METRIC GEOMETRY OF CARNOT-CARATHÉODORY SPACES WITH C^1-SMOOTH VECTOR FIELDS

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Let $\mathbb{M}$ be a connected manifold endowed with a corank one distribution. If there exist two points that can not be connected by a horizontal path then the distribution is integrable.

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$D \subset T\mathbb{M}$ is a *corank one distribution* if $\exists$ a smooth 1-form θ s. t. $D_x = \{v \in T_x\mathbb{M} : \theta(x)\langle v\rangle = 0\}$. An a. c. path γ is called *horizontal* if $\dot\gamma(x) \in D_x$.

• Development

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It follows that $(\mathbb{M}, d_c)$ is a metric space with the subriemannian distance

$$d_c(u,v) = \inf\{L(\gamma) \mid \gamma \text{ is horizontal, } \gamma(0) = u, \gamma(1) = v\}$$

not comparable to Riemannian one.

- **Hörmander, 1967: Hypoelliptic equations**

A problem: *when a distribution solution f to the equation*

$$(X_1^2 + \ldots + X_{n-1}^2 - X_n)f = \varphi \in C^\infty$$

is a smooth function?

Here $X_i \in C^\infty$.

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• **Particular case: Kolmogorov's equations**

$$\frac{\partial^2 u}{\partial x^2} + x\frac{\partial u}{\partial y} - \frac{\partial u}{\partial t} = f$$

• physics (diffusion process), economics (arbitrage theory, some stochastic volatility models of European options), etc.

Hypoelliptic Equations

- **Hörmander (1967)**: sufficient conditions on fields $X_1, \ldots, X_n$:

There exists $M < \infty$ *such that*

- Lie$\{X_1, X_2, \ldots, X_n\} = \text{span}\{X_I(v) \mid |I| \leq M\} = T_v\mathbb{M}$ *for all* $v \in \mathbb{M}$

where

$$X_I(v) = \text{span}\{[X_{i_1}, [X_{i_2}, \ldots, [X_{i_{k-1}}, X_{i_k}] \ldots](v) : X_{i_j} \in H_1\}$$

for $I = (i_1, i_2, \ldots, i_k)$.

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• **Stein (1971)**: The program of studying of geometry of Hörmander vector fields;
description of singularities of fundamental solutions

Quasilinear equations of subelliptic type

Let a function $\mathcal{A} : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$, $\Omega \subset \mathbb{R}^N$ meet the following conditions:

($\mathcal{A}1$) the mapping $\Omega \ni x \mapsto \mathcal{A}(x, \xi)$ is measurable for all $\xi \in \mathbb{R}^n$, the mapping $\mathbb{R}^n \ni \xi \mapsto \mathcal{A}(x, \xi)$ is continuous for a. a. $x \in \Omega$;

there are some constants $0 < \alpha \leq \beta < \infty$ such that

($\mathcal{A}2$) $\langle \mathcal{A}(x, \xi), \xi \rangle \geq \alpha |\xi|^p$;

($\mathcal{A}3$) $|\mathcal{A}(x, \xi)| \leq \beta |\xi|^{p-1}$;

($\mathcal{A}4$) $\langle \mathcal{A}(x, \xi) - \mathcal{A}(x, \eta), \xi - \eta \rangle > 0$;

($\mathcal{A}5$) $\mathcal{A}(x, \lambda\xi) = \lambda |\lambda|^{p-2} \mathcal{A}(x, \xi)$ for all $\lambda \in \mathbb{R} \setminus 0$.

Quasilinear equations of subelliptic type

- $u : \Omega \to \mathbb{R}$ is called an $\mathcal{A}$-solution to the equation

$$-\operatorname{div}_h(\mathcal{A}(x, \nabla_0 u)) = 0 \quad \text{in} \quad \Omega \quad \text{if}$$

$u \in W^1_{p,\mathrm{loc}}$ and

$$\int_\Omega \mathcal{A}(x, \nabla_0 u)\nabla_0 \psi \, dx = 0 \quad \text{for all test functions } \psi \in C^1_0(\Omega).$$

Here $\nabla_0 u = (X_1 u, X_2 u, \ldots, X_n u)$ where $X_1, X_2, \ldots, X_n$ are vector fields meeting Hörmander condition.

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PROBLEM is to prove regularity properties of the $\mathcal{A}$-solution to this equation.

It is known for C^∞-vector fields [1996; Chernikov, V.].

Geometric control theory

$\diamond$ The linear system of ODE ($x \in \mathbb{M}^N$, $n < N$)

$$\dot{x} = \sum_{i=1}^{n} a_i(t) X_i(x), \qquad X_i \in C^\infty.$$

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$$\dot{x} = \sum_{i=1}^{n} u_i(t) X_i(x), \qquad X_i \in C^\infty. \tag{1}$$

• Problem: To find bounded measurable functions $u_i(t)$ such that system (1) has a solution with the initial data $x(0) = p$, $x(1) = q$.

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• It is locally controllable if Lie$\{X_1, X_2, \ldots, X_n\} = T\mathbb{M}$, i.e. the "horizontal" distribution $H\mathbb{M} = \{X_1, X_2, \ldots, X_n\}$ is bracket-generating.

