arXiv:1404.7752v1 [math.OC] 30 Apr 2014

Sub-Riemannian geodesics on the free Carnot group with the growth vector $(2, 3, 5, 8)^*$

Yuri Sachkov
Program Systems Institute
Russian Academy of Sciences
Pereslavl-Zalessky, Russia
E-mail: sachkov@sys.botik.ru

May 1, 2014

To Lena, for the birthday

Abstract

We consider the free nilpotent Lie algebra L with 2 generators, of step 4, and the corresponding connected simply connected Lie group G. We study the left-invariant sub-Riemannian structure on G defined by the generators of L as an orthonormal frame.

We compute two vector field models of L by polynomial vector fields in $\mathbb{R}^8$, and find an infinitesimal symmetry of the sub-Riemannian structure. Further, we compute explicitly the product rule in G, the right-invariant frame on G, linear on fibers Hamiltonians corresponding to the left-invariant and right-invariant frames on G, Casimir functions and co-adjoint orbits on L^*.

Via Pontryagin maximum principle, we describe abnormal extremals and derive a Hamiltonian system $\dot{\lambda} = \vec{H}(\lambda)$, $\lambda \in T^*G$, for normal extremals. We compute 10 independent integrals of $\vec{H}$, of which only 7 are in involution. After reduction by 4 Casimir functions, the vertical subsystem of $\vec{H}$ on L^* shows numerically a chaotic dynamics, which leads to a conjecture on non-integrability of $\vec{H}$ in the Liouville sense.

1 Introduction

In this work we study a variational problem that can be stated equivalently in the following three ways.

*Work supported by Grant of the Russian Federation for the State Support of Researches (Agreement No 14.B25.31.0029).

(1) Geometric statement. Consider two points $a_0, a_1 \in \mathbb{R}^2$ connected by a smooth curve $\gamma_0 \subset \mathbb{R}^2$. Fix arbitrary data $S \in \mathbb{R}$, $c = (c_x, c_y) \in \mathbb{R}^2$, $M = (M_{xx}, M_{xy}, M_{yy}) \in \mathbb{R}^3$. The problem is to connect the points a_0, a_1 by the shortest smooth curve $\gamma \subset \mathbb{R}^2$ such that the domain $D \subset \mathbb{R}^2$ bounded by $\gamma_0 \cup \gamma$ satisfy the following properties:

1. $\text{area}(D) = S$,
2. center of $\text{mass}(D) = c$,
3. second order $\text{moments}(D) = M$.

(2) Algebraic statement. Let L be the free nilpotent Lie algebra with two generators X_1, X_2 of step 4:

$$L = \text{span}(X_1, \dots, X_8), \tag{1}$$
$$[X_1, X_2] = X_3, \tag{2}$$
$$[X_1, X_3] = X_4, \quad [X_2, X_3] = X_5, \tag{3}$$
$$[X_1, X_4] = X_6, \quad [X_1, X_5] = [X_2, X_4] = X_7, \quad [X_2, X_5] = X_8. \tag{4}$$

Let G be the connected simply connected Lie group with the Lie algebra L, we consider $X_1, \dots, X_8$ as a frame of left-invariant vector fields on G. Consider the left-invariant sub-Riemannian structure (G, Δ, g) defined by X_1, X_2 as an orthonormal frame:

$$\Delta_q = \text{span}(X_1(q), X_2(q)), \qquad g(X_i, X_j) = \delta_{ij}.$$

The problem is to find sub-Riemannian length minimizers that connect two given points $q_0, q_1 \in G$:

$$q(t) \in G, \qquad q(0) = q_0, \quad q(t_1) = q_1,$$
$$\dot{q}(t) \in \Delta_{q(t)},$$
$$l = \int_0^{t_1} \sqrt{g(\dot{q}, \dot{q})}\, dt \to \min.$$

(3) Optimal control statement. Let vector fields $X_1, X_2 \in \text{Vec}(\mathbb{R}^8)$ be defined by (14), (15). Given arbitrary points $q_0, q_1 \in \mathbb{R}^8$, it is required to find solutions of the optimal control problem

$$\dot{q} = u_1 X_1(q) + u_2 X_2(q), \qquad q \in \mathbb{R}^8, \quad (u_1, u_2) \in \mathbb{R}^2, \tag{5}$$
$$q(0) = q_0, \qquad q(t_1) = q_1, \tag{6}$$
$$J = \frac{1}{2} \int_0^{t_1} (u_1^2 + u_2^2)\, dt \to \min. \tag{7}$$

The problem stated will be called the nilpotent sub-Riemannian problem with the growth vector $(2, 3, 5, 8)$, or just the $(2, 3, 5, 8)$-problem. There are several important motivations for the study of this problem:

- this problem is a nilpotent approximation of a general sub-Riemannian problem with the growth vector (2,3,5,8) $[2, 5, 7, 13, 20]$,
- this problem is a natural continuation of the basic sub-Riemannian (SR) problems: the nilpotent SR problem on the Heisenberg group (aka Dido's problem, growth vector (2,3)) [6,30], and the nilpotent SR problem on the Cartan group (aka generalized Dido's problem, growth vector (2,3,5)) [21–24],
- this problem is included into a natural infinite chain of rank 2 SR problems with the free nilpotent Lie algebras of step r, $r \in \mathbb{N}$, and more generally into a natural 2-dimensional lattice of rank d SR problems with the free nilpotent Lie algebras of step r, $(d, r) \in \mathbb{N}^2$,
- this problem is the simplest possible SR problem on a step 4 Carnot group, and it is the first SR problem with growth vector of length 4 that should be studied.

To the best of our knowledge, this is the first study of the (2,3,5,8)-problem (although, it was mentioned in [8] as a SR problem with smooth abnormal minimizers).

The structure of this work is as follows.

In Sec. 2 we construct two models ("asymmetric" and "symmetric") of the free nilpotent Lie algebra with 2 generators of step 4 by polynomial vector fields in $\mathbb{R}^8$. For these models, we use respectively an algorithm due to Grayson and Grossman [12] and an original approach. In the symmetric model, a one-parameter group of symmetries leaving the initial point fixed is found.

In Sec. 3 we describe explicitly the product rule in the Lie group $G \cong \mathbb{R}^8$, construct a right-invariant frame on G corresponding naturally to the left-invariant frame given by X_1, X_2 and their iterated Lie brackets, compute the corresponding left-invariant and right-invariant Hamiltonians that are linear on fibers of T^*G, describe Casimir functions and co-adjoint orbits in the dual space L^* of the Lie algebra L.

In Sec. 4 we apply Pontryagin maximum principle to the (2,3,5,8)-problem: we describe abnormal extremals and derive a Hamiltonian system $\dot{\lambda} = \vec{H}(\lambda)$, $\lambda \in T^*G$, for normal extremals.

In Sec. 5 we study integrability of the normal Hamiltonian field $\vec{H}$. We compute 10 independent integrals of $\vec{H}$, of which only 7 are in involution. After reduction by 4 Casimir functions, the vertical subsystem of $\vec{H}$ on L^* shows numerically a chaotic dynamics, which leads to a conjecture on non-integrability of $\vec{H}$.

In Sec. 6 we suggest possible questions for further study.

2 Realisation by polynomial vector fields in $\mathbb{R}^8$

In this section we construct two models of the free nilpotent Lie algebra L(1)–(4) by polynomial vector fields in $\mathbb{R}^8$.

2.1 Free nilpotent Lie algebras

Let $\mathcal{L}_d$ be the real free Lie algebra with d generators [10]; $\mathcal{L}_d$ is the Lie algebra of commutators of d variables. We have $\mathcal{L}_d = \oplus_{i=1}^{\infty} \mathcal{L}_d^i$, where $\mathcal{L}_d^i$ is the space of commutator polynomials of degree i. Then $\mathcal{L}_d^{(r)} := \mathcal{L}_d / \oplus_{i=r+1}^{\infty} \mathcal{L}_d^i$ is the free nilpotent Lie algebra with d generators of step r.

Denote $l_d(i) := \dim \mathcal{L}_d^i$, $l_d^{(r)} := \dim \mathcal{L}_d^{(r)} = \sum_{i=1}^{r} l_d(i)$. The classical expression of $l_d(i)$ is $i l_d(i) = d^i - \sum_{j|i,\ 1 \le j < i} j l_d(j)$.

In this work we are interested in free nilpotent Lie algebras with 2 generators. Dimensions of such Lie algebras for small step are given in Table 1.

i	1	2	3	4	5	6	7	8	9	10
$l_2(i)$	2	1	2	3	6	9	18	30	56	99
$l_2^{(i)}$	2	3	5	8	14	23	41	71	127	226

Table 1: Dimensions of free nilpotent Lie algebras $\mathcal{L}_2^{(i)}$

2.2 Carnot algebras and groups

A Lie algebra L is called a Carnot algebra if it admits a decomposition $L = \oplus_{i=1}^{r} L_i$ as a vector space, such that $[L_i, L_j] \subset L_{i+j}$, $L_s = 0$ for $s > r$, $L_{i+1} = [L_1, L_i]$.

A free nilpotent Lie algebra $\mathcal{L}_d^{(r)}$ is a Carnot algebra with the homogeneous components $L_i = \mathcal{L}_d^i$.

A Carnot group G is a connected, simply connected Lie group whose Lie algebra L is a Carnot algebra. If L is realized as the Lie algebra of left-invariant vector fields on G, then the degree 1 component L_1 can be thought of as a completely nonholonomic (bracket-generating) distribution on G. If moreover L_1 is endowed with a left-invariant inner product g, then (G, L_1, g) becomes a nilpotent left-invariant sub-Riemannian manifold [7]. Such sub-Riemannian structures are nilpotent approximations of generic sub-Riemannian structures [2, 5, 13, 20].

The sequence of numbers

$$(\dim L_1, \dim L_1 + \dim L_2, \dots, \dim L_1 + \cdots + \dim L_r = \dim L)$$

is called the growth vector of the distribution L_1 [30].

For free nilpotent Lie algebras, the growth vector is maximal compared with all Carnot algebras with the bidimension $(\dim L_1, \dim L)$.

2.3 Lie algebra with the growth vector $(2, 3, 5, 8)$

The Carnot algebra with the growth vector (2, 3, 5, 8)

$$\mathcal{L}_2^{(4)} = \operatorname{span}(X_1, \dots, X_8)$$

is determined by the following multiplication table:

$$[X_1, X_2] = X_3, \tag{8}$$

$$[X_1, X_3] = X_4, \quad [X_2, X_3] = X_5, \tag{9}$$

$$[X_1, X_4] = X_6, \quad [X_1, X_5] = [X_2, X_4] = X_7, \quad [X_2, X_5] = X_8, \tag{10}$$

with all the rest brackets equal to zero. This multiplication table is depicted at Fig. 1.

