Geometry structure of neighbourhoods of singular extremals in problems with multidimensional control.

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Layout

Let us consider a general optimal control problem on finite dimensional vector space V:

$$\dot{x} = \varphi(t, x, u).$$

Here $x \in V$, $u \in \Omega$, $\varphi \in C^\infty$ where Ω is a subset of vector space U. We assume that U has finite dimension. Time interval $[T_1, T_2]$ can be bounded or unbounded. The problem is to minimize

$$P\big(T_2, x(T_2)\big) \to \min$$

with some terminal constrains.

If $\widehat{x}(t)$, $\widehat{u}(t)$ is an optimal trajectory then PMP states that

$$H(x,p,u) = p\varphi(t,x,u);$$

$$\begin{cases} \dot{x} &= \quad \dfrac{\partial}{\partial p} H; \\ \dot{p} &= - \quad \dfrac{\partial}{\partial x} H; \\ \widehat{u}(t) \in \underset{u\in\Omega}{\arg\max}\, H\big(t, \widehat{x}(t), \widehat{p}(t), u\big). \end{cases} \tag{1}$$

for some Lagrange multiplier $\widehat{p}(t)$. We consider only normal optimal trajectories. The symbol $\frac{d}{dt}$ means formal differentiation along solutions of system (1) as usual.

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If $\widehat{u}(t) \in \operatorname{Int}\Omega$ then $\frac{\partial}{\partial u}H = 0$. So the Hessian $\frac{\partial^2}{\partial u^2}H$ is a well-defined symmetric bilinear form on $T_{\widehat{u}(t)}\Omega = U$. And it follows form PMP that

$$\frac{\partial^2}{\partial u^2}H \leq 0$$

Definition

An optimal trajectory $\widehat{x}(t), \widehat{u}(t)$ is called singular on (t_1, t_2) if $\widehat{u}(t)$ is an interior point of Ω and $\frac{\partial^2}{\partial u^2}H$ is rank deficient.

Let $\dim U \geq 1$ and let u^i be coordinates in U. Let q_i be the order of 1-dimensional control u^i. Then

$$\frac{\partial}{\partial u_i}\frac{d^k}{dt^k}\frac{\partial}{\partial u_j}H = 0 \quad \forall k \leq q_i + q_j - 1$$

and GLC can be written in terms of Goh-Krener matrix

$$\left((-1)^{q_j}\frac{\partial}{\partial u_i}\frac{d^{q_i+q_j}}{dt^{q_i+q_j}}\frac{\partial}{\partial u_j}H\right)_{i,j=1}^{\dim U}.$$

This matrix must be symmetric non-positive definite.

Layout

Let us consider a simple example

$$\int_0^{+\infty} (x^2 + y^2)\, dt \to \inf$$

$$\begin{cases} \dfrac{d}{dt} x = u \cos t - v \sin t \\ \dfrac{d^2}{dt^2} y = u \sin t + v \cos t \end{cases}$$

where $u^2 + v^2 \leq 1$. The trajectory $x = y = u = v = 0$ is the only one singular. The formal order of u and v is 1 and the Goh-Krener matrix is rank deficient. Put

$$u_1 = u \cos t - v \sin t, \quad v_1 = u \sin t + v \cos t$$

Obviously, u_1 has order 1 and v_1 has order 2.

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Consider the singular control $\widehat{u}(t) \in \operatorname{Int}\Omega$ and the tangent space $T_{\widehat{u}(t)}\Omega$. We want to define an order of directions in $T_{\widehat{u}(t)}\Omega = U$.

Let $\xi \in U$ and $\mathrm{Ord}_t(\xi)$ be 1-dimensional order of the optimal control along ξ.

