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# OPTIMUM ORGANIZATION AND MAXIMUM CAPABILITIES OF HEAT-PUMP HEATING SYSTEMS

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*The authors obtained a lower bound for the energy consumption in heating (maintaining an assigned temperature distribution in the system of intercommunicating chambers) and the corresponding distributions of the total heat-transfer coefficients and the temperature of the working medium of a heat pump in contact with the chambers and the environment.*

***Keywords:*** *heating system, dissipation, temperature distribution, nonequilibrium temperature field.*

**Introduction.** Since its origination, one of the main problems of thermodynamics has been the estimation of the maximum capabilities of thermodynamic systems. With the development of thermodynamics, these estimates have been refined, and their range has expanded. Thus, S. Carnot gave an upper bound for the efficiency of a heat engine [1]. I. Novikov [2] and later F. Curzon and B. Ahlburn [3], separately from him, found the estimate of its maximum output power under the assumption that the cycle consists of two isotherms and two adiabats. L. Rozonoér and A. Tsirlin [4] proved that the Novikov–Curzon–Ahlburn estimate holds even without an assumption of the form of the cycle and found the maximum efficiency of a heat engine at any output power lower than the maximum one, and also the limiting values of the heating and cooling coefficients for irreversible and reverse cycles at an assigned intensity of flows.

The rise in the cost of energy makes obtaining thermodynamic estimates especially relevant for energy consumption in areas where this consumption is particularly high. Mankind spends more energy on heating and air conditioning of buildings and on maintaining an assigned temperature field in cryogenic and high-temperature systems than on chemistry and metallurgy combined.

One possible way of implementing a heating system is to use heat pumps. The expediency of this option depends on the cost of fuel, the environmental temperature, and other factors. In making a decision, it is important to know which minimum energy will have to be consumed. Therefore, the problem of thermostatting [maintaining a nonequilibrium configuration of the temperature field in a system of intercommunicating rooms (chambers) with a minimum energy consumption] is quite complex and relevant.

In the present work, consideration is given to a particular case of this problem, which concerns heating. Here heat fluxes to be distributed in an optimum manner among the chambers are nonnegative. When a peat pump is used for heating, it feeds heat to each chamber so as to maintain an assigned temperature in it. The heat flux fed to each chamber depends on the temperature of the working medium, the heat pump, and the heat-transfer area (coefficient). The total area of heat transfer and hence the corresponding coefficient are bounded.

In designing the system, one should select the contact temperatures and the coefficients of heat transfer of the heat pump for each chamber so that the total power consumed by heating is minimum. Solution of this problem provides a lower bound for the energy to be spent on heating a building in a stationary regime at an assigned environmental temperature.

**Heat-Pump Heating System at a Fixed Temperature of the Chambers.** *Statement of the problem.* We will consider a system consisting of $n$ rooms (chambers), each characterized by the temperature $T_i$ ($i = \overline{1, n}$), a reservoir (environment) with temperature $T_0$, and a heat pump consuming the power $P$ and maintaining an assigned stationary temperature distribution in the system.

We will assume that the heat transfer depends linearly on the temperature difference so that the heat flux between the $i$th and $j$th chambers is equal to

$$q_{ij} = \alpha_{ij}(T_i - T_j) , \tag{1}$$

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with the heat-transfer coefficient $\alpha_{ij}$ being assigned for all $i$ from zero to $n$; here, $\alpha_{ij} = \alpha_{ji}$. If the chambers are not in contact with each other or have no exterior walls, the corresponding heat-transfer coefficient is equal to zero.

All the temperatures $T_i$ ($i = \overline{0, n}$) will be originally considered as being assigned; $T_i > T_0$ at $i > 0$ (problem of heating). In the problem, the sought quantities are the temperatures of the working medium $u_i > 0$ ($i = \overline{0, n}$) in contact with the reservoir and the chambers, and also the coefficients $\alpha_i > 0$ ($i = \overline{0, n}$) related by the condition

$$\sum_{i=0}^{n} \alpha_i = \bar{\alpha} , \tag{2}$$

where the coefficient $\bar{\alpha}$ is determined by the total area of contact of the working medium of the heat pipe during its heat exchange with the environment and the chambers, i.e., in the long run by the dimensions of the heat pump. Selection of the sought variables will be based on the condition of the minimum power consumption $P$. We formalize the stated problem by introducing the following notation for the total heat flux from the $i$th chamber to its surroundings:

$$q_{i\mathrm{v}} = \sum_{j=0}^{n} \alpha_{ij} (T_i - T_j) , \quad i = \overline{1, n} . \tag{3}$$

