Minkowski metrics on solutions of Khokhlov-Zabolotskaya equation

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Nontrivial geometric structures are defined in the natural way on solutions of many equations of mathematical physics. Differential invariants of these structures can be used to investigate the solutions. In this talk, we demonstrate it by the Khokhlov-Zabolotskaya equation.

1. Khokhlov-Zabolotskaya equation

The Khokhlov-Zabolotskaya equation (KZ-equation) is the following nonlinear PDE

$$u_{tx} - (uu_x)_x - u_{yy} - u_{zz} = 0\,.$$

We will consider it as a nonlinear differential operator. To this end consider the trivial bundle

$$\pi : \mathbb{R}^4 \times \mathbb{R} \longrightarrow \mathbb{R}^4, \quad \pi : (t, x, y, z, u) \mapsto (t, x, y, z).$$

By $j^k_p S$ denote a k-jet at p of a section S of π.

$$\pi_k : J^k\pi \longrightarrow \mathbb{R}^4, \quad \pi_k : j^k_p S \mapsto p$$

is the k-jet bundle of sections of π, $k = 1, 2, \ldots$. For $l > m$

$$\pi_{l,m} : J^l\pi \longrightarrow J^m\pi, \quad \pi_{l,m} : j^l_p S \mapsto j^m_p S.$$

Every section S of π generates the section $j_k S$ of π_k

$$j_k S : \mathbb{R}^4 \longrightarrow J^k\pi\,, \quad j_k S : p \mapsto j^k_p S\,.$$

2. Nonlinear differential operator

The nonlinear differential operator identified with the KZ-equation is the following

$$\Delta = \varphi_\Delta \circ j_2 \,,$$

where the function $\varphi_\Delta : J^2\pi \to \mathbb{R}$ is defined by the left hand side of the KZ-equation

$$\varphi_\Delta(t, x, y, z, u, u_t, \ldots, u_{zz}) = u_{tx} - uu_{xx} - u_{yy} - u_{zz} - u_x^2,$$

here $t, x, y, z, u, u_t, \ldots, u_{zz}$ are coordinates on the 2-jet bundle $J^2\pi$.

Obviously, the set of all solutions of KZ-equation coincides with the set of all sections S of π so that $\Delta(S) = 0$.

3. Simbols of Δ

Let $\theta_2 \in J^2\pi$, $\theta_1 = \pi_{2,1}(\theta_2)$, $p = \pi_2(\theta_2)$, and let F_{θ_1} be the fiber of projection $\pi_{2,1}$ over θ_1. That is,

$$F_{\theta_1} = (\pi_{2,1})^{-1}(\theta_1).$$

A symbol of Δ at $\theta_2 \in J^2\pi$ is a restriction of the differential φ_Δ to the tangent space $T_{\theta_2}(F_{\theta_1})$ to F_{θ_1} at $\theta_2 \in F_{\theta_1}$:

$$\begin{aligned} \operatorname{Smbl}_{\theta_2} \Delta &= d\varphi_\Delta\big|_{T_{\theta_2}(F_{\theta_1})}, \\ \operatorname{Smbl}_{\theta_2} \Delta &= du_{tx} - u du_{xx} - du_{yy} - du_{zz}. \end{aligned} \tag{1}$$

Taking into account that F_{θ_1} is an affine space, we get the natural identification

$$T_{\theta_2}(F_{\theta_1}) \cong F_{\theta_1}.$$

It follows that

$$\operatorname{Smbl}_{\theta_2} \Delta \in (F_{\theta_1})^*.$$

4. The exact sequence

$$0 \to \mathbb{R} \cdot (T_p^* \odot T_p^*) \xrightarrow{i} J_p^2 \pi \xrightarrow{\pi_{2,1}} J_p^1 \pi \to 0\,,$$

here T_p^* is the cotangent space at p, and i is defined by

$$i : v\,(df \odot dh) \mapsto j_p^2(fhS),$$

where $f(p) = g(p) = 0$, and $S(p) = v$.
From this sequence, we get

$$F_{\theta_1} \cong \mathbb{R} \cdot T_p^* \odot T_p^* \cong T_p^* \odot T_p^*\,.$$

