Explicit controllability of rolling spheres

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Rolling motions

Suppose M_1, M_2 are submanifolds of the same dimension n in $\mathbb{R}^N$, and $\gamma_1 : I = [0, T] \to M_1$ is piecewise smooth curve in M_1.

A *rolling motion of* M_1 *on* M_2 *along* γ_1 *without twisting or slipping* is

$$X_t = (R(t), s(t)) : I \to \mathrm{SE}_n = \mathrm{SO}_n \ltimes \mathbb{R}^n$$

such that the following rolling constraints hold at almost all $t \in I$:

- **Rolling condition (tangent contact)**

$$X_t(\gamma_1(t)) = \gamma_2(t) \in M_2,$$
$$T_{\gamma_2(t)}(X_t M_1) = T_{\gamma_2(t)} M_2.$$

Rolling motions (continued)

- **No-slip condition ($t \mapsto X_t(M_1)$ has zero velocity at $\gamma_2(t)$)**

$$\dot{X}_t(\gamma_1(t)) = 0,$$

or, equivalently, putting $(X_t)_* = R$, $(X_t)_* \dot{\gamma}_1(t) = \dot{\gamma}_2(t)$.

- **No-twist conditions (tangential and normal)**

$$\dot{R}R^\top (T_{\gamma_2} M_2) \subset (T_{\gamma_2} M_2)^\perp \quad \text{and} \quad \dot{R}R^\top (T_{\gamma_2} M_2)^\perp \subset T_{\gamma_2} M_2.$$

In the above, γ_1 is the *rolling curve* and γ_2 is the *development*.

Theorem

Given a rolling curve γ_1 starting at $p \in M_1$ and $q \in M_2$, there is a unique rolling motion X_t with development γ_2 starting at q.

Rolling systems and controllability

Rolling control system

The rolling constraints define a distribution $\mathcal{D}$ on the space of configurations

$$\Sigma = \{(p, q, R) \in M_1 \times M_2 \times \mathrm{SO}_n : R\,(T_p M_1) = T_q M_2\}\,.$$

Some remarks:

- The tangency condition is holonomic, the other two are not.
- To establish controllability, we may check that $\mathcal{D}$ is non-integrable.
- Controls are velocities. Physically, this is the zero-inertia case.
- In specific cases, simpler equivalent forms of Σ will be used.
- The definition of rolling above follows Sharpe (1996).

If the ambient space is $\mathbb{R}^3$, kinematics of rolling may be described using Darboux frames and the cross-product.

As an example: $M_1 = \mathbb{S}^2(\rho)$ rolls on an horizontal plane $M_2 = P$. If

- $\overrightarrow{n}$ is the upwards unit normal to P,
- $\overrightarrow{v} = \overrightarrow{v}(t)$ is the velocity of the center of the sphere, parallel to P,
- $\overrightarrow{\omega} = \overrightarrow{\omega}(t)$ is the angular velocity of the sphere, the control input,

then

- No slip condition: $\overrightarrow{v} + \overrightarrow{\omega} \times \left(-\rho\overrightarrow{n}\right) = 0$.
- No-twist condition: $\overrightarrow{\omega} \cdot \overrightarrow{n} = 0$. (The other is trivially satisfied)

The rolling motion X_t can be made explicit, if needed.

Let M_1 be an n-sphere centered at the origin in $\mathbb{R}^{n+1}$. Any rolling motion X_t of M_1 on the plane M_2 tangent to it at $q \in M_1 \cap M_2$ satisfies $X_0 = (R(0), s(0)) = (I, 0)$.

The rolling kinematics for $X_t = (R(t), s(t))$ are

$$\dot{R} = AR, \;\; \dot{s} = u$$

for suitable inputs $t \mapsto A(t) \in \mathfrak{so}_n$, $t \mapsto u(t) \in \mathbb{R}^{n+1}$.