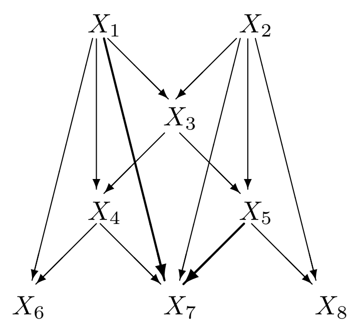

Figure 1: Lie algebra with the growth vector $(2, 3, 5, 8)$

2.4 Hall basis

Free nilpotent Lie algebras have a convenient basis introduced by M. Hall [14]. We describe it using the exposition of [12].

The Hall basis of the free Lie algebra $\mathcal{L}_d$ with d generators $X_1, \ldots, X_d$ is the subset $\mathrm{Hall} \subset \mathcal{L}_d$ that has a decomposition into homogeneous components $\mathrm{Hall} = \cup_{i=1}^{\infty} \mathrm{Hall}_i$ defined as follows.

Each element H_j, $j = 1, 2, \ldots$, of the Hall basis is a monomial in the generators X_i and is defined recursively as follows. The generators satisfy the inclusion $X_i \in \mathrm{Hall}_1$, $i = 1, \ldots, d$, and we denote $H_i = X_i$, $i = 1, \ldots, d$. If we have defined basis elements $H_1, \ldots, H_{N_{p-1}} \in \oplus_{j=1}^{p-1} \mathrm{Hall}_j$, they are simply ordered so that $E < F$ if $E \in \mathrm{Hall}_k$, $F \in \mathrm{Hall}_l$, $k < l$: $H_1 < H_2 < \cdots < H_{N_{p-1}}$. Also if $E \in \mathrm{Hall}_s$, $F \in \mathrm{Hall}_t$ and $p = s + t$, then $[E, F] \in \mathrm{Hall}_p$ if:

1. $E > F$, and

2. if $E = [G, K]$, then $K \in \mathrm{Hall}_q$ and $t \geq q$.

By this definition, one easily computes recursively the first components Hall_i

of the Hall basis for $d = 2$:

$$
\begin{aligned}
&\mathrm{Hall}_1 = \{H_1, H_2\}, \qquad H_1 = X_1, \quad H_2 = X_2,\\
&\mathrm{Hall}_2 = \{H_3\}, \qquad H_3 = [X_2, X_1],\\
&\mathrm{Hall}_3 = \{H_4, H_5\}, \qquad H_4 = [[X_2, X_1], X_1], \quad H_5 = [[X_2, X_1], X_2],\\
&\mathrm{Hall}_4 = \{H_6, H_7, H_8\},\\
&H_6 = [[[X_2, X_1], X_1], X_1],\ H_7 = [[[X_2, X_1], X_1], X_2],\ H_8 = [[[X_2, X_1], X_2], X_2].
\end{aligned}
$$

Consequently, $\mathcal{L}_2^{(4)} = \mathrm{span}\{H_1, \dots, H_8\}$. In the sequel we use a more convenient basis of $\mathcal{L}_2^{(4)} = \mathrm{span}\{X_1, \dots, X_8\}$ with the multiplication table (8)–(10).

2.5 Asymmetric vector field model for $\mathcal{L}_2^{(4)}$

Here we recall an algorithm for construction of a vector field model for the Lie algebra $\mathcal{L}_2^{(r)}$ due to Grayson and Grossman [12]. For a given $r \geq 1$, the algorithm evaluates two polynomial vector fields $H_1, H_2 \in \mathrm{Vec}(\mathbb{R}^N)$, $N = \dim \mathcal{L}_2^{(r)}$, which generate the Lie algebra $\mathcal{L}_2^{(r)}$.

Consider the Hall basis elements $\mathrm{span}\{H_1, \dots, H_N\} = \mathcal{L}_2^{(r)}$. Each element $H_i \in \mathrm{Hall}_j$ is a Lie bracket of length j:

$$
\begin{aligned}
&H_i = [\dots[[H_2, H_{k_j}], H_{k_{j-1}}], \dots, H_{k_1}],\\
&k_j = 1, \qquad k_{n+1} \leq k_n \text{ for } 1 \leq n \leq j-1.
\end{aligned}
$$

This defines a partial ordering of the basis elements. We say that H_i is a direct descendant of H_2 and of each H_{k_l} and write $i \succ 2$, $i \succ k_l$, $l = 1, \dots, j$.

Define monomials $P_{2,k}$ in $x_1, \dots, x_N$ inductively by

$$
P_{2,k} = -x_j \ P_{2,i}/(\deg_j P_{2,i} + 1),
$$

whenever $H_k = [H_i, H_j]$ is a basis Hall element, and where $\deg_j P$ is the highest power of x_j which divides P.

The following theorem gives the properties of the generators.

Theorem 1 (Th. 3.1 [12])**.** *Let $r \geq 1$ and let $N = \dim \mathcal{L}_2^{(r)}$. Then the vector fields $H_1 = \dfrac{\partial}{\partial x_1}$, $H_2 = \dfrac{\partial}{\partial x_2} + \sum_{i \succ 2} P_{2,i} \dfrac{\partial}{\partial x_i}$ have the following properties:*

1. *they are homogeneous of weight one with respect to the grading*

$$
\mathbb{R}^N = \mathrm{Hall}_1 \oplus \dots \oplus \mathrm{Hall}_r;
$$

2. $\mathrm{Lie}(H_1, H_2) = \mathcal{L}_2^{(r)}$.

The algorithm described before Theorem 1 produces the following vector field basis of $\mathcal{L}_2^{(4)}$:

$$
\begin{aligned}
H_1 &= \frac{\partial}{\partial x_1},\\
H_2 &= \frac{\partial}{\partial x_2} - x_1 \frac{\partial}{\partial x_3} - \frac{x_1^2}{2} \frac{\partial}{\partial x_4} - x_1 x_2 \frac{\partial}{\partial x_5} + \frac{x_1^3}{6} \frac{\partial}{\partial x_6} + \frac{x_1^2 x_2}{2} \frac{\partial}{\partial x_7} + \frac{x_1 x_2^2}{2} \frac{\partial}{\partial x_8},\\
H_3 &= \frac{\partial}{\partial x_3} + x_1 \frac{\partial}{\partial x_4} + x_2 \frac{\partial}{\partial x_5} - \frac{x_1^2}{2} \frac{\partial}{\partial x_6} - x_1 x_2 \frac{\partial}{\partial x_7} - \frac{x_2^2}{2} \frac{\partial}{\partial x_8},\\
H_4 &= -\frac{\partial}{\partial x_4} + x_1 \frac{\partial}{\partial x_6} + x_2 \frac{\partial}{\partial x_7},\\
H_5 &= -\frac{\partial}{\partial x_5} + x_1 \frac{\partial}{\partial x_7} + x_2 \frac{\partial}{\partial x_8},\\
H_6 &= -\frac{\partial}{\partial x_6},\\
H_7 &= -\frac{\partial}{\partial x_7},\\
H_8 &= -\frac{\partial}{\partial x_8},
\end{aligned}
$$

with the multiplication table

$$[H_2, H_1] = H_3, \tag{11}$$

$$[H_3, H_1] = H_4, \; [H_3, H_2] = H_5, \tag{12}$$

$$[H_4, H_1] = H_6, \; [H_4, H_2] = H_7, \; [H_5, H_2] = H_8. \tag{13}$$

2.6 Symmetric vector field model of $\mathcal{L}_2^{(4)}$

The vector field model of the Lie algebra $\mathcal{L}_2^{(4)}$ via the fields $H_1, \dots, H_8$ obtained in the previous subsection is asymmetric in the sense that there is no visible symmetry between the vector fields H_1 and H_2. Moreover, no continuous symmetries of the sub-Riemannian structure generated by the orthonormal frame $\{H_1, H_2\}$ are visible, although the Lie brackets (11)–(13) suggest that this sub-Riemannian structure should be preserved by a one-parameter group of rotations in the plane $\operatorname{span}\{H_1, H_2\}$.

One can find a symmetric vector field model of $\mathcal{L}_2^{(4)}$ free of such shortages as in the following statement.

Theorem 2. (1) *The vector fields*

$$X_1 = \frac{\partial}{\partial x_1} - \frac{x_2}{2}\frac{\partial}{\partial x_3} - \frac{x_1^2+x_2^2}{2}\frac{\partial}{\partial x_5} - \frac{x_1 x_2^2}{4}\frac{\partial}{\partial x_7} - \frac{x_2^3}{6}\frac{\partial}{\partial x_8}, \tag{14}$$

$$X_2 = \frac{\partial}{\partial x_2} + \frac{x_1}{2}\frac{\partial}{\partial x_3} + \frac{x_1^2+x_2^2}{2}\frac{\partial}{\partial x_4} + \frac{x_1^3}{6}\frac{\partial}{\partial x_6} + \frac{x_1^2 x_2}{4}\frac{\partial}{\partial x_7}, \tag{15}$$

$$X_3 = \frac{\partial}{\partial x_3} + x_1\frac{\partial}{\partial x_4} + x_2\frac{\partial}{\partial x_5} + \frac{x_1^2}{2}\frac{\partial}{\partial x_6} + x_1 x_2\frac{\partial}{\partial x_7} + \frac{x_2^2}{2}\frac{\partial}{\partial x_8}, \tag{16}$$

$$X_4 = \frac{\partial}{\partial x_4} + x_1\frac{\partial}{\partial x_6} + x_2\frac{\partial}{\partial x_7}, \tag{17}$$

$$X_5 = \frac{\partial}{\partial x_5} + x_1\frac{\partial}{\partial x_7} + x_2\frac{\partial}{\partial x_8}, \tag{18}$$

$$X_6 = \frac{\partial}{\partial x_6}, \tag{19}$$

$$X_7 = \frac{\partial}{\partial x_7}, \tag{20}$$

$$X_8 = \frac{\partial}{\partial x_8} \tag{21}$$

satisfy the multiplication table (8)–(10). *Thus the fields* $X_1, \dots, X_8 \in \mathrm{Vec}(\mathbb{R}^8)$ *model the Lie algebra* $\mathcal{L}_2^{(4)}$.

(2) *The vector field*

$$X_0 = x_2\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial x_2} + x_5\frac{\partial}{\partial x_4} - x_4\frac{\partial}{\partial x_5} + P\frac{\partial}{\partial x_6} + Q\frac{\partial}{\partial x_7} + R\frac{\partial}{\partial x_8}, \tag{22}$$

$$P = -\frac{x_1^4}{24} + \frac{x_1^2 x_2^2}{8} + x_7, \tag{23}$$

$$Q = \frac{x_1 x_2^3}{12} + \frac{x_1^3 x_2}{12} - 2x_6 + 2x_8, \tag{24}$$

$$R = \frac{x_1^2 x_2^2}{8} - \frac{x_2^4}{24} - x_7 \tag{25}$$

satisfies the following relations:

$$[X_0, X_1] = X_2, \qquad [X_0, X_2] = -X_1, \qquad [X_0, X_3] = 0, \tag{26}$$

$$[X_0, X_4] = X_5, \qquad [X_0, X_5] = -X_4, \tag{27}$$

$$[X_0, X_6] = 2X_7, \qquad [X_0, X_7] = X_8 - X_6, \qquad [X_0, X_8] = -2X_7. \tag{28}$$

Thus the field X_0 *is an infinitesimal symmetry of the sub-Riemannian structure generated by the orthonormal frame* $\{X_1, X_2\}$.