Thus

$$\mathrm{Smbl}_{\theta_2} \Delta \in (T_p^* \odot T_p^*)^* \cong T_p \odot T_p.$$

Az a result from (1), we get

$$\mathrm{Smbl}_{\theta_2} \Delta = \partial_t \partial_x - u \partial_x \partial_x - \partial_y \partial_y - \partial_z \partial_z.$$

5. Metrics at the point p

The matrix of $\mathrm{Smbl}_{\theta_2} \Delta$ is nondegenerate

$$\begin{pmatrix} 0 & 1/2 & 0 & 0 \\ 1/2 & -u & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Therefore $\mathrm{Smbl}_{\theta_2} \Delta$ generates the isomorphism $T_p^* \to T_p$. The inverse one has the inverse matrix

$$\begin{pmatrix} 4u & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

and generates the metric of signature $(+,-,-,-)$

$$g(\theta_2) = 4udt^2 + 4dtdx - dy^2 - dz^2 \in T_p^* \odot T_p^* .$$

Thus $\mathrm{Smbl}_{\theta_2} \Delta$ defines the Minkowski metric $g(\theta_2)$ on T_p.

6. Metrics on solutions

Let

$$S : (t, x, y, z) \mapsto \big(t, x, y, z, u(t, x, y, z)\big)$$

be a solution of KZ-equation. Then we get the Minkowski metric on the domain of S:

$$g_S = 4u(t, x, y, z)dt^2 + 4dtdx - dy^2 - dz^2$$

Let L_S be the graph of S. Taking into account that L_S and the domain of S are diffeomorphic, we get that the Minkowski metric g_S is defined on L_S.

Further we use the classical differential invariants of metrics to get classes of explicit solutions of the KZ-equation.

7. Locally-flat solutions

Find solutions S of KZ-equation so that the metrics g_S on $L_S^{(2)}$ are locally-flat.

A metric is locally flat iff its curvature tensor is zero.

In our case, nonzero components of the curvature tensor are:

$$\begin{aligned} &R_{1212} = -2u_{xx}, \ R_{1213} = -2u_{xy}, \ R_{1214} = -2u_{xz}, \\ &R_{1313} = -2u_{yy}, \ R_{1314} = -2u_{yz}, \ R_{1414} = -2u_{zz}\,. \end{aligned} \tag{2}$$

From (2) we observe that the required solution of KZ-equation should also satisfy equations

$$u_{xx} = 0, \quad u_{xy} = 0, \quad u_{xz} = 0, \quad u_{yy} = 0, \quad u_{yz} = 0, \quad u_{zz} = 0\,.$$

Solving these equations together with KZ-equation, we get the following class of explicit KZ-solutions with locally flat metrics:

$$u(t,x,y,z) = h_1(t)yz - \frac{x}{t+c} + h_2(t)y + h_3(t)z + h_4(t)\,,$$

where h_1, h_2, h_3, and h_4 are arbitrary smooth functions and c is an arbitrary constant.

8. Projectively-flat solutions

We look for solutions S of KZ-equation so that metrics g_S on $L_S^{(2)}$ are projectively-flat.

Recall that a metric space (L, g) is projectively-flat if there exist local coordinates in a neighborhood of every point of L such that geodesic lines of g are represented as straight lines in these coordinates.
It is well known that (L, g) is projectively-flat iff (L, g) is a space of constant curvature. Then the curvature tensor is expressed in terms of the metric in the following way

$$R_{lkij} = K(g_{li}g_{kj} - g_{lj}g_{ki})\,, \quad K = \text{constant}.$$

Comparing the curvature tensor of g_S with the tensor $(g_S)_{li}(g_S)_{kj} - (g_S)_{lj}(g_S)_{ki}$, we get that $K = 0$.
Therefore the only locally-flat solutions of KZ-equation are projectively-flat.

