\Theta s + 1 + r) = 1 + rP_2(s) e^{-\frac{\tau_0 s}{r}},$$

whence it follows that the Laplace images of the desired distribution density of the time $\tau_f$ of assembly sojourn in the system is as follows:

$$P_2(s) = \frac{1}{\Theta s + r\left(1 - e^{-\frac{\tau_0 s}{r}}\right) + 1}. \tag{4.21}$$

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## 5. CONCLUSIONS

For systems consisting of multiple assemblies interacting with the uniform environment, formulated was the problem of control and established were the conditions for optimality of solution. Relation was established between the age distribution densities and the time of assembly sojourn in the system. A structural approach to calculation of the distribution of time of assembly sojourn in complex systems with flow branching and merging was developed.

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