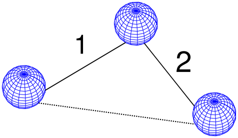
From the first rolling condition, $R\gamma_1 = q$. Then $\gamma_2 = R\gamma_1 + s = q + s$. The no-slip condition then gives $\dot{\gamma}_2 = \dot{s} = R\dot{\gamma}_1 = -\dot{R}\gamma_1 = -AR\gamma_1 = -Aq$. Since $A = \dot{R}R^\top$, the no-twist relations imply that, in appropriate coordinates, where the last vector of the basis is $-q$,

$$A = \left[\begin{array}{c|c} 0 & u \\ \hline -u^\top & 0 \end{array}\right].$$

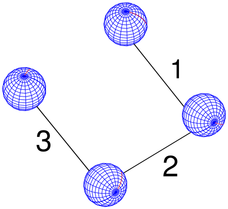
Explicit controllability
2-sphere rolling on a plane (1)

Consider a 2-sphere of radius ρ rolling on the *xy*-plane in $\mathbb{R}^3$, take $\Sigma = \mathbb{R}^2 \times \mathrm{SO}_3$.

- A state transfer $(p, R) \to (q, R)$ is a *slip*.
- A state transfer $(p, R) \to (p, R')$, where R and R' are related a rotation about the z-axis, is a *twist*.
- Controllability follows if we exhibit rolling motions that achieve these state transfers.
- A slip is achievable by two rolling motions along two line segments of large enough integer multiple of $2\pi\rho$ length.



- A twist by an angle θ is achievable by a six rolling step sequence, half of which is as shown:



- Several such maneuvers for achieving slips and twists have been described. For another, see R. Murray et al (1994).

Properties of rolling

If X_t is a rolling map, let $X_t^{-1} : I \to \mathrm{SE}_n$ be given by

$$X_t^{-1} = (X_t)^{-1} = (R, s)^{-1} = \left(R^{-1}, -R^{-1}s\right) \in \mathrm{SE}_n$$

Theorem

Rolling is symmetric: if M_1 rolls on M_2 along γ_1 with rolling motion X_t and development γ_2, then M_2 rolls on M_1 along γ_2 with rolling motion X_t^{-1} and development γ_1.

Theorem

Rolling is transitive. (With the obvious, analogous meaning.)

Kinematics of rolling

n-sphere rolling on another n-sphere, $n \geq 2$

Using the symmetry and transitivity properties, we may find the kinematics of the rolling of one n-sphere on another by taking both such spheres to roll on a common plane $P \subset \mathbb{R}^{n+1}$.

Taking $M_1 = \mathbb{S}^n(\rho_1) + (0, \ldots, 0, -(\rho_1 + \rho_2))$ and $M_2 = \mathbb{S}^n(\rho_2)$, and rolling both M_1, M_2 on the common tangent plane at q, we obtain the kinematic equations for M_1 to roll on M_2:

$$X_t = \left(R_2^\top R_1, R_2^\top (s_1 - s_2)\right),$$

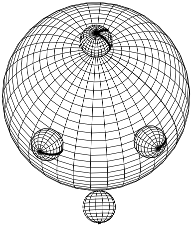
where

$$\begin{cases} \dot{R}_1 & = AR_1 \\ \dot{s}_1 & = -A(q + R_1\tau) \\ \dot{R}_2 & = -\frac{\rho_1}{\rho_2} AR_2 \\ \dot{s}_2 & = -Aq \end{cases}$$

By rescaling, let $M_1 = \mathbb{S}^2(\rho) + (0,0,1+\rho)$, $M_2 = \mathbb{S}^2(1)$. Take $\Sigma = \mathbb{S}^2 \times \mathrm{SO}_3$.

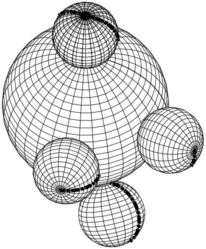
- It is known that the system is not controllable if $\rho = 1$.
- We may assume $0 < \rho < 1$.
- As in the case of a 2-sphere rolling on a plane, controllability follows once we are able to achieve certain state transfers by rolling motions.
- A *twist* is a transfer $(p, R) \to (p, R')$, where R, R' are orientations related by a rotation about the line through the origin and p.

If $0 < \rho < \frac{1}{4}$, we may roll M_1 along four arcs of a spherical quadrangle with four equal sides of arclength $2\pi\rho$ and internal angles α and β. It is proved that the total effect is a twist by an angle of $-(2\alpha + 2\beta)$.



The same maneuver may be used if $\frac{3}{4} < \rho < 1$, by rolling M_1 along the complement of each side of the same quadrangle, relative to the maximal circle it is in.