9. Ricci-flat solutions

We find solutions S of KZ-equation so that the Ricci tensor of g_S on $L_S^{(2)}$ is zero.
In our case, nonzero components of the Ricci tensor are:

$$\begin{aligned} &R_{11} = -2uu_{xx} - 2u_{yy} - 2u_{zz},\ R_{12} = R_{21} = -u_{xx}, \\ &R_{13} = R_{31} = -u_{xy},\ R_{14} = R_{41} = -u_{xz} \end{aligned} \tag{3}$$

From (3) we get that the required solutions of KZ-equation should also satisfy equations

$$u_{xx} = 0, \quad u_{xy} = 0, \quad u_{xz} = 0, \quad u_{yy} + u_{zz} = 0\,.$$

Solving these equations together with KZ-equation, we obtain the following class of explicit solutions of the KZ- equation:

$$u(t,x,y,z) = -\frac{x}{t+C} + h(t,y,z)\,,$$

which are Ricci-flat. Here C is an arbitrary constant and h is a function, satisfying the Laplace equation

$$h_{yy} + h_{zz} = 0\,.$$

10. Einstein manifolds

We find solutions S of KZ-equation so that $L_S^{(2)}$ are Einstein manifolds.

Recall that a Minkowski manifold (M, g) is Einstein iff the Ricci tensor R of metric g is proportional to g. That is

$$R = \lambda g$$

for some constant λ.

Comparing the Ricci tensor of g_S with this metric, we get that the only Ricci-flat solutions of KZ-equation are Einstein manifolds.

11.1. Conformally-flat solutions

We find solutions S of KZ-equation so that metrics g_S on $L_S^{(2)}$ are conformally-flat.

Recall that a metric is called conformally-flat if, in neighborhood of every point, it can be transformed to the form

$$e^f g\,,$$

where f is a smooth function and g is a flat metric.
A metric is conformally-flat iff its Weyl tensor is zero.

In our case, nonzero components of the Weyl tensor are:

$$\begin{aligned} &W_{1212} = -\frac{1}{2}u_{xx},\ W_{1213} = -u_{xy},\ W_{1214} = -u_{xz}, \\ &W_{1313} = -u_{yy} + \frac{1}{3}u_{xx} + u_{zz},\ W_{1314} = -2u_{yz}\,, \end{aligned} \tag{4}$$

and other nonzero components are linear combinations of these ones.

11.2. Conformally-flat solutions

From (4) we get that the required solutions should satisfy the following equations:

$$u_{xx} = 0, \quad u_{xy} = 0, \quad u_{xz} = 0, \quad u_{yz} = 0, \quad u_{yy} - u_{zz} = 0\,.$$

Solving these equations together with KZ-equation, we get the following class of explicit solutions of KZ-equation with conformally-flat metric g_S.

$$u(t,x,y,z) = (\frac{d}{dt}h_1 - h_1^2)(y^2 + z^2) + h_1 x + h_2 y + h_3 z + h_4\,,$$

were h_1, h_2, h_3, and h_4 are arbitrary smooth functions in t.

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Classical differential invariants of nondegenerate metrics

Let g_{ij} be a nondegenerate metric.

Levi-Chivita connection

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} \left(\frac{\partial g_{li}}{\partial x^j} + \frac{\partial g_{lj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l} \right),$$

Curvature tensor

$$R^i_{jkl} = \frac{\partial \Gamma^i_{jl}}{\partial x^k} + \Gamma^i_{kr}\Gamma^r_{jl} - \frac{\partial \Gamma^i_{kl}}{\partial x^j} - \Gamma^i_{jr}\Gamma^r_{kl}.$$

Ricci tensor

$$R_{kl} = R^r_{rkl} \, .$$

Weil tensor

$$W_{ijkl} = R_{ijkl} - \frac{1}{2}(R_{ik}g_{jl} + R_{jl}g_{ik} - R_{il}g_{jk} - R_{jk}g_{il}) - \frac{R_{rs}g^{rs}}{6}(g_{il}g_{jk} - g_{ik}g_{jl}) \, ,$$

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