Proof. In fact, the both statements of the proposition are verified by the direct computation, but we prefer to describe a method of construction of the vector fields $X_1, \dots, X_8$, and X_0.

(1) In the previous work [21] we constructed a similar symmetric vector field model for the Lie algebra $\mathcal{L}_2^{(3)}$, which has growth vector (2, 3, 5):

$$\mathcal{L}_2^{(3)} = \mathrm{span}\{X_1, \dots, X_5\} \subset \mathrm{Vec}(\mathbb{R}^5), \tag{29}$$

$$X_1 = \frac{\partial}{\partial x_1} - \frac{x_2}{2}\frac{\partial}{\partial x_3} - \frac{x_1^2 + x_2^2}{2}\frac{\partial}{\partial x_5}, \tag{30}$$

$$X_2 = \frac{\partial}{\partial x_2} + \frac{x_1}{2}\frac{\partial}{\partial x_3} + \frac{x_1^2 + x_2^2}{2}\frac{\partial}{\partial x_4}, \tag{31}$$

$$X_3 = \frac{\partial}{\partial x_3} + x_1\frac{\partial}{\partial x_4} + x_2\frac{\partial}{\partial x_5}, \tag{32}$$

$$X_4 = \frac{\partial}{\partial x_4}, \tag{33}$$

$$X_5 = \frac{\partial}{\partial x_5}, \tag{34}$$

with the Lie brackets (8), (9). Now we aim to "continue" these relationships to vector fields $X_1, \dots, X_8 \in \mathrm{Vec}(\mathbb{R}^8)$ that span the Lie algebra $\mathcal{L}_2^{(4)}$. So we seek for vector fields of the form

$$X_1 = \frac{\partial}{\partial x_1} - \frac{x_2}{2}\frac{\partial}{\partial x_3} - \frac{x_1^2 + x_2^2}{2}\frac{\partial}{\partial x_5} + \sum_{i=6}^{8} a_1^i \frac{\partial}{\partial x_i}, \tag{35}$$

$$X_2 = \frac{\partial}{\partial x_2} + \frac{x_1}{2}\frac{\partial}{\partial x_3} - \frac{x_1^2 + x_2^2}{2}\frac{\partial}{\partial x_4} + \sum_{i=6}^{8} a_2^i \frac{\partial}{\partial x_i}, \tag{36}$$

$$X_3 = \frac{\partial}{\partial x_3} + x_1\frac{\partial}{\partial x_4} + x_2\frac{\partial}{\partial x_5} + \sum_{i=6}^{8} a_3^i \frac{\partial}{\partial x_i}, \tag{37}$$

$$X_4 = \frac{\partial}{\partial x_4} + \sum_{i=6}^{8} a_4^i \frac{\partial}{\partial x_i}, \tag{38}$$

$$X_5 = \frac{\partial}{\partial x_5} + \sum_{i=6}^{8} a_5^i \frac{\partial}{\partial x_i}, \tag{39}$$

$$X_j = \sum_{i=6}^{8} a_i^j \frac{\partial}{\partial x_j}, \qquad j = 6, 7, 8, \tag{40}$$

such that $\mathrm{span}\{X_1, \dots, X_8\} = \mathcal{L}_2^{(4)}$.

Compute the required Lie brackets:

$$
\begin{aligned}
[X_1, X_2] &= \frac{\partial}{\partial x_3} + x_1 \frac{\partial}{\partial x_4} + x_2 \frac{\partial}{\partial x_5} + \left(\frac{\partial a_2^6}{\partial x_1} - \frac{\partial a_1^6}{\partial x_2} \right) \frac{\partial}{\partial x_6} \\
&\quad + \left(\frac{\partial a_2^7}{\partial x_1} - \frac{\partial a_1^7}{\partial x_2} \right) \frac{\partial}{\partial x_7} + \left(\frac{\partial a_2^8}{\partial x_1} - \frac{\partial a_1^8}{\partial x_2} \right) \frac{\partial}{\partial x_8}, \\
[X_1, X_3] &= \frac{\partial}{\partial x_4} + \frac{\partial a_3^6}{\partial x_1} \frac{\partial}{\partial x_6} + \frac{\partial a_3^7}{\partial x_1} \frac{\partial}{\partial x_7} + \frac{\partial a_3^8}{\partial x_1} \frac{\partial}{\partial x_8}, \\
[X_2, X_3] &= \frac{\partial}{\partial x_5} + \frac{\partial a_3^6}{\partial x_2} \frac{\partial}{\partial x_6} + \frac{\partial a_3^7}{\partial x_2} \frac{\partial}{\partial x_7} + \frac{\partial a_3^8}{\partial x_2} \frac{\partial}{\partial x_8}, \\
[X_1, X_4] &= \frac{\partial a_4^6}{\partial x_1} \frac{\partial}{\partial x_6} + \frac{\partial a_4^7}{\partial x_1} \frac{\partial}{\partial x_7} + \frac{\partial a_4^8}{\partial x_1} \frac{\partial}{\partial x_8}, \\
[X_1, X_5] &= \frac{\partial a_5^6}{\partial x_1} \frac{\partial}{\partial x_6} + \frac{\partial a_5^7}{\partial x_1} \frac{\partial}{\partial x_7} + \frac{\partial a_5^8}{\partial x_1} \frac{\partial}{\partial x_8}, \\
[X_2, X_4] &= \frac{\partial a_4^6}{\partial x_2} \frac{\partial}{\partial x_6} + \frac{\partial a_4^7}{\partial x_2} \frac{\partial}{\partial x_7} + \frac{\partial a_4^8}{\partial x_2} \frac{\partial}{\partial x_8}, \\
[X_2, X_5] &= \frac{\partial a_5^6}{\partial x_2} \frac{\partial}{\partial x_6} + \frac{\partial a_5^7}{\partial x_2} \frac{\partial}{\partial x_7} + \frac{\partial a_5^8}{\partial x_2} \frac{\partial}{\partial x_8}.
\end{aligned}
$$

The vector fields $X_1, \ldots, X_8$ should be independent, thus the determinant constructed of these vectors as columns should satisfy the inequality

$$
D = \det(X_1, \ldots, X_8) = \begin{vmatrix} a_6^6 & a_7^6 & a_8^6 \\ a_6^7 & a_7^7 & a_8^7 \\ a_6^8 & a_7^8 & a_8^8 \end{vmatrix} \neq 0.
$$

We will choose a_i^j such that $D = 1$. It follows from the multiplication table for $X_1, \ldots, X_8$ that

$$
D = \begin{vmatrix} \dfrac{d^2 a_3^6}{dx_1^2} & \dfrac{d^2 a_3^6}{dx_1 dx_2} & \dfrac{d^2 a_3^6}{dx_2^2} \\ \dfrac{d^2 a_3^7}{dx_1^2} & \dfrac{d^2 a_3^7}{dx_1 dx_2} & \dfrac{d^2 a_3^7}{dx_2^2} \\ \dfrac{d^2 a_3^8}{dx_1^2} & \dfrac{d^2 a_3^8}{dx_1 dx_2} & \dfrac{d^2 a_3^8}{dx_2^2} \end{vmatrix}.
$$

In order to get $D = 1$, define the entries of this matrix in the following symmetric way: $a_3^6 = \dfrac{x_1^2}{2}$, $a_3^7 = x_1 x_2$, $a_3^8 = \dfrac{x_2^2}{2}$. Then we obtain from the multiplication table for $X_1, \ldots, X_8$ that $\dfrac{\partial a_2^6}{\partial x_1} - \dfrac{\partial a_1^6}{\partial x_2} = a_3^6 = \dfrac{x_1^2}{2}$, $\dfrac{\partial a_2^7}{\partial x_1} - \dfrac{\partial a_1^7}{\partial x_2} = a_3^7 = x_1 x_2$, $\dfrac{\partial a_2^8}{\partial x_1} - \dfrac{\partial a_1^8}{\partial x_2} = a_3^8 = \dfrac{x_2^2}{2}$. We solve these equations in the following symmetric way: $a_1^6 = 0$, $a_2^6 = \dfrac{x_1^3}{6}$, $a_1^7 = -\dfrac{x_1 x_2^2}{4}$, $a_2^7 = \dfrac{x_1^2 x_2}{4}$, $a_1^8 = -\dfrac{x_2^3}{6}$, $a_2^8 = 0$. Then we substitute these coefficients to (35), (36) and check item (1) of this theorem by direct computation.

Now we prove item (2). We proceed exactly as for item (1): we start from an infinitesimal symmetry [21]

$$X_0 = x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2} + x_5 \frac{\partial}{\partial x_4} - x_4 \frac{\partial}{\partial x_5} \in \operatorname{Vec}(\mathbb{R}^5) \tag{41}$$

of the sub-Riemannian structure on $\mathbb{R}^5$ determined by the orthonormal frame (30), (31) and "continue" symmetry (41) to the sub-Riemannian structure on $\mathbb{R}^8$ determined by the orthonormal frame (14), (15).

So we seek for a vector field $X_0 \in \operatorname{Vec}(\mathbb{R}^8)$ of the form (22) for the functions $P, Q, R \in C^\infty(\mathbb{R}^8)$ to be determined so that the multiplication table (26)–(28) hold.

The first two equalities in (26) yield $X_1 P = -\frac{x_1^3}{6}$, $X_2 P = \frac{x_1^2 x_2}{2}$. Further, $X_3 P = [X_1, X_2] P = X_1 X_2 P - X_2 X_1 P = X_1 \frac{x_1^2 x_2}{2} + X_2 \frac{x_1^3}{6} = x_1 x_2$. Similarly it follows that $X_4 P = x_2$, $X_5 P = x_1$, $X_6 P = 0$, $X_7 P = 1$, $X_8 P = 0$. Since $X_6 P = X_8 P = 0$, then $P = P(x_1, x_2, x_3, x_4, x_5, x_7)$. Moreover, since $X_7 P = 1$, then $P = x_7 + a(x_1, x_2, x_3, x_4, x_5)$. The equality $X_5 P = x_1$ implies that $\frac{\partial a}{\partial x_5} = 0$, i.e., $a = a(x_1, x_2, x_3, x_4)$. Similarly, since $X_4 P = x_2$, then $a = a(x_1, x_2, x_3)$. It follows from the equality $X_3 P = x_1 x_2$ that $\frac{\partial a}{\partial x_3} = x_1 x_2$, i.e., $a = x_1 x_2 x_3 + b(x_1, x_2)$. Moreover, the equality $X_2 P = \frac{x_1^2 x_2}{2}$ implies that $\frac{\partial b}{\partial x_2} = -x_1 x_3 - \frac{x_1^2 x_2}{4}$, i.e., $b = -x_1 x_2 x_3 - \frac{x_1^2 x_2^2}{8} + c(x_1)$. Finally, the equality $X_1 P = -\frac{x_1^3}{2}$ implies that $\frac{d c}{d x_1} = -\frac{x_1^3}{6} + \frac{x_1 x_2^2}{2}$ i.e., $c = -\frac{x_1^4}{24} + \frac{x_1^2 x_2^2}{4}$. Thus equality (23) follows. Similarly we get equalities (24), (25).