If, for a certain chamber, we have the flux $q_{i\mathrm{v}} = 0$, the chamber will be called passive. The temperature of this passive chamber from condition (3) is equal to

$$T_i^{\min} = \frac{\sum_{j=0}^{n} \alpha_{ij} T_j}{\sum_{j=0}^{n} \alpha_{ij}} . \tag{4}$$

The heat pump is not in contact with the passive chambers, since the temperature in them is maintained at an assigned level due to the heat exchange with the remaining chambers.

From the thermal-balance condition, the heat flux from the heat pump into the $i$th chamber is equal to

$$q_i = \alpha_i (u_i - T_i) = q_{i\mathrm{v}} \geq 0 , \quad i = 1, \ldots, n . \tag{5}$$

These fluxes in heating systems are a priori nonnegative.

*Distribution of contact surfaces and selection of working-medium temperatures.* First we consider individual heating systems, when each chamber is heated with its own heat pump. It is necessary to find coefficients $\alpha_{i0}$ and $\alpha_i$ and temperatures $u_{i0}$ and $u_i$ of the working medium of the heat pump such that the condition for the power

$$P_i = q_{i\mathrm{v}} - q_{i0} = q_{i\mathrm{v}} - \alpha_{i0} (T_0 - u_{i0}) \to \min \tag{6}$$

is fulfilled. Since the first term in (6) is fixed, the problem is reduced to maximizing the second term with constraints on the total heat-transfer coefficient and conditions of entropy balance of the working medium

$$\alpha_{i0} + \alpha_i = \bar{\alpha}_i , \tag{7}$$

$$\frac{q_{i0}}{u_{i0}} = \frac{\alpha_{i0} (T_0 - u_{i0})}{u_{i0}} = \frac{q_{i\mathrm{v}}}{u_i} , \tag{8}$$

here, as follows from (5), $\alpha_i(u_i) = \dfrac{q_{i\mathrm{v}}}{u_i - T_i}$.

From (7), we have $\alpha_{i0}(u_i) = \bar{\alpha}_i - \alpha_i(u_i)$. Eliminating $u_{i0}$ from (8), we obtain

$$q_{i0} = \alpha_{i0} (T_0 - u_{i0}) = \frac{q_{i\mathrm{v}} \alpha_{i0} (u_i) T_0}{q_{i\mathrm{v}} + \alpha_{i0} (u_i) u_i} \to \max_{u_i} . \tag{9}$$

The maximum condition of this expression in $u_i$ leads to the equality

$$\frac{dq_{i0}}{du_i} = 0 \rightarrow \alpha_i = \frac{q_{iv}}{u_i - T_i} = \frac{\overline{\alpha}_i}{2} . \tag{10}$$

The optimum solution for each chamber is of the form

$$\alpha_{i0}^* = \alpha_i^* = 0.5\overline{\alpha}_i , \tag{11}$$

$$u_i^* = T_i + 2\frac{q_{iv}}{\overline{\alpha}_i}, \quad q_{i0}^* = \frac{\overline{\alpha}_i q_{iv} T_0}{4q_{iv} + \overline{\alpha}_i T_i}, \quad u_{i0}^* = \frac{T_0(2q_{iv} + \overline{\alpha}_i T_i)}{4q_{iv} + \overline{\alpha}_i T_i}, \tag{12}$$

$$P_{\min^i} = q_{iv} - q_{i0}^* = q_{iv}\frac{4q_{iv} + \overline{\alpha}_i(T_i - T_0)}{4q_{iv} + \overline{\alpha}_i T_i} . \tag{13}$$

Thus, at assigned temperatures in all the rooms and in the environment, coefficients of heat exchange between them, and total coefficient (area) of heat transfer of the heat pump, the minimum heat flux necessary for heating each chamber $q_{iv}$ and the minimum power consumption are determined from expression (13).