APPLICATIONS of SUBRIEMANNIAN GEOMETRY

- Thermodynamics
- Non-holonomic mechanics
- Geometric Control Theory
- Subelliptic equation
- Geometric measure theory
- Quasiconformal analysis
- Analysis on metric spaces
- Contact geometry
- Complex variable
- Economics
- Transport problem
- Quantum control
- Neurobiology
- Tomography
- Robotecnics

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- $\forall v \in \mathbb{M}\ \exists U(v)$ and vector fields $X_1, X_2, \ldots, X_N \in C^1$ such that

$H_i\mathbb{M}(v) = \text{span}\{X_1(v), \ldots, X_{\dim H_i}(v)\}$, $\dim H_i\mathbb{M}(v) = \dim H_i$;

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- $[H_i, H_j] \subset H_{i+j}$, $i, j = 1, \ldots, M-1$.

It is equivalent to $\quad [X_i, X_j](v) = \displaystyle\sum_{k\colon \deg X_k \leq \deg X_i + \deg X_j} c_{ijk}(v) X_k(v)$

where $\deg X_k = \min\{m : X_k \in H_m\}$;

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◇ *If* $H_{j+1} = \text{span}\{H_j, [H_1, H_j], [H_2, H_{j-1}], \ldots, [H_k, H_{j+1-k}]\}$ where $k = \lfloor \frac{j+1}{2} \rfloor$, $j = 1, \ldots, M-1$, then $\mathbb{M}$ is called the Carnot manifold.

Classical example.

$\mathbb{M}$ is connected smooth manifold, $\dim \mathbb{M} = N$

$T\mathbb{M}$ is a tangent bundle;

"horizontal" subbundle is

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There is a filtration $H\mathbb{M} = H_1 \subseteq H_2 \subseteq \dots \subseteq H_M = T\mathbb{M}$ such that

$$\operatorname{span}\{H_1, [H_1, H_i]\} = H_{i+1}, \quad \dim H_i = \text{const}.$$

$\Longrightarrow$ $(\mathbb{M}, H\mathbb{M}, \langle \cdot, \cdot \rangle_{H\mathbb{M}})$ defines a subriemannian geometry

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- Sub-Riemannian geometry describes changing of physical location when the movement is possible in some prescribed directions.

Examples

1. Heisenberg group $\mathbb{H}^n$

$$\mathbb{M} = \mathbb{R}^{2n+1} : X_i = \frac{\partial}{\partial x_i} - \frac{x_{n+i}}{2}\frac{\partial}{\partial t},\ \ X_{n+i} = \frac{\partial}{\partial x_i} - \frac{x_i}{2}\frac{\partial}{\partial t},\ \ X_{2n+1} = \frac{\partial}{\partial t}$$

$$H_1 = \mathrm{span}\{X_1, X_2, \ldots, X_{2n}\},\ \ H_2 = [H_1, H_1] = \mathrm{span}\{X_{2n+1}\}$$

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$H_1 = \mathrm{span}\{X_1, X_2, \ldots, X_{2n}\}$, $H_2 = [H_1, H_1] = \mathrm{span}\{X_{2n+1}\}$

2. Carnot group is a connected simply connected group Lie G with stratified Lie algebra V:

$$V = V_1 \bigoplus V_2 \bigoplus \ldots \bigoplus V_M; \; [V_1, V_i] = V_{i+1}$$

! A Carnot group is a tangent cone to a subriemannian space in a regular point (Mitchell 1985; Gromov, Bellaiche 1996)

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- 1996 M. Gromov, A. Bellaïche approximation theorem on local behavior of metrics in the given space and in a local tangent cone.

Basic Concepts

Exponential mapping: $u \in \mathbb{M}$, $(v_1, \ldots, v_N) \in \mathbb{R}^N$,

$$\begin{cases} \dot{\gamma}(t) = \sum\limits_{i=1}^{N} v_i X_i(\gamma(t)), & t \in [0,1], \\ \gamma(0) = u. \end{cases}$$

Then $\exp\Big(\sum\limits_{i=1}^{N} v_i X_i\Big)(u) = \gamma(1)$. For each point u, define $\theta_u : U(0) \to \mathbb{M}$ as $\theta_u(v_1, \ldots, v_N) = \exp\Big(\sum\limits_{i=1}^{N} v_i X_i\Big)(u)$.

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Dilatations Δ_τ^u: if $u \in \mathbb{M}$ и $v = \exp\Big(\sum\limits_{i=1}^{N} v_i X_i\Big)(u)$ then

$$\Delta_\tau^u(v) = \exp\Big(\sum_{i=1}^{N} v_i \tau^{\deg X_i} X_i\Big)(u)$$

The New Approach to regular CC-spaces: a Local Lie Group at $u \in \mathbb{M}$ for C^1-Smooth Case

$$[X_i, X_j](v) = \sum_{k\colon \deg X_k \leq \deg X_i + \deg X_j} c_{ijk}(v) X_k(v).$$

Theorem 1 (2009; Karmanova, V.). Coefficients $\{c_{ijk}(u)\}_{\deg X_k = \deg X_i + \deg X_j} = \{\bar{c}_{ijk}\}$ satisfy Jacobi identity:

$$\sum_k \bar{c}_{ijk}(u)\bar{c}_{kml}(u) + \sum_k \bar{c}_{mik}(u)\bar{c}_{kjl}(u) + \sum_k \bar{c}_{jmk}(u)\bar{c}_{kil}(u) = 0$$

for all $i, j, m, l = 1, \ldots, N$, and

$\bar{c}_{ijk} = -\bar{c}_{jik}$ for all $i, j, k = 1, \ldots, N$.