Using the Goh-Krener theorem we can prove that

$$\mathrm{Ord}_t(\lambda_1\xi_1 + \lambda_2\xi_2) \geq \min\Big\{\mathrm{Ord}_t(\xi_1), \mathrm{Ord}_t(\xi_2)\Big\}.$$

Let $f : U \to \mathbb{R}$ be an arbitrary map with the following property

$$f(\lambda_1\xi_1 + \lambda_2\xi_2) \geq \min\Big\{f(\xi_1), f(\xi_2)\Big\}$$

for any $\lambda_i \in \mathbb{R}$ and $\xi_i \in U$. Then its set of values is finite and contains no more than $\dim U + 1$ values:

$$f(U) = \Big\{q_0 < q_1 < \ldots < q_k\Big\} \quad k \leq \dim U.$$

Moreover, if $U_i = f^{-1}\{x \geq q_i\}$ then U_i is a linear subspace of U and

$$U_k \subset U_{k-1} \subset \ldots \subset U_0 = U.$$

So U is represented as a flag $\{U_i(t)\}_{i=0}^k$ and a direction $\xi \in U$ has the order q at t iff $\xi \in U_q(t)$ and $\xi \notin U_{q+1}$.

Layout

Let $\widehat{u}(t)$ be a singular extremal on (t_1, t_2). Put by definition

$$U_0(t) = U \quad \text{and} \quad B_0(t) = \frac{\partial^2}{\partial u^2} H$$

and then by iteration process if B_q is rank deficient then put

$$U_{q+1}(t) = \ker B_q(t);$$

$$B_{q+1}(t) = (-1)^q \frac{\partial}{\partial u} \frac{d^{\,2(q+1)}}{dt^{\,2(q+1)}} \frac{\partial}{\partial u} H \Big|_{U_{q+1}(t)}.$$

Theorem (A)

The bilinear forms $B_q(t)$ are well-defined and are independent on any coordinate system on U. Moreover $B_q(t)$ is symmetric non-positive definite on singular extremal and

$$\frac{\partial}{\partial u}\frac{d^{2q+1}}{dt^{2q+1}}\frac{\partial}{\partial u}H\bigg|_{U_{q+1}(t)} = 0 \quad \forall q \geq 0.$$

Definition

Let the dimensions of $U_q(t)$ be constant on (t_1, t_2). Then local order of the singular extremal $\widehat{x}(t)$, $\widehat{u}(t)$ is a sequence $(q_0, q_1, \ldots)$ such that

$$q_i = \dim U_i(t) - \dim U_{i+1}(t) = \mathrm{rk}\, B_i(t).$$

Notice that if classical local order in problem with 1-dimensional input is q then the previous definition gives the sequence of orders $(0, \ldots, 0, 1, 0, \ldots)$ where 1 is put on q-th place.

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If $\dim U = 1$ then necessary conditions of junction of a singular extremal with non-singular one is described in terms of intrinsic order of the control problem in some neighbourhood of the singular extremal.

However, $\frac{\partial}{\partial u} H$ does not vanish on any neighbourhood of the singular extremal in general. So $\frac{\partial^2}{\partial u^2} H$ is not well-defined on any neighbourhood except the situation when only linear (affine) changes of coordinates on U are allowed.

Let the Hamiltonian system be affine in u:

$$H(t,x,p,u) = H^0(t,x,p) + \big\langle H^u(t,x,p), u \big\rangle;$$
$$\begin{cases} \dot{x} & = \quad \dfrac{\partial}{\partial p} H^0 + \langle \dfrac{\partial}{\partial p} H^u, u \rangle; \\ \dot{p} & = - \quad \dfrac{\partial}{\partial x} H^0 - \langle \dfrac{\partial}{\partial x} H^u, u \rangle. \end{cases} \tag{2}$$

If only affine coordinates $u^i = u^i(t)$ on U are allowed then the forms $B_q(t,x,p)$ and flag $\{U_q(t,x,p)\}_{i=1}^{\infty}$ are well-defined at any point (t,x,p).

Definition

Let the singular trajectory $\widehat{x}(t), \widehat{u}(t)$ has the local order $(q_1, q_2, \ldots)$. If statements of theorem A holds on a neighbourhood of the singular trajectory then we say that the system (2) has the intrinsic order $(q_1, q_2, \ldots)$ in the neighbourhood.