*Interconnected systems*. We will assume that restrictions on the total cost of the heating system dictate constraints on the total heat-transfer surface of heat pumps:

$$\sum_i \overline{\alpha}_i = \overline{\alpha} , \quad \overline{\alpha}_i \geq 0 . \tag{14}$$

Let us find a distribution $\overline{\alpha}_i$ such that

$$P = \sum_i P_{\min^i}(\overline{\alpha}_i) \rightarrow \min\left(\sum_i \overline{\alpha}_i = \overline{\alpha}\right) . \tag{15}$$

Optimality conditions of this problem lead to the equality [5]

$$\frac{dP_{\min}^i}{d\overline{\alpha}_i} = \lambda_0 , \quad i = 1, 2, \dots ,$$

whence we have

$$\frac{q_{iv}}{4q_{iv} + \overline{\alpha}_i T_i} = \lambda_0 , \quad i = 1, 2, \dots . \tag{16}$$

Relation (16) in combination with equality (14) determine the optimum distribution of the heat-transfer surfaces among individual heaters and, after the substitution into (15), the minimum total power. Here and in what follows we denote for brevity the total entropy flow due to the heat transfer in the thermostatted system as

$$A = \sum_{i=1}^{n} \frac{q_{iv}}{T_i} . \tag{17}$$

With account taken of the introduced notation, we obtain

$$\overline{\alpha}_i^* = \overline{\alpha}\frac{q_{iv}}{AT_i} . \tag{18}$$

Thus, the optimum total heat-transfer coefficients of the heat pump for each chamber must be proportional to the entropy flow during the exchange of this chamber with the environment. Note that the value of the total entropy flow due to the heat transfer $A$ has been determined by the conditions of the problem. Substitution of the values of $\overline{\alpha}_i^*$ into conditions (12) and (13) leads o the expressions

$$u_i^* = T_i\left(1 + 2\frac{A}{\alpha_i}\right), \quad u_{i0}^* = T_0\frac{2 + \overline{\alpha}/A}{4 + \overline{\alpha}/A} = u_0^* , \quad \forall i . \tag{19}$$

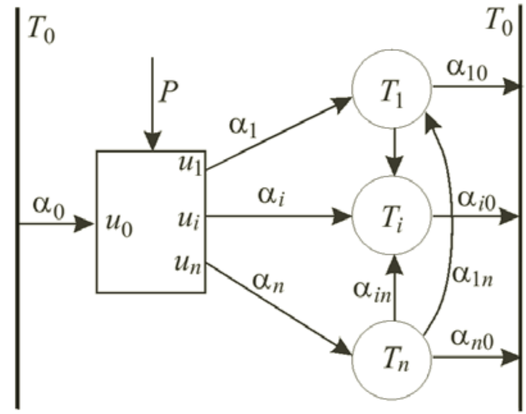


Fig. 1. Structure of the heating system.

Thus, when the distribution of the heat-transfer surfaces is optimum, the temperatures of contact of the heat pumps with the environment must be identical. This means that in the system with a shared heat pump (Fig. 1), at an optimum distribution of the surfaces of contact of the working medium with the chambers, the area of contact of its working medium with the environment must be such that the coefficient $\alpha_0$ is equal to half the total coefficient $\overline{\alpha}$, and the temperature of contact of the working medium with the reservoir $u_0^*$ must be selected from formula (19). The temperatures and areas of contact of the working medium with the chambers are the same as for individual heating systems.

The minimum power consumption in the system with an optimum surface distribution is equal to

$$P_{\min}^* = \sum_{i=1}^{n} q_{i\nu} - \frac{\overline{\alpha} T_0 A}{\overline{\alpha} + 4A} = \sum_{i=1}^{n} q_{i0} - \frac{\overline{\alpha} T_0 A}{\overline{\alpha} + 4A} . \tag{20}$$

*Possibility of selecting the temperatures of part of the chambers.* In actual heating systems, the temperatures are only fixed in part of the rooms. In this situation the temperatures of the remaining rooms (free temperatures) should be maintained at such a level that the power consumed by the heat pump is minimum. It should be taken into account that the temperature cannot be lower than its minimum value determined by condition (4).

Let $T_\nu$ be the free temperature; its change will produce a change in the flux $q_{\nu\nu}$ on which both the first term and the second term in (20) depend. The first term grows with temperature and related heat flux in the $\nu$th chamber, whereas the second decreases but in such a manner that the total value of the consumed power grows. Therefore, to the minimum power consumption there corresponds the minimum value of $T_\nu$ consistent with the condition of nonnegativeness of $q_{\nu\nu}$. Thus, the temperature in the chamber must be selected from condition (4), and in the expressions for $A$ and for the minimum power the heat flux is equal to zero: $q_{\nu\nu} = 0$. In the distribution of the contact surface, it turns out to be zero for rooms with free temperatures.