Then the collection $\{\bar{c}_{ijk}\}$ defines nilpotent graded Lie algebra.

The New Approach to regular CC-spaces: a Local Lie Group at $u \in \mathbb{M}$ for C^1-Smooth Case

According to the second Lie theorem we take *basis vector fields* $\{(\widehat{X}_i^u)'\}_{i=1}^N$ *in* $\mathbb{R}^N$ *constituting a Lie algebra in such a way that*

$$[(\widehat{X}_i^u)', (\widehat{X}_j^u)'](v) = \sum_{k\colon \deg X_k = \deg X_i + \deg X_j} \bar{c}_{ijk} (\widehat{X}_k^u)'(v),$$

$(\widehat{X}_i^u)' = e_i$, $i = 1, \ldots, N$,

and $\exp = \mathrm{Id}$.

The corresponding Lie group is nilpotent graded Lie group $\mathbb{G}_u\mathbb{M}$

A Local Lie Group $\mathcal{G}^u\mathbb{M}$

In a neighborhod $\mathcal{G}_u \subset \mathbb{M}$ of u push-forwarded vector fields

$\widehat{X}_i^u = D\theta_u(\widehat{X}_i^u)'$ define a structure of local Lie group

in such a way that

$$\theta_u : \mathbb{G}_u\mathbb{M} \to \mathcal{G}_u\mathbb{M}$$

is a local isomorphism of Lie groups.

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- vector fields $\widehat{X}_i^u$ are left-invariant

Then $(\mathcal{G}, \widehat{X}_1^u, \ldots, \widehat{X}_N^u, \cdot) = \mathcal{G}^u\mathbb{M}$ **is a local Lie group**

- In the case of Carnot manifolds it is called **the local Carnot group**

Quasimetric

Let $v = \exp\Big(\sum_{i=1}^{N} v_i \widehat{X}_i^u\Big)(w)$. Then

$$d_\infty^u(v,w) = \max_{i=1,\dots,N}\{|v_i|^{\frac{1}{\deg X_i}}\}$$

- $d_\infty^u(v,w) \geq 0$; $d_\infty^u(v,w) = 0 \Leftrightarrow v = w$

- $d_\infty^u(v,w) = d_\infty^u(w,v)$

- generalized triangle inequality: for a neighborhood $U \Subset \mathbb{M}$, there exists a constant $c = c(U)$ such that for any $v, s, w \in U$ we have

$$d_\infty^u(v,w) \leq c(d_\infty^u(v,s) + d_\infty^u(s,w))$$

Quasimetric

- d_∞ is defined similarly (with X_i instead of $\widehat{X}_i^u$, $i = 1, \ldots, N$): if $v = \exp\left(\sum_{i=1}^{N} v_i X_i\right)(w)$ then

$$d_\infty(v, w) = \max_{i=1,\ldots,N} \{|v_i|^{\frac{1}{\deg X_i}}\}.$$

- $d_\infty(v, w) \geq 0$; $d_\infty(v, w) = 0 \Leftrightarrow v = w$.

- $d_\infty(v, w) = d_\infty(w, v)$.

- generalized triangle inequality: Do we have locally

$$d_\infty(v, w) \leq c(d_\infty(v, s) + d_\infty(s, w)) \quad \text{for some constant } c?$$

Gromov type nilpotentization theorem

Theorem 2 [2012; Greshnov]. For $x \in \mathrm{Box}(g, r_g)$ consider

$$X_i^\varepsilon(x) = (\Delta^g_{\varepsilon^{-1}})_* \varepsilon^{\deg X_i} X_i(\Delta^g_\varepsilon x), \quad i = 1, \dots, N.$$

Then the following expansion holds:

$$X_i^\varepsilon(x) = \widehat{X}_i^g(x) + \sum_{j=1}^{N} a_{ij}(x) \widehat{X}_j^g(x)$$

where $a_{ij}(x) = o(\varepsilon^{\max\{0, \deg X_j - \deg X_i\}})$ for $x \in \mathrm{Box}(g, \varepsilon r_g)$ and $o(\cdot)$ is uniform in g belonging to some compact set of $\mathbb{M}$ as $\varepsilon \to 0$.

Gromov type nilpotentizaton theorem

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Corollary 1 (**Gromov Type Theorem**): *We have* $X_i^{\varepsilon} \to \widehat{X}_i^g$ *as* $\varepsilon \to 0$, $i = 1, \dots, N$, *at the points of* $\mathrm{Box}(g, r_g)$ *and this convergence is uniform in* g *belonging to some compact neighborhood.*

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Corollary 2 [2009; Karmanova, V.]. Generalized triangle inequality holds locally for some constant c: $d_\infty(v, w) \leq c(d_\infty(v, s) + d_\infty(s, w))$.

MAIN RESULT: Comparison of Local Geometries

Let $\mathcal{U} \subset \mathbb{M}$ where $\mathbb{M} \in C^1$:

- $\theta_v(B(0, r_v)) \supset \mathcal{U}$ for all $v \in \mathcal{U}$,
- $\mathcal{G}^u\mathbb{M} \supset \mathcal{U}$ for all $u \in \mathcal{U}$,
- $\theta_v^u(B(0, r_{u,v})) \supset \mathcal{U}$ for all $u, v \in \mathcal{U}$.

