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Stability of Inflectional Elasticae Centered at Vertices or Inflection Points

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Abstract—Stability conditions for inflectional Euler's elasticae centered at vertices or inflection points are obtained. Theoretical results are compared with experimental data for elastic rods.

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1. INTRODUCTION

In 1744, Leonhard Euler published a solution to the following problem on stationary configurations of an elastic rod [12]. Given an elastic rod in a plane with prescribed positions of the endpoints and tangents at the endpoints, it is required to determine possible configurations of the rod under these boundary conditions. Euler derived differential equations for stationary configurations of the rod and described their possible qualitative types. These configurations are called Euler's elasticae.

Euler's elasticae are critical points of the elastic energy functional on the space of curves with fixed endpoints and tangents at the endpoints. The goal of this paper is to study the local optimality of elasticae: whether a critical point is a point of local minimum of the energy functional? We are interested in elasticae that provide a minimum value of the energy functional among sufficiently close curves that satisfy the boundary conditions (local optimality). This question has been little studied despite its obvious importance for mechanics and engineering applications.

The local optimality is essential for elasticity theory since it corresponds to the stability of Euler's elasticae under small perturbations that preserve the boundary conditions. In calculus of variations and optimal control, a point at which an extremal trajectory loses its local optimality is called a conjugate point. We provide a precise description of conjugate points confining our attention to elasticae whose center is a vertex, i.e., an extremum of the curvature, or an inflection point.

Euler derived a differential equation now known as the Euler–Lagrange equation for the corresponding optimal control problem and reduced this equation to quadratures. In the modern terms, Euler studied the qualitative behavior of Jacobian functions that parameterize the elastic curves via the qualitative study of ODEs that determine these curves. Euler described all possible types of elastic curves and found values of parameters for which these types can occur. Studying elastica infinitesimally close to a straight line, Euler derived the famous formula for the critical compressive load which causes an initially straight strut to buckle.

The elastica problem has long been of only theoretical interest and has been used as an example of application of elliptic functions (see, e.g., [22, 6]). With the advent of steel in the 19th century and widespread design of thin-wall structures, which stimulated the development of the stability theory of deformable systems, the solution of the elastica problem acquired practical significance. In particular, the following practically important issues arose: How does a strut behave under compressive loads higher than the Euler critical load? What is the postcritical configuration of the strut? Is this configuration unique and is it stable? These issues have been a subject of

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numerous publications [3, 8, 20, 14, 28, 42, 35, 39, 34, 40] (see also the references therein) in which equilibrium states of inextensible unshearable rods subject to various loading and boundary conditions have been studied. In the last decades, elasticae have attracted increasing interest in connection with application of the rod theories to the analysis of micro- and nanostructures in biology and nanotechnology [21, 41, 33, 23]. It is well known that a flexible rod can exhibit multiple equilibrium configurations under given terminal loads. It is therefore of interest to infer which of these configurations can occur in reality or, in other words, which of them are stable. However, the stability question of possible curvilinear states of the rod has received little attention.

Important mathematical results on stability of elasticae were obtained in 1906 by future Nobel prize winner Max Born in his PhD thesis "Stability of Elastic Curves in the Plane and Space" [15]. Born showed that an elastic arc without inflection points is stable. For the general case, he wrote down the Jacobian vanishing at conjugate points. Because of the complexity of the functions entering this Jacobian, Born restricted himself to numerical investigation of conjugate points. He was the first to plot elasticae on the basis of numerical computations and to check the correspondence between theory and experimental data for elastic rods.

The stability question of planar elasticae was discussed in [8, 9]. Maddocks [32] studied the stability of elasticae under dead (conservative) loads by examining the sign of the second variation of the energy functional. To this end, he used the nodal properties of various functions rather than exact analytical solutions governing the equilibrium of the rod. He was the first to find a secondary bifurcation of a rod under loads higher than the Euler load. In [25, 30], the case of a rod with clamped ends under compressive load was considered using the notion of a conjugate point in the calculus of variations.

To the authors' knowledge, the other theoretical results on the stability of curvilinear states have mainly been obtained with numerical methods by reducing a continuous elastic rod to discrete models, which makes it possible to apply the well-known criteria for determining the sign of the Hessian matrix of the energy of a system with a finite number of degrees of freedom. In [19, 36], the simplest case of a cantilevered beam subject to the tip load is considered. In [4, 5], the finite element method is used to determine postcritical equilibrium states, study the secondary bifurcation [2, 17], and investigate the stability in the postcritical range. In [27, 31], a semi-analytical method is proposed based on the series expansion of the second variation of the strain energy functional in terms of eigenfunctions of an auxiliary Sturm–Liouville problem. A review of the literature shows that the stability question for elasticae has been studied for terminate loads. Little attention has been given to the case of arbitrary kinematic boundary conditions. The reason is that serious mathematical difficulties arise in determining possible equilibrium states for this case. Some experimental results on elasticae which support the existence of multiple equilibria can be found in [23, 24]. An experimental–theoretical analysis of the elastica stability is performed in [18, 16].