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Consider a junction of a singular extremal $(\widehat{x}(t), \widehat{p}(t), \widehat{u}(t))$ on interval (t_1, t_2) with non-singular one $(\widetilde{x}(t), \widetilde{p}(t), \widetilde{u}(t))$ at some point $\tau \in (t_1, t_2)$. Let for example the non-sigular extremal be defined on $(\tau; \tau + \varepsilon)$. Naturally, $\widetilde{u}(t) \in \partial\Omega$.

We will say that the junction at τ is regular if

1. Controls $\widehat{u}(t)$ and $\widetilde{u}(t)$ are smooth on (t_1, t_2) and $(\tau; \tau + \varepsilon)$ respectively.
2. There exists $\lim_{t \to \tau + 0} \widetilde{u}(t) = u^0$

Theorem (B)

Let τ be a regular junction point. Consider curve

$$\xi(t) = \widetilde{u}(t) - \widehat{u}(t).$$

If q is the order of $\xi(t)$ (i.e. maximum number q such that $\xi(t) \in U_q(t)$ for $t \in [\tau; \tau + \varepsilon)$) and strong GLC satisfied

$$(-1)^q \frac{\partial}{\partial u} \frac{d^{2q}}{dt^{2q}} \frac{\partial}{\partial u} H \Big|_{U_q(t, \widetilde{x}(t), \widetilde{p}(t))} = B_q(t, \widetilde{x}(t), \widetilde{p}(t)) < 0$$

then q is odd or $q = \infty$.

Corollary

If the sequence $(q_1, q_2, \dots)$ of the intrinsic order of the affine on u system (2) is consist of only even orders i.e.

$$q_{2k+1} = 0 \quad \forall k$$

$$\sum_k q_{2k} = \dim U$$

and strong GLC satisfied for direction $\xi(t) = \widetilde{u}(t) - \widehat{u}(t)$ then no regular junction is possible.

Layout

Consider the following optimal control problem with multi-dimensional input

$$\begin{gathered}\frac{1}{2}\int\limits_0^{+\infty}\langle Cx, x\rangle\, dt \to \min;\\ x^{(q)} = u, |u| \le 1.\end{gathered} \tag{3}$$

Here x and u is in $\mathbb{R}^n$, $q \in \mathbb{N}$ and C is a symmetric bilinear form. We can assume that C is positive definite. Terminal conditions:

$$x(0) = x_0^0;\ \dot{x}(0) = x_1^0;\ \ldots;\ x^{(q-1)}(0) = x_{q-1}^0.$$

There is only one singular extremal $x(t) = u(t) \equiv 0$ and its order is q (i.e. flag of orders is $(0, \ldots, 0, q, 0, \ldots)$)).

Let $\mathbb{R}^n = \bigoplus_k V_k$ where V_k eigenspaces of C.

Theorem (C)

If initial conditions $x_0^0, \ldots, x_{q-1}^0$ belong to a spase V_k then the optimal trajectory $\widehat{x}(t)$ belongs to the subspace spanned by initial conditions

$$\widehat{x}(t) \in \mathrm{span}(x_0^0, \ldots, x_{q-1}^0) \subseteq V_k.$$

Layout

Let us consider the optimal synthesis on V_k. It consides with the synthesis in the following reduced problem

$$\begin{gathered}\frac{1}{2}\int_0^{+\infty}\langle x,x\rangle\,dt \to \min; \\ x^{(q)}=u, |u|\leq 1;\end{gathered} \tag{4}$$

where $x, u \in \mathbb{R}^q$. Put

$$x_1 = x, \quad x_2 = \dot{x}, \ldots, \ x_q = x^{(q-1)},$$

Let $p_1, \ldots, p_q$ be conjugate variables in the corresponding hamilton system:

$$H = -\frac{1}{2}x_1^2 + p_1x_2 + p_2x_3 + \ldots + p_qu.$$

The reduced problem (4) has the following symmetries:

1. The group $SO_n(q)$ acts on the variables by simultaneous rotation of all variables x_i, p_i and u.
2. The large-scale group $\mathbb{R}_+$ acts in the following way:

$$x_k \mapsto \lambda^{q-k} x_k;$$
$$p_k \mapsto \lambda^{q+k+1} p_k;$$
$$u \mapsto u; t \mapsto \lambda t.$$

Thus the problem (4) has the symmery group $G = SO_n(q) \times \mathbb{R}_+$. This simple fact allows us to construct some explicit solutions.