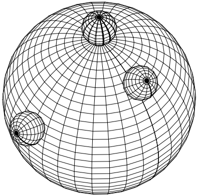
To perform a twist in the cases $\frac{1}{4} < \rho < \frac{1}{2}$ and $\frac{1}{2} < \rho < \frac{3}{4}$, we again roll M_1 along four arcs of a spherical lozange, but now with sides of arclength $\pi\rho$ and internal angles α and β. The total effect is that of a twist by an angle of $-2\alpha + 2\beta$.



In the cases $\rho = \frac{1}{4}$, $\rho = \frac{1}{2}$, $\rho = \frac{3}{4}$, twists are easy to obtain by rolling either to the equator or to the opposite pole.

- A *slip* is a transfer $(p, R) \to (q, R')$, where R' is obtained from R by the same rotation that takes p to q.

In order to perform a slip, we may roll M_1 along two suitable arcs of length integer multiple of $2\pi\rho$ and then perform a suitable twist.



To achive a given end-state, decide which is the contact point of M_1 at that end-state, make any rolling motion to achieve that contact point, then perform a slip to achive the desired contact point of M_2 and finally perform a twist to achieve the desired final orientation.

M_1, M_2 are n-spheres in $\mathbb{R}^{n+1}$, of radii $0 < \rho < 1$ and 1, centered at c and the origin, $q = M_1 \cap M_2$, $L = \operatorname{span}\{q\}$.

A *twist at q* is $(\exp M, 0) \in \mathrm{SE}_n$, , $M \in \mathfrak{so}_n$ and $Mq = 0$.
A *slip from q* is $(\exp N, 0) \in \mathrm{SE}_n$, $N \in \mathfrak{so}_n$, $N(L) \subset L^{\perp}$, and $N\left(L^{\perp}\right) \subset L$.
In a suitable basis,

$$M = \left[\begin{array}{c|c} \tilde{M} & 0 \\ \hline 0 & 0 \end{array}\right], \quad N = \left[\begin{array}{c|c} 0 & b \\ \hline -b^{\top} & 0 \end{array}\right].$$

Controllability follows if we can obtain these Euclidean motions by rolling.

Proposition

If $\overline{X} = (\exp M, 0)$ is a twist at p_0, there is a rolling X_t such that $X_T = \overline{X}$.

Express $\exp \tilde{M}$ as a product of Givens rotations $\exp(tA_{ij})$. The twist is achieved by a sequence of rolling motions using only two control inputs.

Proposition

If $\overline{X} = (\exp M, 0)$ is a slip at q, there is a rolling X_t such that $X_T = \overline{X}$.

By conjugation with $n-1$ twists at p_0, the problem reduces achieving

$$\left(\exp\left[\begin{array}{c|c} 0 & p \\ \hline -p^\top & 0 \end{array}\right], 0\right), \quad p = (0, \ldots, 0, t) \in \mathbb{R}^n,$$

similar to a slip in the case $n = 2$ with respect to the three last coordinates.

Theorem

The rolling curve γ_1 is a geodesic of iff γ_2 is a geodesic of M_2.

A rolling motion along a geodesic is *a pure (rolling motion)*.

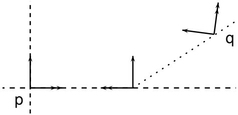
Kendall's problem (1950's)

What is the minimum number of pure rolling motions that are sufficient to control a 2-sphere moving on a plane?

This question was settled by Hammersley (1984).

Rolling a 2-sphere on a plane in four pure motions

We may assume the sphere has radius one and $\Sigma = \mathbb{R}^2 \times SO_3$.
To achieve a slip, perform two pure motions of length $k2\pi$, as before.
A simultaneous twist by angle θ and forced slip $(p, R) \to (q, R')$ is achieved in two pure motions of length π thus:



If the initial contact point of the sphere is as desired, achieve the final state in four pure motions. If it is antipodal, a single motion corrects twist and contact point and two further place the sphere. Otherwise, a single motion reduces to the previous case. This is a simplified version of work of L. Biscolla (2005).

Known results and open problems

- A 2-sphere rolling on a plane is controllable in three pure motions, obtained by Hammersley. The proof is not simple.
- A 2-sphere rolling on another 2-sphere is controllable in no more than four pure motions, proved by L. Frankel (2007).
- We believe it is open whether three motions are sufficient in this last case. Work is in progress by the group of W. Oliva.
- The higher dimensional analogues and many other generalizations were already suggested by Hammersley himself in 1984 and have remained open.

References

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