In [38], two-sided bounds for the first conjugate point on elasticae were obtained and stability conditions in terms of inflection points were proved. These results are mentioned below in Section 3.

From the theoretical viewpoint, this study is a continuation of papers [37, 38]. It has the following structure. In Section 2, the elastic problem is stated as an optimal control problem, and the parameterization of elasticae by Jacobian functions obtained in [37] is recalled. In Section 3, we give some necessary results of [38] on conjugate points of elasticae with inflection points (these curves are referred to as inflectional elasticae). In Sections 4 and 5, we prove stability conditions for inflectional elasticae whose midpoint is their vertex or inflection point, respectively. We obtain expressions for the supremum of lengths of inflectional elasticae centered at a vertex or inflection point. In Section 6, theoretical results on the stability of elasticae are supported experimentally using specimens made of a thin celluloid film. A series of photos showing the stability or instability of stationary configurations of the elastic film is presented. In the final Section 7, we sum up the results obtained and list some open problems for further research.

2. OPTIMAL CONTROL PROBLEM AND INFLECTIONAL ELASTICAE

The problem on equilibrium configurations of an elastic inextensible rod γ under given kinematic conditions at its endpoints a_0 and a_1 is stated as the following optimal control problem [26, 37]:

$$\dot{x} = \cos\theta, \qquad \dot{y} = \sin\theta, \qquad \dot{\theta} = u, \tag{1}$$

$$q = (x, y, \theta) \in M = \mathbb{R}^2_{x,y} \times S^1_\theta, \qquad u \in \mathbb{R}, \tag{2}$$

$$q(0) = q_0 = (x_0, y_0, \theta_0), \qquad q(t_1) = q_1 = (x_1, y_1, \theta_1), \qquad t_1 \ \text{ fixed}, \tag{3}$$

$$J = \frac{1}{2}\int_0^{t_1} u^2(t)\,dt \to \min, \tag{4}$$

where x and y are the coordinates of a point of an elastica, θ is the angle between the tangent to the elastica and the x axis (see Fig. 1), t is the arclength, the dot denotes the derivative with respect to t, and the integral J is proportional to the bending energy of the elastica.

When studying the stability of configurations of a rod described by solutions to the variational problem (1)–(4), we use the terminology of optimal control theory [1, 7, 26]. In particular, in view of Kirchhoff's kinetic analogy between elasticae and a pendulum [6], the parameter t is called time.

Euler's problem has obvious symmetries: parallel translations and rotations of the two-dimensional plane; thus, it is a left-invariant problem on the group of rotations of the plane. In view of these symmetries, we assume below that the initial point is

$$q_0 = (x_0, y_0, \theta_0) = (0, 0, 0),$$

i.e., at $t = 0$ elasticae pass through the origin in the x direction.

The elasticae (the projections of extremal trajectories in problem (1)–(4)) are divided into two classes:

(i) inflectional elasticae (with inflection points),

(ii) non-inflectional elasticae (without inflection points).

In this paper, we confine our attention to inflectional elasticae (as mentioned above, all non-inflectional elasticae are stable [15]).

In [37], the following parameterization was obtained for inflectional elasticae:

$$\begin{aligned} x_t = {}& \frac{2}{\sqrt{r}}\operatorname{dn}^2(\sqrt{r}\varphi)\big(\mathrm{E}(\sqrt{r}\varphi_t) - \mathrm{E}(\sqrt{r}\varphi)\big) + \frac{4k^2}{\sqrt{r}}\operatorname{dn}(\sqrt{r}\varphi)\operatorname{sn}(\sqrt{r}\varphi)\big(\operatorname{cn}(\sqrt{r}\varphi) - \operatorname{cn}(\sqrt{r}\varphi_t)\big) \\ & + \frac{2k^2}{\sqrt{r}}\operatorname{sn}^2(\sqrt{r}\varphi)\big(\sqrt{r}t + \mathrm{E}(\sqrt{r}\varphi) - \mathrm{E}(\sqrt{r}\varphi_t)\big) - t, \end{aligned} \tag{5}$$

$$\begin{aligned} y_t = {}& \frac{2k}{\sqrt{r}}(2\operatorname{dn}^2(\sqrt{r}\varphi) - 1)(\operatorname{cn}(\sqrt{r}\varphi) - \operatorname{cn}(\sqrt{r}\varphi_t)) \\ & - \frac{2k}{\sqrt{r}}\operatorname{sn}(\sqrt{r}\varphi)\operatorname{dn}(\sqrt{r}\varphi)\big(2\big(\mathrm{E}(\sqrt{r}\varphi_t) - \mathrm{E}(\sqrt{r}\varphi)\big) - \sqrt{r}t\big), \end{aligned} \tag{6}$$

$$\sin\frac{\theta_t}{2} = k\operatorname{dn}(\sqrt{r}\varphi)\operatorname{sn}(\sqrt{r}\varphi_t) - k\operatorname{sn}(\sqrt{r}\varphi)\operatorname{dn}(\sqrt{r}\varphi_t), \tag{7}$$