Consider any 2-plane $L \subseteq \mathbb{R}^q$, $0 \in L$. Introduce a complex structure on L.

Theorem (B)

The 2-plane $L \ni 0$ contains $[q/2]$ pairs of optimal trajectories of the reduced problem (4):

$$x = x_1 = A_k\, t^q \exp\Big\{\pm i\alpha_k \ln|t|\Big\},\ u = \exp\Big\{\pm i\alpha_k \ln|t|\Big\},$$

where $\alpha_k \in \mathbb{R}_+$ can be found from the initial conditions

$$\mathrm{Im}\, P_q(\alpha) = 0, \quad (-1)^{q+1}\mathrm{Re}\, P_q(\alpha) > 0,$$

and $1/A_k = (-1)^{q+1}\mathrm{Re} P_q(\alpha_k)$. Here

$$P_q(\alpha) = \big(2q + i\alpha\big)\big((2q-1) + i\alpha\big) \dots \big(1 + i\alpha\big).$$

Also if q is odd then there is a trajectory $x = \frac{t^q}{q!}$ and $u = 1$.

For example, if $q = 4$ then we have two different logarithmic spirals with different "rotation speeds" disparate over the field of rational numbers.

$$u^1 = \exp(\alpha_1 i \ln|t|), \qquad u^2 = \exp(\alpha_2 i \ln|t|);$$

$$x^1 = A_1 t^4 \exp(\alpha_1 i \ln|t|), \quad x^2 = A_2 t^4 \exp(\alpha_2 i \ln|t|).$$

Here $\alpha_1 \cong 1.36$, $\alpha_2 \cong 10,4$. Values α_1 and α_2 are disparate over $\mathbb{Q}$.

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Now we can construct explicit solutians of the original problem (3) by combining solutions of the reduced problem (4). For example, if $q = 4$ and two different eigen values λ_1 and λ_2 of the bilinear form C are related as

$$\frac{\lambda_1}{\lambda_2} = \frac{P_4(\alpha_1)}{P_4(\alpha_2)},$$

then the linear combination $x(t) = ax^1(t) + bx^2(t)$, $u(t) = au^1(t) + bu^2(t)$ is also a solution of (3) when

$$a^2 + b^2 = 1.$$

So the optimal control moves along an irrational winding of the Clifford torus $\mathbb{T}_{ab}$. The whole winding is traversed in a finite time. After that there is a non-reguler junction with the singular trajectory $x = u = 0$.

In general case $q \geq 4$ we can construct explicit optimal controls in k-torus $\mathbb{T}^k$ if some eigen valus $\lambda_1, \dots, \lambda_k$ of C are related as

$$\lambda_1 : \lambda_2 : \dots : \lambda_k = P_q(\alpha_1) : P_q(\alpha_2) : \dots : P_q(\alpha_k).$$

If the roots $\{\alpha_j\}_{j=1}^k$ are lieanly independent over $\mathbb{Q}$ then the constructed optimal control belongs to irrational winding of $\mathbb{T}^k$ and the whole winding is traversed in a finite time.
But we can not prove the linearly independece of the roots $\{\alpha_j\}_{j=1}^k$ over $\mathbb{Q}$ and this question is still open.

Note that the polynomial $P_q(\alpha)$ has very specific structure:

$$P_q(\alpha) = \big(2q + i\alpha\big)\big((2q-1) + i\alpha\big)\dots\big(1 + i\alpha\big).$$

This gives us the courage to state the following hypothesis.

Hypothsis

The roots $\{\alpha_{2j-1}\}_{j=1}^{[q/2]}$ of the polynomial $\mathrm{Im}P_q(\alpha)$ are linearly independent over the field $\mathbb{Q}$ of rational numbers.