Then multiplication table (26)–(28) for the vector field (22)–(25) is verified by a direct computation. □

3 Carnot group

In this section we study the Carnot group G with the Lie algebra $L = \mathcal{L}_2^{(4)}$.

3.1 Product rule in G

In this subsection we compute the product rule in the connected simply connected Lie group G with the Lie algebra $L = \mathcal{L}_2^{(4)}$ on which the vector fields $X_1, \ldots, X_8$ given by (14)–(21) are left-invariant.

Our algorithm for computation of the product rule in a Lie group G with a known left-invariant frame $X_1, \ldots, X_n \in \operatorname{Vec}(G)$ follows from the next argument. Let $g_1, g_2 \in G$, and let $g_2 = e^{t_n X_n} \circ \ldots \circ e^{t_1 X_1}(\mathrm{Id})$, $t_1, \ldots, t_n \in \mathbb{R}$, where we denote by $e^{tX} : G \to G$ the flow of the vector field X. Then $g_1 \cdot g_2 = g_1 \cdot e^{t_n X_n} \circ \ldots \circ e^{t_1 X_1}(\mathrm{Id}) = e^{t_n X_n} \circ \ldots \circ e^{t_1 X_1}(g_1)$ by left-invariance of X_i. So an algorithm for computation of $g_1 \cdot g_2$ is the following:

1. Compute $e^{t_i X_i}(g), \quad t_i \in \mathbb{R},\ g \in G$.
2. Compute $e^{t_n X_n} \circ \ldots \circ e^{t_1 X_1}(g), \quad t_i \in \mathbb{R},\ g \in G$.
3. Solve the equation $e^{t_n X_n} \circ \ldots \circ e^{t_1 X_1}(\mathrm{Id}) = g_2$ for $t_1, \ldots, t_n \in \mathbb{R}$ (we assume that this is possible in a unique way).
4. Compute $g_1 \cdot g_2 = e^{t_n X_n} \circ \ldots \circ e^{t_1 X_1}(g_2)$.

By this algorithm, we compute the product $z = x \cdot y$ in the coordinates on G (notice that as a manifold $G = \mathbb{R}^8$), as follows:

$$
\begin{aligned}
& x = (x_1, \ldots, x_8),\ y = (y_1, \ldots, y_8),\ z = (z_1, \ldots, z_8) \in G = \mathbb{R}^8, \\
& z_1 = x_1 + y_1, \\
& z_2 = x_2 + y_2, \\
& z_3 = x_3 + y_3 + \frac{1}{2}(x_1 y_2 - x_2 y_1), \\
& z_4 = x_4 + y_4 + \frac{1}{2}(x_1(x_1 + y_1) + x_2(x_2 + y_2) + x_1 y_3), \\
& z_5 = x_5 + y_5 - \frac{1}{2} y_1(x_1(x_1 + y_1) + x_2(x_2 + y_2)) + x_2 y_3, \\
& z_6 = x_6 + y_6 + \frac{x_1}{12}(2x_1^2 y_2 + 3x_1 y_1 y_2 - 2y_2^3 + 6x_1 y_3 + 12 y_4), \\
& z_7 = x_7 + y_7 + \frac{1}{24}(3x_1^2 y_2(2x_2 + y_2) - x_2(3x_2 y_1^2 + 6y_1^2 y_2 + 4(y_2^3 - 6y^4)) \\
& \qquad + x_1(-6x_2^2 y_1 + 4y_1^3 + 6y_1 y_2^2 + 24 x_2 y_3 + 24 y_5)), \\
& z_8 = x_8 + y_8 + \frac{x_2}{2}(-2x_2^2 y_1 + 2y_1^3 - 3x_2 y_1 y_2 + 6x_2 y_3 + 12 y_5).
\end{aligned}
$$

3.2 Right-invariant frame on G

Computation of the right-invariant frame on G corresponding to a left-invariant frame can be done via the following simple lemma. Denote the inversion on a Lie group G as $i : G \to G$, $i(g) = g^{-1}$.

Lemma 1. *Let $X_1, X_2, X_3 \in \mathrm{Vec}(G)$ and $Y_1, Y_2, Y_3 \in \mathrm{Vec}(G)$ be respectively left-invariant and right-invariant vector fields on a Lie group G such that $Y_j(\mathrm{Id}) = -X_j(\mathrm{Id})$, $j = 1, 2, 3$. Then*

$$
i_* X_j = Y_j, \quad i = 1, 2, 3, \tag{42}
$$

$$
[X_1, X_2] = X_3 \quad \Leftrightarrow \quad [Y_1, Y_2] = Y_3. \tag{43}
$$

Proof. Equality (42) follows by the left-invariance and right-invariance of the fields X_i and Y_i respectively. Equality (43) follows since the diffeomorphism $i : G \to G$ preserves Lie bracket of vector fields (see e.g. [1]). □

Thus if $X_1, \dots, X_n \in \operatorname{Vec} G$ is a left-invariant frame on a Lie group G, then $Y_1, \dots, Y_n \in \operatorname{Vec} G$, $Y_j = i_* X_j$, is the right-invariant frame such that $Y_j(\mathrm{Id}) = -X_j(\mathrm{Id})$, $j = 1, \dots, n$, and the same product rules as for $X_1, \dots, X_n$.

Immediate computation using the product rule in G given in Subsec. 3.1 gives the following right-invariant frame on the Lie group $G = \mathbb{R}^8$:

$$
\begin{aligned}
Y_1 &= -\frac{\partial}{\partial x_1} - \frac{x_2}{2}\frac{\partial}{\partial x_3} - \frac{x_1x_2 + 2x_3}{2}\frac{\partial}{\partial x_4} + \frac{x_1^2}{2}\frac{\partial}{\partial x_5} \\
&\quad + \frac{x_2^3 - 6x_4}{6}\frac{\partial}{\partial x_6} - \frac{2x_1^3 + 3x_1x_2^2 + 12x_5}{12}\frac{\partial}{\partial x_7}, \\
Y_2 &= -\frac{\partial}{\partial x_2} - \frac{x_1}{2}\frac{\partial}{\partial x_3} - \frac{x_2^2}{2}\frac{\partial}{\partial x_4} + \frac{x_1x_2 - 2x_3}{2}\frac{\partial}{\partial x_5} \\
&\quad + \frac{3x_1^2x_2 + 2x_2^3 - 12x_4}{12}\frac{\partial}{\partial x_6} - \frac{x_1^3 + 6x_5}{6}\frac{\partial}{\partial x_8}, \\
Y_i &= -\frac{\partial}{\partial x_i}, \qquad i = 3, \dots, 8.
\end{aligned}
$$

3.3 Left-invariant and right-invariant Hamiltonians on T^*G

Using the expressions for the left-invariant and right-invariant frames given in Subsec. 2.6 and Subsec. 3.2, we define the corresponding left-invariant and right-invariant Hamiltonians, linear on fibers in T^*G:

$$
h_i(\lambda) = \langle \lambda, X_i \rangle, \qquad g_i(\lambda) = \langle \lambda, Y_i \rangle \qquad \lambda \in T^*G, \quad i = 1, \dots, 8.
$$

In the canonical coordinates $(x_1, \dots, x_8, \psi_1, \dots, \psi_8)$ on T^*G [1] we have the following:

$$
\begin{aligned}
h_1 &= \psi_1 - \frac{x_2}{2}\psi_3 - \frac{x_1^2 + x_2^2}{2}\psi_5 - \frac{x_1x_2^2}{4}\psi_7 - \frac{x_2^3}{6}\psi_8, \\
h_2 &= \psi_2 + \frac{x_1}{2}\psi_3 + \frac{x_1^2 + x_2^2}{2}\psi_4 + \frac{x_1^3}{6}\psi_6 + \frac{x_1^2x_2}{4}\psi_7, \\
h_3 &= \psi_3 + x_1\psi_4 + x_2\psi_5 + \frac{x_1^2}{2}\psi_6 + x_1x_2\psi_7 + \frac{x_2^2}{2}\psi_8, \\
h_4 &= \psi_4 + x_1\psi_6 + x_2\psi_7, \\
h_5 &= \psi_5 + x_1\psi_7 + x_2\psi_8, \\
h_i &= \psi_i, \qquad i = 6, 7, 8,
\end{aligned}
$$

and

$$
\begin{aligned}
g_1 &= -\psi_1 - \frac{x_2}{2}\psi_3 - \frac{x_1x_2+2x_3}{2}\psi_4 + \frac{x_1^2}{2}\psi_5 \\
&\quad + \frac{x_2^3-6x_4}{6}\psi_6 - \frac{2x_1^3+3x_1x^2+12x_5}{12}\psi_7, \qquad (44)\\
g_2 &= -\psi_2 - \frac{x_1}{2}\psi_3 - \frac{x_2^2}{2}\psi_4 + \frac{x_1x_2-2x_3}{2}\psi_5 \\
&\quad + \frac{3x_1^2x_2+2x_2^3-12x_4}{12}\psi_6 - \frac{x_1^3+6x_5}{6}\psi_8, \qquad (45)\\
g_i &= -\psi_i, \qquad i = 3,\dots,8. \qquad (46)
\end{aligned}
$$

3.4 Casimir functions on L^*

In this subsection we compute Casimir functions on the dual space L^* to the Lie algebra $L = \mathcal{L}_2^{(4)}$, i.e., the smooth functions

$$f : L^* \to \mathbb{R} \text{ such that } \{f, h_i\} = 0, \qquad i = 1,\dots,8.$$

Simultaneously we characterize orbits of the co-adjoint action of the Lie group G on L^*

$$\{\mathrm{Ad}^*_{q^{-1}}(h) \mid q \in G\}. \qquad (47)$$

Theorem 3. *The functions*

$$h_6, \quad h_7, \quad h_8, \quad C = h_5^2h_6 - 2h_4h_5h_7 + h_4^2h_8 - 2h_3(h_6h_8 - h_7^2) \qquad (48)$$

are Casimir functions on L^*, $L = \mathcal{L}_2^{(4)}$.

If $h_6h_8 - h_7^2 \neq 0$, then these functions are independent, and any Casimir function depends functionally of them.