Theorem 3 (2009; Karmanova, V.). *Let $u, u', v \in \mathcal{U} \Subset \mathbb{M}$. Assume that $d_\infty(u, u') = O(\varepsilon)$ and $d_\infty(u, v) = O(\varepsilon)$, and consider points*

$$w_\varepsilon = \exp\Bigl(\sum_{i=1}^{N} w_i \varepsilon^{\deg X_i} \widehat{X}_i^u\Bigr)(v) \text{ and } w'_\varepsilon = \exp\Bigl(\sum_{i=1}^{N} w_i \varepsilon^{\deg X_i} \widehat{X}_i^{u'}\Bigr)(v).$$

Then

$$\max\{d_\infty^u(w_\varepsilon, w'_\varepsilon), d_\infty^{u'}(w_\varepsilon, w'_\varepsilon)\} = o(\varepsilon)$$

where $o(\varepsilon)$ is uniform in $u, u', v \in \mathcal{U}$.

Corollaries

4) Local Approximation Theorem for d_∞-quasimetric (2009; Karmanova, V.):

Let $v, w \in \mathrm{Box}(g, \varepsilon) \subset \mathbb{M}$. Then

$$|d_\infty(v, w) - d^u_\infty(v, w)| = o(\varepsilon).$$

Corollaries

Assumption: Suppose that $\mathbb{M}$ is a Carnot manifold.

5) Rashevsky–Chow type Theorem (2012; Basalaev, V.): *Any two points* $u, v \in \mathbb{M}$ *can be connected by a horizontal curve* γ *(i. e.,* $\dot{\gamma}(t) \in H_{\gamma(t)}\mathbb{M}$ *for almost all* $t \in [0, 1]$*).*

The intrinsic metric on Carnot–Carathéodory space

$$d_c(u, v) = \inf_{\substack{\gamma \text{ is horizontal} \\ \gamma(0)=u, \gamma(1)=v}} \{L(\gamma)\}$$

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The intrinsic metric on Carnot–Carathéodory space

$$d_{cc}(u, v) = \inf_{\substack{\gamma \text{ is horizontal} \\ \gamma(0)=u, \gamma(1)=v}} \{L(\gamma)\}$$

6) Local Approximation Theorem for d_{cc}-metric (2009; Karmanova, V.): *For* $v, w \in B_{cc}(u, \varepsilon)$*, we have*

$$|d_{cc}(v, w) - d^u_{cc}(v, w)| = o(\varepsilon).$$

Corollaries (Ball-Box Theorem)

7) Mitchell-Gershkovich-Nagel-Stein-Wainger theorem type Ball–Box Theorem (2012). *For* $\mathcal{U} \Subset \mathbb{M}$, *there exist constants* $c(\mathcal{U}) \leq C(\mathcal{U})$ *such that*

$$c(\mathcal{U})d_\infty(x,y) \leq d_{cc}(x,y) \leq C(\mathcal{U})d_\infty(x,y),$$

where $x, y \in \mathcal{U}$, *and* $d_{cc}(x,y)$ *is a Carnot–Carathéodory metric.*

Corollaries

7) Mitchell-Gershkovich-Nagel-Stein-Wainger theorem type Ball–Box Theorem (2012). *For $\mathcal{U} \Subset \mathbb{M}$, there exist constants $c(\mathcal{U}) \leq C(\mathcal{U})$ such that*

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where $x, y \in \mathcal{U}$, and $d_{cc}(x,y)$ is a Carnot–Carathéodory metric.

Proof: [2011, V.] $d^u_{cc}(u,w)(1-o(1)) \leq d_{cc}(u,w) \leq d^u_{cc}(u,w)(1+o(1))$;

$d^u_\infty(u,w)(1-o(1)) \leq d_\infty(u,w) \leq d^u_\infty(u,w)(1+o(1))$;

$d^u_{cc}(u,w) \sim d^u_\infty(u,w)$.

Application to Quasilinear equations of subelliptic type

THEOREM [1996 : Chernikov, V.]. *Let* $X_1, X_2, \ldots, X_n$ *are* C^1-*vector fields in* $\Omega \subset \mathbb{R}^N$ *extended to a collection of* C^1-*vector fields constituting a structure of a Carnot manifold.*

Then any $\mathcal{A}$-*solution* $u : \Omega \to \mathbb{R}$ *to the equation*

$$-\operatorname{div}_h(\mathcal{A}(x, \nabla_0 u)) = 0$$

is Hölder continuous: $|u(x) - u(y)| \leq M d_{cc}^{\lambda}(x, y)$, $\lambda \in (0, 1)$.

Application to Geometric control theory

◇ The linear system of ODE ($x \in \mathbb{M}^N$, $n < N$)

$$\dot{x} = \sum_{i=1}^{n} u_i(t) X_i(x), \quad X_i \in C^1.$$

• Problem: To find measurable functions $u_i(t)$ such that system (1) has a solution with the initial data $x(0) = p$, $x(1) = q$.

If system (1) has a solution for every $q \in U(p)$ then it is called *locally controllable.*

• (1) locally controllable if "horizontal" vector fields $\{X_1, \dots, X_n\}$ can be extended to the system of vector fields constituting a structure of a Carnot manifold.

More Applications

• *sub-Riemannian differentiability theory*: Rademacher-type and Stepanov-type Theorems on sub-Riemannian differentiability of mappings of Carnot manifolds (S. Vodopyanov)

• *geometric measure theory on sub-Riemannian structures*: area formula for intrinsically Lipschitz mappings of Carnot manifolds, coarea formula for C^{M+1}-smooth mappings of Carnot manifolds (M. Karmanova; S. Vodopyanov)

• *geometry of non-equiregular Carnot–Carathéodory spaces* (S. Selivanova)

Sub-Riemannian Differentiability [2007; V.]