**Heating System at a Working-Medium Temperature Identical for All the Chambers.** We consider the problem of heating with the aid of a heat-transfer agent with the same temperature $u_1$ for all the chambers. An example can be hot-water or warm-air heating, when the heat-transfer agent is initially cooled to a temperature $u_1$ and is then distributed by the chambers. Since in this type of heating one can only feed heat to rooms rather than take it up, not all the temperature fields can be realized. For definiteness, we will assume that $u_1 \geq T_i \forall i$ and will consider the problem of heating. For the selected configuration to be implemented, it is necessary to fulfill the following condition:

$$q_{i\nu} = \sum_{j=0}^{n} \alpha_{ij}(T_i - T_j) \geq 0 , \quad i = \overline{1, n} . \tag{21}$$

We formalize the problem on minimum power for this system by assuming that all $q_{i\nu} \geq 0$:

$$P = \sum_{i=1}^{n} \alpha_i (u_1 - T_i) - \alpha_0 (T_0 - u_0) = \sum_{i=1}^{n} q_{i\nu} - \alpha_0 (T_0 - u_0) \to \min_{\alpha_i, u_1, u_0} , \tag{22}$$

with the condition of entropy balance of the heat-pump′s working medium

$$\frac{1}{u_1}\sum_{i=1}^{n} q_{iv} - \frac{q_0}{u_0} = 0 , \tag{23}$$

of energy balance for each room

$$q_{iv} = \alpha_i (u_1 - T_i) = \sum_{j=0}^{n} \alpha_{ij}(T_i - T_j) , \quad i = \overline{1, n} , \tag{24}$$

and of total constraint on the heat-transfer surface

$$\sum_{i=0}^{n} \alpha_i = \overline{\alpha} . \tag{25}$$

Since at fixed temperatures of the chambers $T_i$ and coefficients of heat exchange between them $\alpha_{ij}$, by virtue of (24), the first term on the right-hand side of (22) is fixed, the problem is reduced to maximizing the flux taken up from the reservoir

$$q_0 = \left(\overline{\alpha} - \sum_{i=1}^{n} \alpha_i\right)\left(T_0 - u_0\right) \to \max_{\alpha_i, u_1, u_0} , \tag{26}$$

with conditions (23)–(25).

*Optimality conditions.* We consider problem (26) with conditions (23)–(25). We eliminate $u_0$ from (23) and (25):

$$u_0 = T_0 \frac{\alpha_0}{\frac{1}{u_1}\overline{q} + \alpha_0} , \tag{27}$$

where $\overline{q} = \sum_{i=1}^{n} q_{iv}$ is the total heat flux. The quality $\alpha_0$ depends on $u_1$ by virtue of the fact that it is equal to $\overline{\alpha} - \sum_{i=1}^{n} \alpha_i$ . We express $\alpha_i$ from (24):

$$\alpha_i(u_1) = \frac{q_{iv}}{u_1 - T_i} . \tag{28}$$

We introduce the notation $\sigma(u_1) = \frac{\overline{q}}{u_1}$ and represent the heat flux $q_0$ by $\alpha_0$ in the form

$$q_0 = \alpha_0(u_1)(T_0 - u_0) = T_0 \frac{\alpha_0(u_1)\sigma(u_1)}{\alpha_0(u_1) + \sigma(u_1)} . \tag{29}$$

This expression depends on the variable $u_1$; the condition of maximum $q_0$ for it is of the form

$$\frac{\partial q_0}{\partial u_1} = 0 \Rightarrow \frac{\partial \alpha_0}{\partial u_1}\sigma^2(u_1) + \frac{\partial \sigma}{\partial u_1}\alpha_0^2(u_1) = 0 . \tag{30}$$