Proof. For all $i = 6,7,8$, $j = 1,\dots,8$, we have $[X_i, X_j] = 0$, thus $\{h_i, h_j\} = 0$. The equality $\{C, h_j\} = 0$ for $j = 1,\dots,8$ is verified immediately. Thus h_6, h_7, h_8, C are Casimir functions. Now we prove that there are no other Casimir functions on L^*.

Let $f \in C^\infty(L^*)$ be a Casimir function, then

$$
\begin{aligned}
\{f, h_1\} &= -h_3\frac{\partial f}{\partial h_2} - h_4\frac{\partial f}{\partial h_3} - h_6\frac{\partial f}{\partial h_4} - h_7\frac{\partial f}{\partial h_5} = 0, \qquad (49)\\
\{f, h_2\} &= h_3\frac{\partial f}{\partial h_1} - h_5\frac{\partial f}{\partial h_3} - h_7\frac{\partial f}{\partial h_4} - h_8\frac{\partial f}{\partial h_5} = 0, \qquad (50)\\
\{f, h_3\} &= h_4\frac{\partial f}{\partial h_1} + h_5\frac{\partial f}{\partial h_2} = 0, \qquad (51)\\
\{f, h_4\} &= h_6\frac{\partial f}{\partial h_1} + h_7\frac{\partial f}{\partial h_2} = 0, \qquad (52)\\
\{f, h_5\} &= h_7\frac{\partial f}{\partial h_1} + h_8\frac{\partial f}{\partial h_2} = 0. \qquad (53)
\end{aligned}
$$

These equalities are conveniently rewritten in terms of the following vector fields $V_i \in \operatorname{Vec} L^*$:

$$V_1 = -h_3 \frac{\partial}{\partial h_2} - h_4 \frac{\partial}{\partial h_3} - h_6 \frac{\partial}{\partial h_4} - h_7 \frac{\partial}{\partial h_5}, \tag{54}$$

$$V_2 = h_3 \frac{\partial}{\partial h_1} - h_5 \frac{\partial}{\partial h_3} - h_7 \frac{\partial}{\partial h_4} - h_8 \frac{\partial}{\partial h_5}, \tag{55}$$

$$V_3 = h_4 \frac{\partial}{\partial h_1} + h_5 \frac{\partial}{\partial h_2}, \tag{56}$$

$$V_4 = h_6 \frac{\partial}{\partial h_1} + h_7 \frac{\partial}{\partial h_2}, \tag{57}$$

$$V_5 = h_7 \frac{\partial}{\partial h_1} + h_8 \frac{\partial}{\partial h_2}. \tag{58}$$

Namely, equalities (49)–(53) have the form $V_i f = 0$, $i = 1, \dots, 5$.

The vector fields V_i, $i = 1, \dots, 5$, form a Lie algebra with the product table $[V_1, V_2] = -V_3$, $[V_1, V_3] = -V_4$, $[V_2, V_3] = -V_5$. Denote for any $h \in L^*$ by O_h the orbit of the fields $V_1, \dots, V_5$ passing through the point h [1]. It is easy to see that O_h is the orbit (47) of the co-adjoint action of the Lie group G on L^* [16, 18].

By the Orbit Theorem [1], O_h is an immersed submanifold of L^* of dimension

$$\dim O_h = \dim \operatorname{Lie}_h(V_1, \dots, V_5) = \dim \operatorname{span}(V_1(h), \dots, V_5(h)) = \operatorname{rank} J(h),$$

where

$$J(h) = (V_1, \dots, V_5) = \begin{pmatrix} 0 & h_3 & h_4 & h_6 & h_7 \\ -h_3 & 0 & h_5 & h_7 & h_8 \\ -h_4 & -h_5 & 0 & 0 & 0 \\ -h_6 & -h_7 & 0 & 0 & 0 \\ -h_7 & -h_8 & 0 & 0 & 0 \end{pmatrix}. \tag{59}$$

Further, since O_h is a co-adjoint orbit, it is a symplectic, thus even-dimensional manifold, i.e., $\dim O_h \in \{0, 2, 4\}$.

Denote $\Delta = h_6 h_8 - h_7^2$, and let $\Delta \neq 0$. Since

$$\det \begin{pmatrix} 0 & h_3 & h_6 & h_7 \\ -h_3 & 0 & h_7 & h_8 \\ -h_6 & -h_7 & 0 & 0 \\ -h_7 & -h_8 & 0 & 0 \end{pmatrix} = -\Delta^2 \neq 0, \tag{60}$$

then $\operatorname{rank} J(h) = \dim O_h = 4$. We have

$$O_h \subset \{h' \in L^* | C(h') = C(h),\ h_i(h') = h_i(h),\ i = 6, 7, 8\}. \tag{61}$$

The subset in the right-hand side of inclusion (61) is arcwise connected, thus this inclusion is in fact an equality. In greater detail:

$$O_h = \mathbb{R}^2_{h'_1, h'_2} \times Q, \tag{62}$$

$$Q = \left\{(h'_3, h'_4, h'_5) \in \mathbb{R}^3 | h'_3 = \left(h_6 \left(h'_5\right)^2 - 2h_7 h'_4 h'_5 + h_8 \left(h'_4\right)^2 - C\right) / (2\Delta)\right\}. \tag{63}$$

If $\Delta > 0$, then Q is an elliptic paraboloid; and if $\Delta < 0$, then Q is a hyperbolic paraboloid.

So in the case $\Delta \neq 0$ the orbits O_h are common level sets of the functions (48). Any Casimir function is constant on the orbits O_h, thus it depends functionally on the functions (48). □

The next description of co-adjoint orbits follows from the previous proof.

Corollary 1. *let $h \in L^*$. Denote $\Delta = h_6h_8 - h_7^2$, $\Delta_1 = h_5h_7 - h_4h_8$, $\Delta_2 = h_5h_6 - h_4h_7$.*

(1) *The co-adjoint orbit $\{\mathrm{Ad}^*_{q^{-1}}(h) \mid q \in G\}$ coincides with the orbit O_h of vector fields* (54)–(58) *through the point h.*

(2) *The orbits O_h have the following dimensions:*

(2.1) $\Delta^2 + \Delta_1^2 + \Delta_2^2 \neq 0 \quad \Rightarrow \quad \dim O_h = 4,$

(2.2) $\Delta^2 + \Delta_1^2 + \Delta_2^2 = 0, \ h_3^2 + \cdots + h_8^2 \neq 0 \quad \Rightarrow \quad \dim O_h = 2,$

(2.3) $h_3^2 + \cdots + h_8^2 = 0 \quad \Rightarrow \quad \dim O_h = 0.$

(3) *If $\Delta \neq 0$, then the orbit O_h is described explicitly as* (62), (63).

In Subsec. 5.3 we consider the restriction of the vertical part of the Hamiltonian vector field $\vec{H}$ to the orbit O_h, $\Delta \neq 0$.

4 Pontryagin maximum principle

In this section we apply a necessary optimality condition — Pontryagin Maximum Principle (PMP) [1,9] to the sub-Riemannian problem (5)–(7) and derive ODEs for the geodesics of this problem. To this end introduce the Hamiltonian of PMP

$$h_u^\nu(\lambda) = u_1h_1(\lambda) + u_2h_2(\lambda) + \frac{\nu}{2}(u_1^2 + u_2^2),$$
$$\lambda \in T^*G, \qquad u \in \mathbb{R}^2, \qquad \nu \in \mathbb{R}.$$

Theorem 4 (PMP, [1])**.** *Let $q(t)$, $t \in [0, t_1]$, be a SR minimizer corresponding to a control $u(t)$, $t \in [0, t_1]$. Then there exists a Lipschitzian curve $\lambda(t) \in T^*G$, $t \in [0, t_1]$, $\pi(\lambda(t)) = q(t)$, and a number $\nu \in \{-1, 0\}$ such that the following conditions hold:*

1. *the Hamiltonian system of PMP*

$$\dot{\lambda}(t) = \overrightarrow{h}^{\nu}_{u(t)}(\lambda(t)) \quad a.e.\ t \in [0, t_1], \tag{64}$$

2. *the maximality condition $h^\nu_{u(t)}(\lambda(t)) = \max_{v \in \mathbb{R}^2} h^\nu_v(\lambda(t))$, $t \in [0, t_1]$,*

3. *and the nontriviality condition $(\lambda(t), \nu) \neq (0, 0)$, $t \in [0, t_1]$.*

In view of the product rule (8)–(10), the Hamiltonian system (64) reads in the parametrization $T^*G \ni \lambda = (h_1, \dots, h_8, q)$ as follows:

$$\begin{aligned}
\dot h_1 &= -u_2 h_3,\\
\dot h_2 &= u_1 h_3,\\
\dot h_3 &= u_1 h_4 + u_2 h_5,\\
\dot h_4 &= u_1 h_6 + u_2 h_7,\\
\dot h_5 &= u_1 h_7 + u_2 h_8,\\
\dot h_6 &= \dot h_7 = \dot h_8 = 0,\\
\dot q &= u_1 X_1 + u_2 X_2.
\end{aligned}$$

In the next subsections we specialize the conditions of PMP for the abnormal ($\nu = 0$) and normal ($\nu = -1$) cases.

4.1 Abnormal case

Let $\nu = 0$. Then the maximality condition $h_u^0(\lambda) = u_1 h_1(\lambda) + u_2 h_2(\lambda) \to \max\limits_{u \in \mathbb{R}^2}$ yields the identities along abnormal extremals: $h_1(\lambda) = h_2(\lambda) = 0$. Then $0 = \dot h_1 = -u_2 h_3$ and $0 = \dot h_2 = u_1 h_3$. Since any minimizer can be reparametrized to have a constant velocity ($u_1^2 + u_2^2 \equiv \text{const}$), we have $u_1^2 + u_2^2 \neq 0$ along non–constant trajectory, thus abnormal extremals satisfy one more identity: $h_3(\lambda) = 0$. Then $0 = \dot h_3 = u_1 h_4 + u_2 h_5$, thus $(u_1(t), u_2(t)) = k(t)(-h_5(t), h_4(t))$ along abnormal extremals. After reparametrization of time we get the abnormal controls $u_1 = -h_5$, $u_2 = h_4$. Summing up, abnormal extremals $\lambda(t)$ are described as follows.

Proposition 1. *Abnormal extremals of the* $(2, 3, 5, 8)$ *sub-Riemannian problem* (5)–(7) *are reparameterizations of curves* $\lambda(t) \in T^*G$ *that satisfy the conditions*

$$\begin{aligned}
&h_1(\lambda(t)) = h_2(\lambda(t)) = h_3(\lambda(t)) = 0,\\
&\begin{pmatrix} \dot h_4 \\ \dot h_5 \end{pmatrix} = D \begin{pmatrix} h_4 \\ h_5 \end{pmatrix}, \qquad D = \begin{pmatrix} h_7 & -h_6 \\ h_8 & -h_7 \end{pmatrix},\\
&\dot h_6 = \dot h_7 = \dot h_8 = 0,\\
&\dot q = -h_5 X_1 + h_4 X_2.
\end{aligned} \tag{65}$$

We have $\operatorname{tr} D = 0$, $\Delta = \det D = h_6 h_8 - h_7^2$, and the following cases are possible:

(1) $\Delta < 0$, then system (65) has the saddle phase portrait,

(2) $\Delta > 0$, then system (65) has the center phase portrait,

(3) $\Delta = 0$, $D \neq 0$, then the phase portrait of (65) consists of lines and fixed points,

(4) $D = 0$, then the phase portrait of (65) consists of fixed points.