Definition. A mapping $\varphi : (\mathbb{M}, d_{cc}) \to (\widetilde{\mathbb{M}}, \tilde{d}_{cc})$ *is hc-differentiable* at $u \in \mathbb{M}$ if there exists a horizontal homomorphism

$$L_u : (\mathcal{G}^u, d^u_{cc}) \to (\mathcal{G}^{\varphi(u)}, d^{\varphi(u)}_{cc})$$

of local Carnot groups such that

$$\tilde{d}_{cc}(\varphi(w), L_u(w)) = o(d_{cc}(u, w)),\ \ E \cap \mathcal{G}^u \ni w \to u.$$

- For mappings of Carnot groups, this notion coincides with the definition of $\mathcal{P}$-differentiability in the sense of P. Pansu.

- Denote the hc-differential of φ at u by the symbol $\widehat{D}\varphi(u)$

Sub-Riemannian Differentiability [2007; V.]

Rademacher-Type Theorem. *Suppose that a mapping* $\varphi : (\mathbb{M}, d_{cc}) \to (\widetilde{\mathbb{M}}, \tilde{d}_{cc})$ *is Lipschitz. Then* φ *is* hc*-differentiable almost everywhere.*

Stepanov-Type Theorem. *Suppose that a mapping* $\varphi : (\mathbb{M}, d_{cc}) \to (\widetilde{\mathbb{M}}, \tilde{d}_{cc})$ *is such that*

$$\varlimsup_{y \to x} \frac{\tilde{d}_{cc}(\varphi(y), \varphi(x))}{d_{cc}(y, x)} < \infty$$

almost everywhere. Then φ *is* hc*-differentiable almost everywhere.*

Theorem. *Suppose that* $\varphi : (\mathbb{M}, d_{cc}) \to (\widetilde{\mathbb{M}}, \tilde{d}_{cc})$ *is* C^1_H*-smooth and contact (i. e.,* $D_H\varphi[H\mathbb{M}] \subset H\widetilde{\mathbb{M}}$*). Then* φ *is continuously* hc*-differentiable everywhere.*

Definition of Approximate Sub-Riemannian Differentiability [2000; V.]

Let $E \subset \mathcal{M}$ be a measurable subset of $\mathcal{M}$ and $\varphi : E \to \widetilde{\mathcal{M}}$ be a measurable mapping.

An approximate differential of a mapping φ at a point g is the horizontal homomorphism $L : \mathcal{G}^g \to \mathcal{G}^{\varphi(g)}$ of the local Carnot groups such that the set

$$\{v \in B_{cc}(g,r) \cap \mathcal{G}^g : \tilde{d}_{cc}^{\varphi(g)}(\varphi(v), L(v)) > d_{cc}^g(g,v)\varepsilon\}$$

has $\mathcal{H}^\nu$-density zero at the point g for any $\varepsilon > 0$.

Whitney Type Theorem [2012; Basalaev, V.]

Theorem. *Let $\mathcal{M}$, $\widetilde{\mathcal{M}}$ be Carnot manifolds, $E \subset \mathcal{M}$ be a measurable subset of $\mathcal{M}$ and $f : E \to \widetilde{\mathcal{M}}$ be a measurable mapping. The following conditions are equivalent:*

1) *the mapping f is approximately differentiable almost everywhere in E;*

2) *the mapping f has approximate derivatives along the basic horizontal vector fields almost everywhere in E;*

3) *there is a sequence of the disjoint sets $Q_1, Q_2, \dots$ such that $\mathcal{H}^{\nu}\Big(E \setminus \bigcup\limits_{i=1}^{\infty} Q_i\Big) = 0$ and every restriction $f|_{Q_i}$ is a Lipschitz mapping;*

4) $f : E \to \widetilde{\mathcal{M}}$ *meets the condition* $\operatorname{ap}\varlimsup\limits_{x \to g} \frac{\widetilde{d}_{cc}(f(g), f(x))}{d_{cc}(g, x)} < \infty$.

Sub-Riemannian Area Formula [2011; Karmanova]

- the sub-Riemannian Jacobian

$$\mathcal{J}^{SR}(\varphi, y) = \sqrt{\det(\widehat{D}\varphi(y)^*\widehat{D}\varphi(y))}.$$

Theorem. *Let $\varphi : \mathbb{M} \to \widetilde{\mathbb{M}}$ be a Lipschitz mapping of Carnot manifolds with respect to cc-metrics. Then, the area formula holds:*

$$\int\limits_{\mathbb{M}} f(y)\mathcal{J}^{SR}(\varphi, y)\, d\mathcal{H}^{\nu}(y) = \int\limits_{\widetilde{\mathbb{M}}} \sum_{y\colon y\in\varphi^{-1}(x)} f(y)\, d\mathcal{H}^{\nu}(x),$$

where $f : \mathbb{M} \to \mathbb{E}$ ($\mathbb{E}$ is an arbitrary Banach space) is such that the function $f(y)\sqrt{\det(\widehat{D}\varphi(y)^*\widehat{D}\varphi(y))}$ is integrable. Here Hausdorff measures are constructed with respect to quasimetrics d_2 (in the preimage) and $\widetilde{d}_2$ (in the image) with the normalizing factor ω_{ν}.

Sub-Riemannian Coarea Formula [2009; Karmanova, V.]