Since

$$\frac{\partial \alpha_0}{\partial u_1} = \sum_{i=1}^{n} \frac{q_i}{(u_1 - T_i)^2} , \quad \frac{\partial \sigma}{\partial u_1} = -\frac{1}{u_1^2}\overline{q} ,$$

optimality condition (30) leads to the equation

$$\overline{q}\sum_{i=1}^{n} \frac{q_{iv}}{(u_1 - T_i)^2} = \left(\overline{\alpha} - \sum_{i=1}^{n} \frac{q_{iv}}{u_1 - T_i}\right)^2 . \tag{31}$$

Solving Eq. (31), we find the optimum value $u_1^*$ and, from formulas (27) and (28), the corresponding values $u_0^*$ and $\alpha_i^*$, which determines, after the substitution into equalities (26) and (22), the value of minimum power necessary for maintaining an assigned temperature field. Clearly, the minimum required power in this case is higher than that found in the previous section.

*Rooms with a free temperature.* If the temperature of part of the rooms $T_\nu$ ($\nu = \overline{1, m}$) is not assigned, it should be selected from the condition of minimization of the power consumption (22) with account of (26)–(28). This minimum, as in the problem with individual selection of heat-transfer-agent temperatures, corresponds to the free-temperature minimum consistent with the condition of nonnegativeness of the fluxes. Thus, the temperatures $T_\nu$ ($\nu = \overline{1, m}$) are selected from condition (4), and the contact surfaces are equal to zero and hence $\alpha_\nu = 0$.

**Examples of Solution of Problems of Optimization of Heating for Heat-Pump Systems.** *Example 1. System of two chambers with individual converters and a shared environment.* The considered system is presented in Fig. 1. We have assigned the temperatures of the chambers $T_1 = 300$ K and $T_2 = 295$ K, the environmental temperature $T_0 = 270$ K, the coefficient of heat exchange between the chambers $\alpha_{12} = 100$, the coefficients of heat exchange of the chambers with the environment $\alpha_{10} = 20$, $\alpha_{20} = 30$, and the total heat-transfer coefficients (contact surfaces) for the working medium of the heat pumps during the exchange with the environment and the chambers $\alpha_1 + \alpha_1^0 = \overline{\alpha}_1 = 40$, $\alpha_2 + \alpha_2^0 = \overline{\alpha}_2 = 80$, and $\overline{\alpha} = \overline{\alpha}_1 + \overline{\alpha}_2 = 120$. It is necessary to find the power-consumption minimum and the corresponding distributions of the heat-transfer coefficients and the temperatures of the working medium in contact with the environment $u_{10}$ and $u_{20}$ and in contact with the chambers $u_1$ and $u_2$.

The heat flux from the heat pump into the first chamber is equal to

$$q_{1\nu} = \alpha_{12}(T_1 - T_2) + \alpha_{10}(T_1 - T_0) = 100 \cdot 5 + 20 \cdot 30 = 1100 ,$$

and the heat flux from the heat pump into the second chamber is equal to

$$q_{2\nu} = \alpha_{12}(T_2 - T_1) + \alpha_{20}(T_2 - T_0) = -100 \cdot 5 + 30 \cdot 25 = 250 .$$

Thus, as follows from formula (19), the value of the total entropy production due to the heat transfer is

$$A = \frac{1100}{300} + \frac{250}{295} = 4.51 .$$

The temperatures of contact of the heat-pumps′ working medium with the chambers are equal, as follows from (19), to

$$u_1 = 300\left(1 + 2\,\frac{4.51}{40}\right) = 367.65, \quad u_2 = 295\left(1 + 2\,\frac{4.51}{80}\right) = 328.26 .$$

and the temperatures of contact of the heat-pumps′ working medium with the environment (they are identical for the two heat pumps) are equal to

$$u_{10}^* = u_{20}^* = 270\left(\frac{2 + \dfrac{120}{4.51}}{4 + \dfrac{120}{4.51}}\right) = 252.35 .$$

The minimum power consumptions in the system, which are determined from formula (20), are

$$P_1 = 1100\,\frac{4 \cdot 1100 + 40(300 - 270)}{4 \cdot 1100 + 40 \cdot 300} = 375.61 ,$$

$$P_2 = 250\,\frac{4 \cdot 250 + 80(295 - 270)}{4 \cdot 250 + 80 \cdot 295} = 30.49 , \quad P = 375.61 + 30.49 = 406.09 .$$

The optimum distribution of the heat-transfer surface for the heat pumps, which is determined from (20), is equal to