Thus follows that abnormal extremals are analytic (this is related to the famous open question on smoothness of sub-Riemannian minimizers [7, 8]).

One can show that projections of abnormal extremal trajectories to the plane $\mathbb{R}^2_{x_1x_2}$ in these cases are respectively the following:

(1) hyperbolas, their separatrices, and center,

(2) homothetic ellipses and their center,

(3) parabolas,

(4) fixed points.

Trajectories that project to hyperbolas and parabolas are strictly abnormal (i.e., abnormal trajectories that are not normal trajectories [1, 29]). Moreover, one can parameterize the abnormal variety, i.e., the submanifold of G filled by abnormal trajectories [8]. These results will appear in a forthcoming work [28].

4.2 Normal case

Let $\nu = -1$. Then the maximality condition $h_u^{-1}(\lambda) = u_1h_1(\lambda) + u_2h_2(\lambda) - \frac{1}{2}\left(u_1^2 + u_2^2\right) \to \max\limits_{u\in\mathbb{R}^2}$ yields the normal controls $u_1 = h_1$, $u_2 = h_2$. Thus the normal extremals are trajectories of the Hamiltonian system

$$\dot{\lambda} = \overrightarrow{H}(\lambda), \quad \lambda \in T^*G, \tag{66}$$

with the normal Hamiltonian

$$H = \frac{1}{2}\left(h_1^2 + h_2^2\right). \tag{67}$$

In the parametrization $T^*G \ni \lambda = (h_1, \ldots, h_8, q)$, system (66) reads as follows:

$$\dot{h}_1 = -h_2h_3, \tag{68}$$
$$\dot{h}_2 = h_1h_3, \tag{69}$$
$$\dot{h}_3 = h_1h_4 + h_2h_5, \tag{70}$$
$$\dot{h}_4 = h_1h_6 + h_2h_7, \tag{71}$$
$$\dot{h}_5 = h_1h_7 + h_2h_8, \tag{72}$$
$$\dot{h}_6 = \dot{h}_7 = \dot{h}_8 = 0, \tag{73}$$
$$\dot{q} = h_1X_1 + h_2X_2.$$

5 Integrability of the normal Hamiltonian field

In this section we study integrability of the Hamiltonian field $\vec{H}$. We compute 10 independent integrals of $\vec{H}$, of which only 7 are in involution. Recall that for the Liouville integrability of the Hamiltonian system $\dot{\lambda} = \vec{H}(\lambda)$ with 8 degrees of freedom we need 8 independent integrals in involution [4]. After reduction by Casimir functions (48), the vertical subsystem of $\vec{H}$ shows numerically a chaotic dynamics, which leads to Conjecture 1 below on non-integrability of $\vec{H}$.

5.1 Algebra of integrals of $\vec{H}$

The normal Hamiltonian system $\dot{\lambda} = \vec{H}(\lambda)$ reads in the canonical coordinates $(\psi_1, \dots, \psi_8; x_1, \dots x_8)$ on T^*G as follows:

$$\dot{\psi}_1 = h_1 \left(x_1\psi_5 + \frac{x_2^2}{2}\psi_7 \right) - h_2 \left(\frac{1}{2}\psi_3 + x_1\psi_4 + \frac{x_1^2}{2}\psi_6 + \frac{x_1 x_2}{2}\psi_7 \right),$$

$$\dot{\psi}_2 = h_1 \left(\frac{1}{2}\psi_3 + x_2\psi_5 + \frac{x_1 x_2}{2}\psi_7 + \frac{x_2^2}{2}\psi_8 \right) - h_2 \left(x_2\psi_4 + \frac{x_1^2}{2}\psi_7 \right),$$

$$\dot{\psi}_i = 0, \qquad i = 3, \dots, 8,$$

$$\dot{q} = h_1 X_1(q) + h_2 X_2(q),$$

$$h_1 = \psi_1 - \frac{x_2}{2}\psi_3 - \frac{x_1^2 + x_2^2}{2}\psi_5 - \frac{x_1 x_2^2}{4}\psi_7 - \frac{x_2^3}{6}\psi_8, \qquad (74)$$

$$h_2 = \psi_2 + \frac{x_1}{2}\psi_3 + \frac{x_1^2 + x_2^2}{2}\psi_4 + \frac{x_1^3}{6}\psi_6 + \frac{x_1^2 x_2}{4}\psi_7. \qquad (75)$$

In view of results of Secs. 2, 3, the Hamiltonian field $\vec{H}$ has the following integrals:

- the system Hamiltonian H (67),
- right-invariant Hamiltonians $g_1, \dots, g_8$ (44)–(46),
- the Hamiltonian of rotation $h_0(\lambda) = \langle \lambda, X_0 \rangle$ (22),
- Casimir functions h_6, h_7, h_8, C (48),
- the cyclic variables $\psi_3, \dots, \psi_8$ of the Hamiltonian H (67), (74), (75).

Of these integrals, only 10 are functionally independent, thus we get an algebra of integrals

$$I = \operatorname{span}(H, g_1, \dots, g_8, h_0) \qquad (76)$$

with the nonzero brackets

$$\{h_0, g_4\} = g_5, \qquad \{h_0, g_5\} = -g_4, \qquad (77)$$

$$\{h_0, g_6\} = 2g_7, \qquad \{h_0, g_7\} = g_8 - g_6, \qquad \{h_0, g_8\} = -2g_7. \qquad (78)$$

So we have an Abelaian algebra generated by 7 independent integrals:

$$A = \operatorname{span}(H, g_3, \dots, g_8). \tag{79}$$

We proved the following statement.

Theorem 5. *The normal Hamiltonian vector field $\vec{H}$ has an algebra I (76)–(78) of 10 independent integrals, and an Abelian algebra A (79) of 7 independent integrals.*

Thus there lacks just one integral commuting with the integrals in A in order to have Liouville integrability of $\vec{H}$.

5.2 Homogeneous integrals of $\vec{H}$

A natural source of integrals of $\vec{H}$ are homogeneous polynomials in the momenta h_i: $P_k = P_k(h_1, \dots, h_8)$, $\deg P_k = k$. Although, for $k = 1, 2, 3$ we get no new integrals in this way, i.e., P_1, P_2, P_3 are expressed through the Casimir functions and the Hamiltonian H.

Theorem 6. *Let a homogeneous polynomial $P_k(h_1, \dots, h_8)$ be an integral of the field $\vec{H}$. Then:*

(1) $P_1 = \sum_{i=6}^{8} a_i h_i, \qquad a_i \in \mathbb{R},$

(2) $P_2 = \sum_{i,j=6}^{8} a_{ij} h_i h_j + bH, \qquad a_{ij},\ b \in \mathbb{R},$

(3) $P_3 = \sum_{i,j,l=6}^{8} a_{ijl} h_i h_j h_l + H \sum_{i=6}^{8} b_i h_i + aC, \qquad a_{ijl},\ b_i,\ a \in \mathbb{R}.$

Proof. (1) Let $P_1 = \sum_{i=1}^{8} a_i h_i$, $a_i \in \mathbb{R}$, be an integral of $\vec{H}$, then

$$\begin{aligned} 0 = \{H, P_1\} = {} & -a_1 h_1 h_3 + a_2 h_1 h_3 + a_3(h_1 h_4 + h_2 h_5) + a_4(h_1 h_6 + h_2 h_7) \\ & + a_5(h_1 h_7 + h_2 h_8), \end{aligned}$$

thus $a_1 = \dots = a_5 = 0$, so $P_1 = \sum_{i=6}^{8} a_i h_i$.

Statements (2) and (3) are proved similarly. □

In addition to attempts to prove Liouville integrability of $\vec{H}$, we tried also to apply noncommutative integrability theory [19], but failed.

On the other hand, in the next subsection we present a numerical evidence of chaotic dynamics for the (reduction of) the Hamiltonian field $\vec{H}$, which suggests thet this field is not Liouville integrable.

5.3 Reduction of the vertical subsystem

The Hamiltonian field $\vec{H}$ on T^*G has a vertical part $\vec{H}_{\text{vert}}$ defined on L^* as follows (see e.g. [1]):

$$\vec{H}_{\text{vert}}(\lambda) = (\operatorname{ad} dH)^* \lambda, \qquad \lambda \in L^*.$$

In the coordinates $(h_1, \ldots, h_8)$ on L^*, the ODE $\dot\lambda = \vec{H}_{\mathrm{vert}}(\lambda)$ reads just as equations (68)–(73).

For any $p = (h_6^0, h_7^0, h_8^0, C^0) \in \mathbb{R}^4$, consider the common level surface of the Casimir functions (48)

$$O_p = \{\lambda \in L^* \mid h_i(\lambda) = h_i^0,\ i = 6, 7, 8,\ C(\lambda) = C^0\}.$$

By Corollary 1, in the generic case $\Delta^0 = h_6^0 h_8^0 - (h_7^0)^2 \neq 0$, the level set O_p is an orbit of co-adjoint action of the Lie group G on L^*, it is 4-dimensional, and is parameterized by the coordinates (h_1, h_2, h_4, h_5) as (62), (63). In these coordinates, the restriction of the vertical subsystem $\dot\lambda = \vec{H}_{\mathrm{vert}}(\lambda)$ to O_p reads as follows:

$$\begin{aligned}
&\dot h_1 = -h_2\ h_3(h_4, h_5),\\
&\dot h_2 = h_1\ h_3(h_4, h_5),\\
&\dot h_4 = h_1 h_6^0 + h_2 h_7^0,\\
&\dot h_5 = h_1 h_7^0 + h_2 h_8^0,\\
&h_3(h_4, h_5) = (h_8^0 h_4^2 - 2h_7^0 h_4 h_5 + h_6^0 h_5^2 - C^0)/(2\Delta^0).
\end{aligned}$$

Restriction of this system to the level surface $\{H = 1/2\}$ gives, in the coordinates

$$h_1 = \cos\theta, \quad h_2 = \sin\theta, \quad h_3 = c,$$
$$h_4 = a, \quad h_5 = b, \quad h_6 = m, \quad h_7 = p, \quad h_8 = n,$$

the following 3 equations:

$$\dot\theta = (2pab - na^2 - mb^2)/(2\Delta) + k, \tag{80}$$
$$\dot a = m\cos\theta + p\sin\theta, \tag{81}$$
$$\dot b = p\cos\theta + n\sin\theta, \qquad m,\ n,\ p,\ k = \mathrm{const}\,. \tag{82}$$