- the sub-Riemannian coarea factor

$$\mathcal{J}^{SR}_{\widetilde{N}}(\varphi, x) = \sqrt{\det(\widehat{D}\varphi(x)\widehat{D}\varphi(x)^*)} \cdot \frac{\omega_N}{\omega_\nu}\frac{\omega_{\widetilde{\nu}}}{\omega_{\widetilde{N}}}\frac{\omega_{\nu-\widetilde{\nu}}}{\prod\limits_{k=1}^{M}\omega_{n_k-\tilde{n}_k}}.$$

Theorem. *Suppose that* $\varphi \in C^{M+1}(\mathbb{M}, \widetilde{\mathbb{M}})$ *is a contact mapping of two Carnot manifolds,* $\dim H_1\mathbb{M} \geq \dim \widetilde{H}_1\widetilde{\mathbb{M}}$, $\dim H_i\mathbb{M} - \dim H_{i-1}\mathbb{M} \geq \dim \widetilde{H}_i\widetilde{\mathbb{M}} - \dim \widetilde{H}_{i-1}\widetilde{\mathbb{M}}$, $i = 2, \dots, M$. *Then the following coarea formula*

$$\int\limits_{\mathbb{M}} \mathcal{J}^{SR}_{\widetilde{N}}(\varphi, x) f(x)\, d\mathcal{H}^{\nu}(x) = \int\limits_{\widetilde{\mathbb{M}}} d\mathcal{H}^{\widetilde{\nu}}(z) \int\limits_{\varphi^{-1}(z)} f(u)\, d\mathcal{H}^{\nu-\widetilde{\nu}}(u)$$

holds, where $f : \mathbb{M} \to \mathbb{E}$ *(*$\mathbb{E}$ *is an arbitrary Banach space) is such that the product* $\mathcal{J}^{SR}_{\widetilde{N}}(\varphi, x) f(x) : \mathbb{M} \to \mathbb{E}$ *is integrable.*

Weighted Carnot-Carathéodory spaces [2011; Selivanova]

- $\mathbb{M}$, $\dim \mathbb{M} = N$ is a smooth connected manifold;

- $X_1, X_2, \ldots, X_q \in C^{2M+1}$ span $T\mathbb{M}$; $\deg X_i := d_i$, $d_1 \leq \ldots \leq d_q$;

- $X_I = [X_{i_1}, [\ldots, [X_{i_{k-1}}, X_{i_k}] \ldots]$, where $I = (i_1, \ldots, i_k)$; $|I|_h := d_{i_1} + \ldots + d_{i_k}$;

- $H_j = \operatorname{span}\{X_I \mid |I|_h \leq j\}$.

$$H\mathbb{M} = H_1 \subseteq H_2 \subseteq \ldots \subseteq H_M = T\mathbb{M}$$

$$[H_i, H_j] \subseteq H_{i+j}.$$

Here $[H_i, H_j]$ is the linear span of commutators of the vector field generating H_i and H_j.

W.l.o.g. assume $d_1 := 1,\ d_q := M$.

• M is the depth of the Carnot-Carathéodory space $\mathbb{M}$.

• $u \in \mathbb{M}$ is regular, if $\dim(H_k(v)) = \text{const},\ v \in U,\ k = 1, \ldots, M$ in some neighborhood $U = U(u) \subset \mathbb{M}$. Otherwise, $u \in \mathbb{M}$ is nonregular.

Pecularity 1

Different choices of weights may lead to different combinations of regular and nonregular points.

Example

$\mathbb{M} = \mathbb{R}^3$; vector fields $\{X_1 = \partial_y, X_2 = \partial_x + y\partial_t, X_3 = \partial_x\}$.

Nontrivial commutator: $[X_1, X_2] = \partial_t$.

1. Let $\deg(X_i) := 1$, $i = 1, 2, 3$. Then $\deg([X_1, X_2]) = 2$ and

$$H_1 = \text{span}\{X_1, X_2, X_3\},\ H_2 = H_1 \cup \text{span}\{[X_1, X_2]\}.$$

In this case $\{y = 0\}$ is a plane consisting of nonregular points.

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In this case $\{y = 0\}$ is a plane consisting of nonregular points.

2. Let $\deg(X_1) := a$, $\deg(X_2) := b$, $\deg(X_3) := a + b$, $a \leq b$. Then $\deg([X_1, X_2]) = a + b \;\Rightarrow H_a = \operatorname{span}\{X_1\}$, $H_b = H_a \cup \operatorname{span}\{X_2\}$,$H_{a+b} = H_a \cup H_b \cup \operatorname{span}\{X_3, [X_1, X_2]\}$.
In this case all points of $\mathbb{R}^3$ are regular.

Pecularity 2

The intrinsic Carnot-Carathéodory metric d_c might not exist.

Example

$\mathbb{M} = \mathbb{R}^N$ with standard basis $\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_N}$.

Let $\deg(\partial_{x_i}) = 1$ for $1 \leq i \leq m$; $\deg(\partial_{x_i}) > 1$ for $i > m$.

Definitely, $H_i = \text{span}\{\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_i}\}$ satisfy $[H_i, H_j] \subseteq H_{i+j}$, since $[H_i, H_j] = \{0\}$.

But $H_1 = \text{span}\{\partial_{x_i}\}_{i=1}^{m}$ (for any $m < N$) does not span.