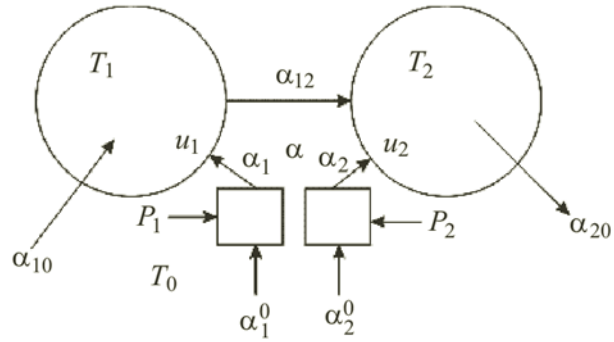


Fig. 2. Two-chamber system with a shared environment.

$$\overline{\alpha}_1^* = 120 \frac{1100}{4.51 \cdot 300} = 97.56 , \quad \overline{\alpha}_2^* = 120 \frac{250}{4.51 \cdot 295} = 22.54 .$$

The specific heat-transfer coefficient of the heat pump depends on the coefficient of heat transfer between the air and the heat-exchanger metal, on the thermal conductivities of the metal walls, and on the coefficient of heat transfer between the metal wall and the heat-transfer agent. The last two quantities are usually much higher than the first quantity, and the heat-transfer coefficient is not much smaller than the heat-transfer coefficient in contact with the air. It is approximately 35 W/(K·m$^2$). Thus, the surface of contact of the heat pump with the environment is about 97.57/35 = 2.8 m$^2$.

*Example 2. Temperature of the first chamber is fixed and the temperature of the second chamber is free.* We consider the same system as in the previous example with the only difference that the temperature $T_2$ is not fixed. We have assigned the following parameters: $T_1$ = 300 K, environmental temperature $T_0$ = 270 K, coefficient of heat exchange between the chambers $\alpha_{12} = 100$, coefficient of heat exchange of the chambers with the environment $\alpha_{10} = 20$ and $\alpha_{20} = 30$, and total heat-transfer coefficients (contact surfaces) for the heat-pumps′ working medium during the exchange with the environment and the chambers $\alpha_1 + \alpha_1^0 = \overline{\alpha}_1 = 40$, $\alpha_2 + \alpha_2^0 = \overline{\alpha}_2 = 80$, and $\overline{\alpha} = \overline{\alpha}_1 + \overline{\alpha}_2 = 120$. It is necessary to find the free temperature of the chamber $T_2$ and the heat-pump capacity corresponding to this selection.

From condition (4), we have the temperature $T_2$ = 293 K. The value of the total entropy production due to the heat transfer is equal, according to (19), to

$$A(T_2) = \frac{100(300 - 293) + 20(300 - 270)}{300} = 4.33 .$$

The power required to maintain the temperatures decreases compared to the case of fixed temperatures and is $P^* = 278.36$.

**Conclusions.** It has been shown what conditions must be satisfied for the optimum distribution of the heat-transfer coefficients and of the temperature of contact of the working medium with the heated rooms and with the environment in the problem of heating with heat pumps. The value of the consumed power, which has been obtained on fulfillment of these conditions, can be used as a lower bound for the implemented heating system.

## NOTATION

$A$, entropy production; $P_i$, power necessary for heating the $i$th chamber; $q$, heat flux; $T_0$, temperature of the reservoir; $T_i$, temperature of the $i$th chamber; $u_i$, temperature of the working medium in contact with the $i$th chamber; $\alpha_{ij}$, coefficient of heat exchange between the $i$th and $j$th chambers. Subscripts: ν, total external heat flux.

## REFERENCES


1. S. Carnot, Reflection on the motive power of fire, in: *The Second Law of Thermodynamics* [Russian translation], Gostekhizdat, Moscow–Leningrad (1934).
2. I. I. Novikov, The efficiency of atomic power stations, *Atomic Energy*, **3**, No. 11, 409–412 (1957)
3. F. L. Curzon and B. Ahlburn, Efficiency of a Carnot engine at maximum power output, *Am. J. Phys*., **43**, 22–24 (1975).
4. L. I. Rozonoér and A. M. Tsirlin, Optimum control of thermodynamic systems, *Avtomat. Telemekh.*, Nos. 1–3 (1983).
5. W. I. Zangwill, *Nonlinear Programming* [Russian translation], Sov. Radio, Moscow (1966).