If $\theta(t)$ is increasing (or decreasing), then system (80)–(82) defines a Poincarè mapping

$$\begin{aligned}
&P : \mathbb{R}^2 \to \mathbb{R}^2, \qquad P(a, b) = (a', b'),\\
&(\theta(t), a(t), b(t))|_{t=0} = (0, a, b),\\
&(\theta(t), a(t), b(t))|_{t=T>0} = (2\pi, a', b').
\end{aligned}$$

We computed numerically the orbits $\{P^i(a, b) \mid i \in \mathbb{N}\}$, and for various values of the parameters (m, n, p, k) and initial points (a, b), we get regular or chaotic bahaviour, see Figs. 2–7. This numeric evidence leads to the following

Conjecture 1. (1) *The Hamiltonian vector field $\vec{H}$ is not Liouville integrable on T^*G.*

(2) *There exist symplectic submanifolds $S \subset T^*G$, $0 < \dim S < \dim T^*G$, such that $\vec{H}$ is Liouville integrable on S.*

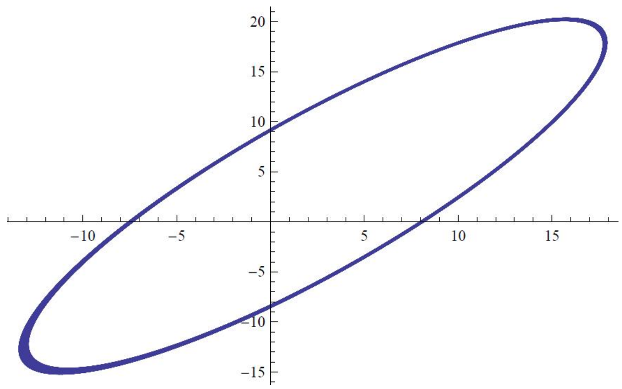

Figure 2: Regular orbit of Poincaré map ($5 \cdot 10^5$ points)

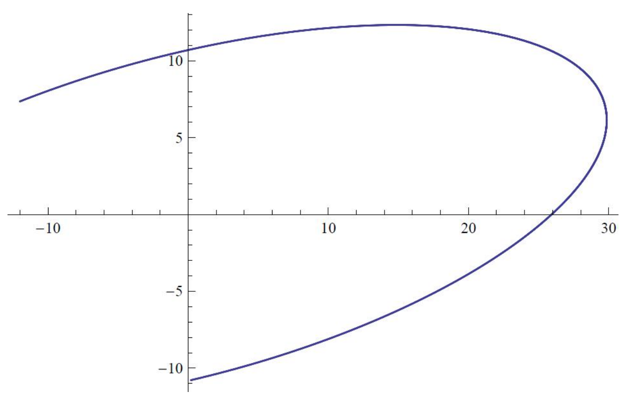

Figure 3: Regular orbit of Poincaré map ($5 \cdot 10^5$ points)

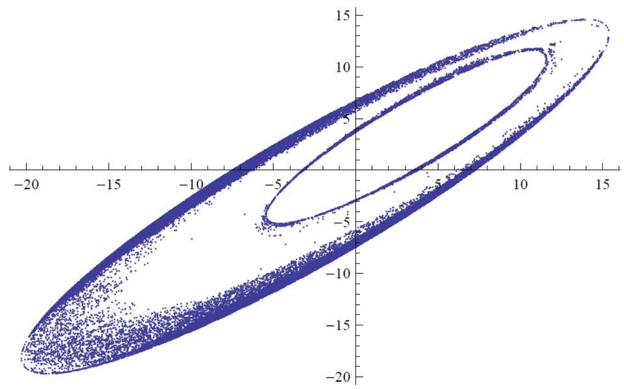

Figure 4: Chaotic orbit of Poincaré map ($5 \cdot 10^6$ points)

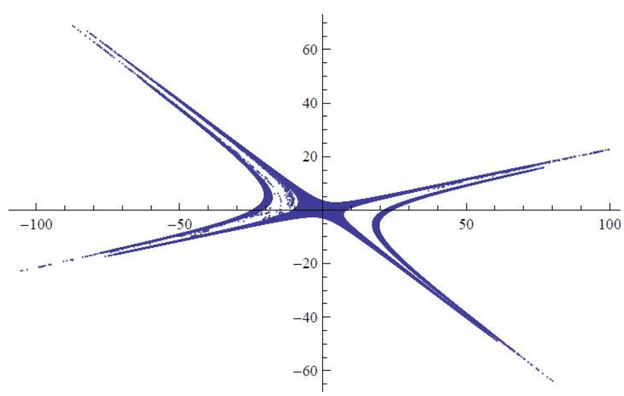

Figure 5: Chaotic orbit of Poincaré map ($5 \cdot 10^5$ points)

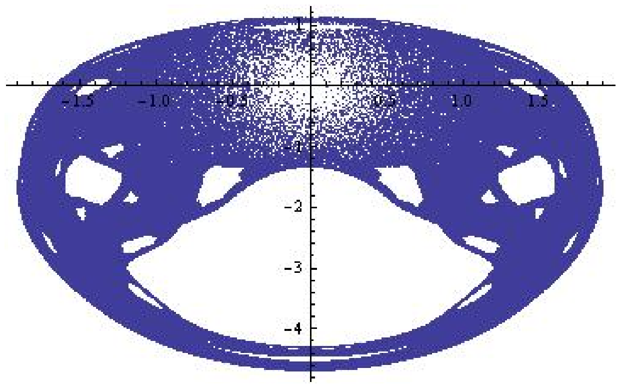

Figure 6: Chaotic orbit of Poincaré map ($5 \cdot 10^5$ points)

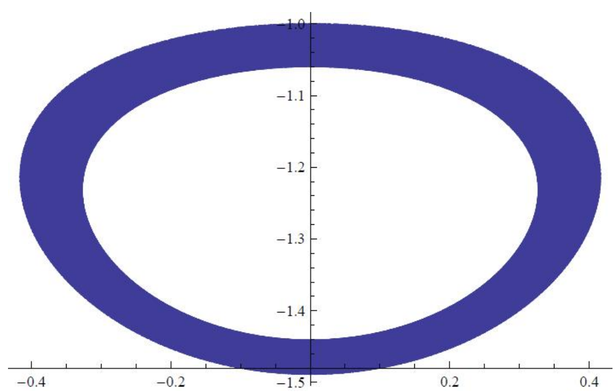

Figure 7: Chaotic orbit of Poincaré map ($5 \cdot 10^5$ points)

5.4 Lower-dimensional projections

For special initial values of $\lambda \in L^*$, projections of normal geodesics $q(t)$ of the $(2,3,5,8)$-problem to certain subspaces of the state space $\mathbb{R}^8$ yield geodesics of lower-dimensional sub-Riemannian problems since there is an obvious nested chain of nilpotent SR problems on Carnot groups, like Russian Matryoshka:

$$(2) \subset (2,3) \subset (2,3,5) \subset (2,3,5,8),$$

corresponding to the chain of subspaces:

$$\mathbb{R}^2_{x_1 x_2} \subset \mathbb{R}^3_{x_1 x_2 x_3} \subset \mathbb{R}^5_{x_1 \dots x_5} \subset \mathbb{R}^8_{x_1 \dots x_8}.$$

Multiplication table in the Heisenberg algebra (growth vector $(2,3)$) is

$$[X_1, X_2] = X_3, \tag{83}$$

and in the Cartan algebra (growth vector $(2,3,5)$) is

$$[X_1, X_2] = X_3, \quad [X_1, X_3] = X_4, \quad [X_2, X_3] = X_5. \tag{84}$$

Multiplication tables (83) and (84) are depicted resp. in Figs. 8 and 9 (compare with Fig. 1 for the (2,3,5,8) Carnot algebra).

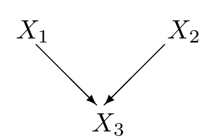

Figure 8: The Heisenberg algebra

If $h_3(\lambda) = \dots = h_8(\lambda) = 0$, then $(x_1(t), x_2(t))$ is a Riemannian geodesic in the Euclidean plane $\mathbb{R}^2_{x_1 x_2}$, i.e., a straight line.

If $h_4(\lambda) = \dots = h_8(\lambda) = 0$, then $(x_1(t), x_2(t), x_3(t))$ is a sub-Riemannian geodesic in the Heisenberg group $\mathbb{R}^3_{x_1 x_2 x_3}$, thus the curve $(x_1(t), x_2(t))$ is a straight line or a circle [6, 30].

If $h_6(\lambda) = h_7(\lambda) = h_8(\lambda) = 0$, then $(x_1(t), \dots, x_5(t))$ is a sub-Riemannian geodesic in the Carnot group $\mathbb{R}^5_{x_1 \dots x_5}$, thus the curve $(x_1(t), x_2(t))$ is an Euler elastica — a stationary configuration of elastic rod in the plane [11, 17, 21–27], see the plots of elasticae for various values of elastic energy at Figs. 10–13.

For generic $\lambda \in L^*$, the curves $(x_1(t), x_2(t))$ look like "elasticae of variable elastic energy", see Figs. 14, 15.

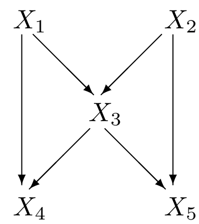

Figure 9: The Cartan algebra

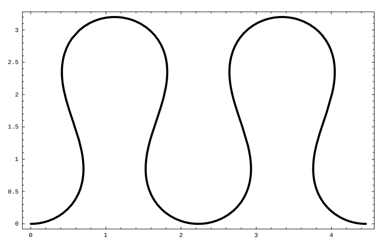

Figure 10: Inflexional elastica

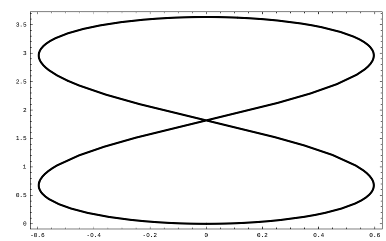

Figure 11: Inflexional elastica

There is an obvious relation of optimality of trajectories of the (2,3,5,8)-problem and its lower-dimensional projections due to the following simple statement.

Proposition 2 ([3]). *Consider two optimal control problems:*

$$\begin{aligned} &\dot{q}^i = f^i(q^i, u), \quad q^i \in M^i, \quad u \in U, \\ &q^i(0) = q^i_0, \quad q^i(t_1) = q^i_1, \\ &J = \int_0^{t_1} \varphi(u)\, dt \to \min, \\ &i = 1, 2. \end{aligned}$$

Suppose that there exists a smooth map $G : M^1 \to M^2$, s. t. if $q^1(t)$ is the trajectory of the first system corresponding to a control $u(t)$, then $q^2(t) = G(q^1(t))$ is the trajectory of the second system with the same control.

Further assume that $q^1(t)$ and $q^2(t)$ are such trajectories. If $q^2(t)$ is locally (globally) optimal for the second problem, then $q^1(t)$ is locally (globally) optimal for the first problem.