Metric structure

We obtain all estimates for the following quasimetric **Nagel, Stein, Wainger 1985:**

$$\rho(v,w) = \inf\{\delta > 0 \mid \text{ there is a curve } \gamma : [0,1] \to U \text{ такая, что}$$

$$\gamma(0) = v, \gamma(1) = w, \dot{\gamma}(t) = \sum_{|I|_h \leq M} w_I X_I(\gamma(t)), |w_I| < \delta^{|I|_h}\}.$$

Here $X_I = [X_{i_1}, [\ldots, [X_{i_{k-1}}, X_{i_k}] \ldots]$, where $I = (i_1, \ldots, i_k)$; $|I|_h = d_{i_1} + \ldots + d_{i_k}$.

Quasimetric space (X, d_X)

X is a topoogical space; $d_X : X \times X \to \mathbb{R}^+$ is such that

(1) $d_X(u,v) \geq 0$; $d_X(u,v) = 0 \Leftrightarrow u = v$;

(2) $d_X(u,v) \leq c_X d_X(v,u)$, where $1 \leq c_X < \infty$ uniformly on $u, v \in X$ (generalized symmetry property);

(3) $d_X(u,v) \leq Q_X(d_X(u,w) + d_X(w,v))$, where $1 \leq Q_X < \infty$ uniformly on all $u, v, w \in X$ (generalized triangle inequality);

(4) $d_X(u,v)$ upper semicontinuous on the first argument

$Q_X = c_X = 1 \Rightarrow (X, d_X)$ **metric space**

Questions

1) Are some analogs of classical results of sub-Riemannian geometry true for weighted C-C spaces equipped with the quasimetric ρ?

2) Which objects are tangent cones?

• How to define the tangent cone to a quasimetric space? (Gromov's theory does not work)

• What is the algebraic structure of the tangent cone to a weighted C-C space?

Results on local geometry

Theorem 1 (Estimate of divergence of integral lines).

Let $u, v \in U$, $\rho(u,v) = O(\varepsilon)$, $r = O(\varepsilon)$ and $B^{\rho}(v,r) \cup B^{\rho^u}(v,r) \subseteq U$. Then the following estimate on the divergence of integral lines holds: $R(u,v,r) = O(\varepsilon^{1+\frac{1}{M}})$.

Theorem 2 (Local approximation theorem).

If $u, v, w \in U$, $\rho(u,v) = O(\varepsilon)$ and $\rho(u,w) = O(\varepsilon)$, then

$$|\rho(v,w) - \rho^u(v,w)| = O(\varepsilon^{1+\frac{1}{M}}).$$

Theorem 3 (Tangent cone theorem).

The quasimetric space (U, ρ^u) is the tangent cone to the quasimetric space (U, ρ) at $u \in U$; the tangent cone is isomorphic to G/H, where G is a nilpotent graded group.

Basic considerations

- Choice of basis $\{Y_1, Y_2, \ldots, Y_N\}$ among $\{X_I\}_{|I|_h \leq M}$:

$*$ $Y_1, Y_2, \ldots, Y_N$ are linearly independent at u (hence in some neighborhood $U(u)$);

$*$ $\sum\limits_{i=1}^{N} \deg Y_i$ is minimal;

$*$ $\sum\limits_{j=1}^{N} |I_j|$ is minimal, where $Y_j = X_{I_j}$.

- Coordinates of the second kind $\Phi^u : \mathbb{R}^N \to U$

$$\Phi^u(x_1, \ldots, x_N) = \exp(x_1 Y_1) \circ \exp(x_2 Y_2) \circ \ldots \circ \exp(x_N Y_N)(u)$$

Basic considerations

- $\{\widehat{X}_I^u\}_{|I|_h \le M}$ – nilpotent approximations of $\{X_I\}_{|I|_h \le M}$ at $u \in U$.

$H_j(u) = \widehat{H}_j(u)$, where $H_j = \mathrm{span}\{\widehat{X}_I^u\}_{|I|_h \le j}$, $\widehat{H}_j = \mathrm{span}\{\widehat{X}_I^u\}_{|I|_h \le j}$.

- Quasimetic

$$\rho^u(v,w) = \inf\{\delta > 0 \mid \text{ there is a curve } \gamma : [0,1] \to U,$$

$$\gamma(0) = v, \gamma(1) = w, \dot{\gamma}(t) = \sum_{|I|_h \le M} w_I \widehat{X}_I^u(\gamma(t)), |w_I| < \delta^{|I|_h}\}.$$

Conical property:

$$\rho^u(\Delta_\varepsilon^u v, \Delta_\varepsilon^u w) = \varepsilon \rho^u(v,w).$$

Divergence of integral lines

Let $u, v \in U$, $r > 0$. *Divergence of integral lines* with the center of nilpotentization u on $B(v,r)$ is

$$R(u,v,r) = \max\{ \sup_{\widehat{y} \in B^{\rho^u}(v,r)} \{\rho^u(y,\widehat{y})\}, \sup_{y \in B^{\rho}(v,r)} \{\rho(y,\widehat{y})\}\}$$

Here the points y and $\widehat{y}$ are defined as follows. Let $\gamma(t)$ be an arbitrary curve such that

$$\begin{cases} \dot{\gamma}(t) = \sum\limits_{|I|_h \le M} b_I \widehat{X}_I^u(\gamma(t)), \\ \gamma(0) = v, \gamma(1) = \widehat{y}, \end{cases}$$

and

$$\rho^u(v,\widehat{y}) \le \max_{|I|_h \le M} \{|b_I|^{1/|I|_h}\} \le r.$$

$y = \exp(\sum\limits_{|I|_h \le M} b_I X_I(v)$. So sup is taken over infinite set of points $\widehat{y} \in B^{\rho^u}(v,r)$ and reals $\{b_I\}_{|I|_h \le M}$,

Remarks about methods of proofs

• Generalization and synthesis of some methods from

∗ Hermes 1991;

∗ Bellaiche 1996;

∗ Christ, Nagel, Stein, Wainger 1999;

∗ Jean 2001.

• Results on regular C-C spaces (Vodopyanov, Karmanova 2007–2009; Karmanova 2010–2011 $X_i \in C^{1,\alpha}$: without using the Backer-Campbell-Hausdorff formula);

• Study of geometric properties of the quasimetrics ρ and ρ^u;

Main geometric properties of ρ and ρ^u

- Generalized triangle inequalities for ρ and ρ^u;

- The “Rolling-of-the-box Lemma” For all $u, v \in U$ and $r, \xi > 0$

$$\bigcup_{x \in B^{\rho^u}(v,r)} B^{\rho^u}(x, \xi) \subseteq B^{\rho^u}(v, r + C\xi),$$

$$\bigcup_{x \in B^{\rho}(v,r)} B^{\rho}(x, \xi) \subseteq B^{\rho}(v, r + C\xi + O(r^{1+\frac{1}{M}}) + O(\xi^{1+\frac{1}{M}})).$$

- Let $u, v \in U$, $r > 0$. Then

$$B^{\rho}(v, r) \subseteq B^{\rho^u}(v, r + CR(u, v, r)),$$

$$B^{\rho^u}(v, r) \subseteq B^{\rho}(v, r + CR(u, v, r) + O(r^{1+\frac{1}{M}}) + O(R(u, v, r)^{1+\frac{1}{M}})),$$

where $R(u, v, r)$ is the divergence of integral lines.

Metrical aspect [2010; Selivanova]

• We introduce a theory of convergence of quasimetric spaces such that

1) For metric spaces, it is equivalent to Gromov's theory;

2) For boundedly compact quasimetric spaces the limit is unique up to isometry;

3) It gives an adequate notion of the tangent cone.

Quasimetric space (X, d_X)

X is a topoogical space; $d_X : X \times X \to \mathbb{R}^+$ is such that

(1) $d_X(u,v) \geq 0$; $d_X(u,v) = 0 \Leftrightarrow u = v$;

(2) $d_X(u,v) \leq c_X d_X(v,u)$, where $1 \leq c_X < \infty$ uniformly on $u, v \in X$ (generalized symmetry property);

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(4) $d_X(u,v)$ upper semicontinuous on the first argument

Gromov's theory for metric spaces does not work!

We introduce the distance

$$d_{qm}(X,Y) = \text{inf}\{\rho > 0 \mid \exists f : X \to Y, g : Y \to X, \text{such that}$$

$$\max\left\{\text{dis}(f),\ \text{dis}(g),\ \sup_{x\in X} d_X(x, g(f(x))),\ \sup_{y\in Y} d_Y(y, f(g(y)))\right\} \le \rho\}$$

where $\text{dis}(f) = \sup_{u,v\in X} |d_Y(f(u), f(v)) - d_X(u,v)|$.

Property. For metric spaces d_{qm} is equivalent to d_{GH}:

$$d_{GH}(X,Y) \le d_{qm}(X,Y) \le 2d_{GH}(X,Y).$$

• For noncompact quasimetric spaces we say that $(X_n, p_n) \underset{qm}{\to} (X, p)$, if there is such $\delta_n \to 0$, that for all $r > 0$ there exist mappings $f_{n,r} : B^{d_{X_n}}(p_n, r + \delta_n) \to X$, $g_{n,r} : B^{d_X}(p, r + 2\delta_n) \to X_n$ such that

(1) $f_{n,r}(p_n) = p$, $g_{n,r}(p) = p_n$;

(2) $\mathrm{dis}(f_{n,r}) < \delta_n$, $\mathrm{dis}(g_{n,r}) < \delta_n$;

(3) $\sup\limits_{x \in B^{d_{X_n}}(p_n, r+\delta_n)} d_{X_n}(x, g_{n,r}(f_{n,r}(x))) < \delta_n$.

• $T_x X = \lim\limits_{\lambda \to \infty} (X, x, \lambda \cdot d)$ is the tangent cone to X at $x \in X$

For quasimetric spaces with dilations, in particular Carnot-Carathéodory spaces, we can take
$f_n = \delta^x_{\lambda_n}$, $g_n = \delta^x_{\lambda_n^{-1}}$ where $\lambda \to \infty$, and prove the main result.

Theorem (Existence and structure of the tangent cone to a nonreguar weighted quasimetric Carnot-Caratheodory space).

At $u \in \mathbb{M}$ the tangent cone to $(\mathbb{M}, \rho)$ (in the sense of our definition) is G/H; the Lie algebra V of G is graded and nilpotent:

$$V = V_1 \bigoplus V_2 \bigoplus \ldots \bigoplus V_M; \ [V_1, V_i] \subseteq V_{i+1}.$$

• When Hörmander's condition holds, the tangent cone to $(\mathbb{M}, d_c)$ (in the Gromov's sense) at a regular point is a stratified Lie group, i.e. $[V_1, V_i] = V_{i+1}$, at nonregular point it is G/H.

THANK YOU FOR YOUR ATTENTION!