This proposition provides lower bounds for the cut time

$$t_{\text{cut}}(\lambda) = \sup\{t > 0 \mid \pi \circ e^{s\vec{H}}(\lambda) \text{ is globally optimal for } s \in [0, t]\}$$

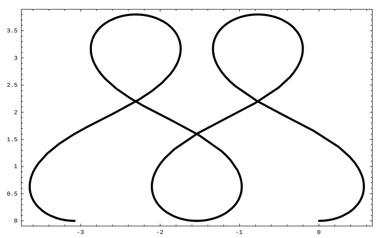

Figure 12: Inflexional elastica

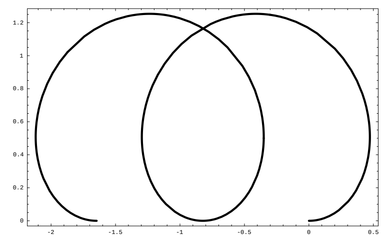

Figure 13: Non-inflexional elastica

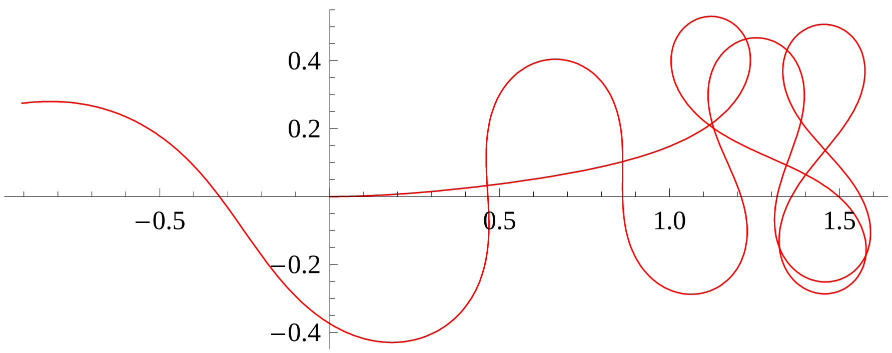

Figure 14: Elastica of variable elastic energy

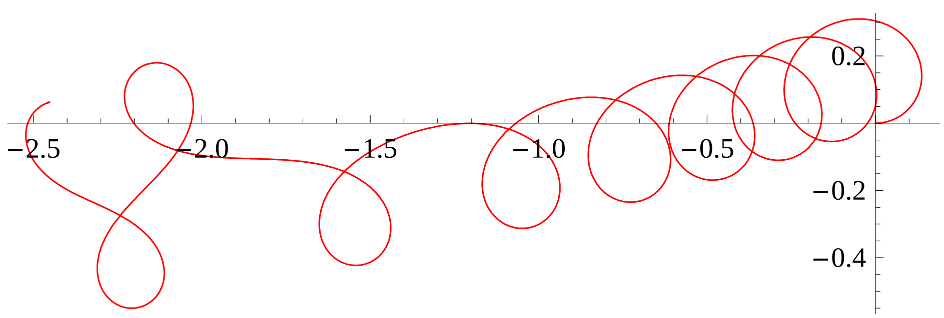

Figure 15: Elastica of variable elastic energy

and the first conjugate time

$$t^1_{\text{conj}}(\lambda) = \sup \left\{ t > 0 \mid \pi \circ e^{s\vec{H}}(\lambda) \text{ is locally optimal for } s \in [0, t] \right\}$$

of the (2,3,5,8)-problem in terms of the same functions for its lower-dimensional projections.

For the Riemannian problem on the plane, the straight lines are optimal forever, so the cut and first conjugate times are $+\infty$, thus for the (2,3,5,8)-problem

$$h_3(\lambda) = \dots h_8(\lambda) = 0 \quad \Rightarrow \quad t_{\text{cut}}(\lambda) = t^1_{\text{conj}}(\lambda) = +\infty.$$

For the sub-Riemannian problem on the Heisenberg group, the circles are locally and globally optimal up to the first loop, thus for the (2,3,5,8)-problem

$$h_3(\lambda) \neq 0, \quad h_4(\lambda) = \dots = h_8(\lambda) = 0 \quad \Rightarrow \quad t^1_{\text{conj}}(\lambda) \geq t_{\text{cut}}(\lambda) \geq \frac{2\pi}{|h_3(\lambda)|}.$$

Similar, but much more complicated bounds hold for the case $h_6(\lambda) = h_7(\lambda) = h_8(\lambda) = 0$ via comparison with the cut and first conjugate times for the sub-Riemannian problem on the Cartan group [21–24].

6 Conclusion

We see the following interesting questions for the (2,3,5,8)-problem:

1. study optimality of abnormal geodesics,

2. describe all cases where the normal Hamiltonian vector field $\vec{H}$ is Liouville intergable, integrate and study the corresponding normal geodesics,

3. describe precisely the chaotic dynamics of the normal Hamiltonian vector field $\vec{H}$.

We plan to address these questions in forthcoming works.

References

[1] A.A. Agrachev, Yu. L. Sachkov, *Geometric control theory*, Fizmatlit, Moscow 2004; English transl. *Control Theory from the Geometric Viewpoint*, Springer-Verlag, Berlin 2004.

[2] A. A. Agrachev, A. A. Sarychev, Filtration of a Lie algebra of vector fields and nilpotent approximation of control systems, *Dokl. Akad. Nauk SSSR*, **295** (1987), English transl. in *Soviet Math. Dokl.*, **36** (1988), 104–108.

[3] A.Ardentov, Yu. Sachkov, Conjugate points in nilpotent sub-Riemannian problem on the Engel group, *Journal of Mathematical Sciences*, Vol. 195, No. 3, December, 2013, 369–390.

[4] V.I. Arnold, *Mathematical Methods of Classical Mechanics*, Springer, 1997.

[5] A. Bellaïche, The tangent space in sub-Riemannian geometry, *in: Sub-Riemannian geometry*, vol. 144 of Progr. Math., Birkhäuser, Basel, 1996, pp. 1–78.

[6] R. Brockett, Control theory and singular Riemannian geometry, In: *New Directions in Applied Mathematics*, (P. Hilton and G. Young eds.), Springer-Verlag, New York, 11–27.

[7] R. Montgomery, *A Tour of Subriemannian Geometries, Their Geodesics and Applications*. American Mathematical Society (2002).

[8] R.Monti, The regularity problem for sub-Riemannian geodesics, *Geometric Control Theory and sub-Riemannian Geometry*, Springer, INdAM Series 5 (2013), G. Stefani, U. Boscain, J.-P. Gauthier, A. Sarychev, M. Sigalotti (eds.).

[9] L.S. Pontryagin, V.G. Boltyanskii, R.V. Gamkrelidze, E.F. Mishchenko, *The mathematical theory of optimal processes*, Wiley Interscience, 1962.

[10] Ch. Reutenauer, *Free Lie algebras*, London Mathematical Society Monographs. New Series, 7, The Clarendon Press Oxford University Press, 1993.

[11] L.Euler, *Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes, sive Solutio problematis isoperimitrici latissimo sensu accepti*, Lausanne, Geneva, 1744.

[12] M. Grayson, R. Grossman, Nilpotent Lie algebras and vector fields, *Symbolic Computation: Applications to Scientific Computing*, R.Grossman, Ed., SIAM, Philadelphia, 1989, pp. 77–96.

[13] M. Gromov, Carnot-Carathéodory spaces seen from within, *in: Sub-Riemannian geometry*, vol. 144 of Progr. Math., Birkhäuser, Basel, 1996, pp. 79—323.

[14] M. Hall, A basis for free Lie rings and higher commutators in free groups, *Proc. Amer. Math. Soc.*, 1 (1950), 575—581.

[15] H. Hermes, Nilpotent approximations of control systems and distributions, *SIAM J. Control Optim.*, 24 (1986), 731—736.

[16] A. A. Kirillov, *Lectures on the orbit method*, Graduate Studies in Mathematics 64, Providence, RI: American Mathematical Society, (2004).

[17] A.E.H.Love, *A Treatise on the Mathematical Theory of Elasticity*, 4th ed., New York: Dover, 1927.

[18] J. Marsden, T. Ratiu, *Introduction to Mechanics and Symmetry*, Springer, 1999.

[19] A. S. Mishchenko, A. T. Fomenko, Generalized Liouville method of integration of Hamiltonian systems, *Functional Analysis and Its Applications*, 12 (1978), No. 2, pp 113–121.

[20] J. Mitchell, On Carnot-Carathéodory metrics, *J. Differential Geom.*, 21 (1985), pp. 35—45.

[21] Yu. L. Sachkov, Exponential mapping in generalized Dido's problem, *Mat. Sbornik*, 194 (2003), 9: 63–90 (in Russian). English translation in: *Sbornik: Mathematics*, **194** (2003).

[22] Yu. L. Sachkov, Discrete symmetries in the generalized Dido problem (in Russian), *Matem. Sbornik*, **197** (2006), 2: 95–116. English translation in: *Sbornik: Mathematics*, **197** (2006), 2: 235–257.

[23] Yu. L. Sachkov, The Maxwell set in the generalized Dido problem (in Russian), *Matem. Sbornik*, **197** (2006), 4: 123–150. English translation in: *Sbornik: Mathematics*, **197** (2006), 4: 595–621.

[24] Yu. L. Sachkov, Complete description of the Maxwell strata in the generalized Dido problem (in Russian), *Matem. Sbornik*, **197** (2006), 6: 111–160. English translation in: *Sbornik: Mathematics*, **197** (2006), 6: 901–950.

[25] Yu. L. Sachkov, Maxwell strata in the Euler elastic problem. *J. Dynam. Control Systems* **14** (2008), No. 2, 169–234.

[26] Yu.Sachkov, Conjugate points in Euler's elastic problem, *Journal of Dynamical and Control Systems*, 2008 Vol. 14 (2008), No. 3 (July), 409–439.

[27] Yu.Sachkov, E. F. Sachkova, Exponential mapping in Euler's elastic problem, *Journal of Dynamical and Control Systems*, Vol. 21 (2014), in print.

[28] Yu.Sachkov, E. F. Sachkova, Abnormal extremals in the $(2,3,5,8)$ sub-Riemannian problem, *submitted*.

[29] H.Sussmann, W.Liu, Shortest parts for sub-Riemannian metrics on rank-2 distributions, *Memoirs of the American Mathematical Society*, No. 564, Vol. 118, November 1995.

[30] A.M. Vershik, V.Y. Gershkovich, Nonholonomic Dynamical Systems. Geometry of distributions and variational problems. (Russian) In: *Itogi Nauki i Tekhniki: Sovremennye Problemy Matematiki, Fundamental'nyje Napravleniya*, Vol. 16, VINITI, Moscow, 1987, 5–85. (English translation in: *Encyclopedia of Math. Sci.* **16**, Dynamical Systems 7, Springer Verlag.)