$$\cos\frac{\theta_t}{2} = \operatorname{dn}(\sqrt{r}\varphi)\operatorname{dn}(\sqrt{r}\varphi_t) + k^2\operatorname{sn}(\sqrt{r}\varphi)\operatorname{sn}(\sqrt{r}\varphi_t), \tag{8}$$

where $\operatorname{cn}(p)$, $\operatorname{sn}(p)$, and $\operatorname{dn}(p)$ are the Jacobian elliptic functions, $\mathrm{E}(p)$ is the Jacobian epsilon function, $k \in (0,1)$ is the modulus of Jacobian functions, $F(p,k)$ and $E(p,k)$ are the elliptic

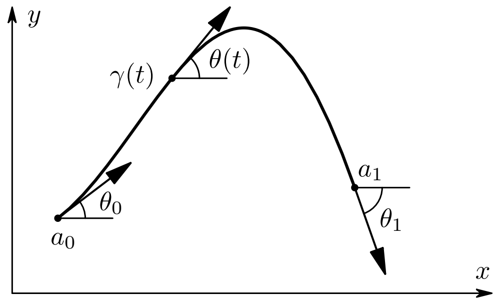


Fig. 1. Elastic arc and notation.

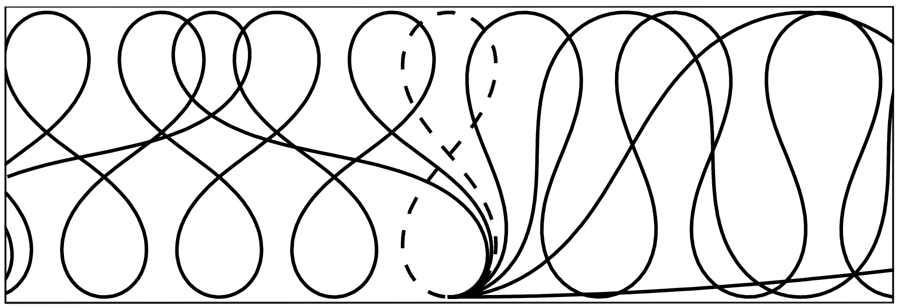


Fig. 2. Inflectional elasticae.

integrals of the first and second kinds, respectively, $K(k) = F(\pi/2, k)$ and $E(k) = E(\pi/2, k)$ are the complete elliptic integrals of the first and second kinds, respectively, [11, 29], and $\varphi_t = \varphi + t$.

The parameters φ, k, and r have the following meaning:

- the modulus k determines the shape of an elastica;
- the parameter r characterizes the magnitude of resulting reactive forces at the fixed endpoints of the rod;
- the ratio $\frac{k}{\sqrt{r}}$ determines the size of an elastica: the amplitude of an elastica (the maximum deviation from the straight line passing through inflection points) is equal to $\frac{2k}{\sqrt{r}}$;
- φ is the initial phase of an elastica.

Some inflectional elasticae of different shapes and the same amplitude are shown in Fig. 2. The dashed curve depicts the closed 8-shaped elastica corresponding to the modulus $k = k_0$ (see Lemma 3.1 below).

The family of inflectional elasticae is parameterized by the triples

$$\lambda = (\varphi, k, r) \in N_1 = \big\{(\varphi, k, r) \;\big|\; r > 0,\ k \in (0, 1),\ \sqrt{r}\varphi \pmod{4K(k)} \in [0, 4K(k)]\big\}.$$

Thus, for any $t \in \mathbb{R}$ one can define an exponential mapping

$$\mathrm{Exp}_t\colon\ \lambda = (\varphi, k, r) \mapsto q_t = (\theta_t, x_t, y_t), \qquad \lambda \in N_1, \quad q_t \in M,$$

that transforms a triple $\lambda = (\varphi, k, r)$ into the endpoint of the corresponding extremal trajectory projecting to the inflectional elastica.

3. CONJUGATE POINTS

Now we recall the results of [10, 37] related to the description of conjugate points in Euler's problem for inflectional elasticae. It is known that an instant t is a conjugate time if and only if the mapping Exp_t is degenerate, i.e., its Jacobian $J = \frac{\partial(x_t, y_t, \theta_t)}{\partial(\varphi, k, r)}$ vanishes. Direct computation based on the parameterization of extremal trajectories (5)–(8) yields

$$J = \frac{\partial(x_t, y_t, \theta_t)}{\partial(\varphi, k, r)} = -\frac{32k}{(1-k^2)r^{3/2}\Delta^2} J_1, \tag{9}$$

$$J_1 = a_0 + a_1 z + a_2 z^2, \qquad z = \operatorname{sn}^2 \tau \in [0,1], \tag{10}$$

$$a_2 = -k^2 \operatorname{sn} p \cdot x_1, \tag{11}$$

$$a_2 + a_1 + a_0 = (1-k^2) \operatorname{sn} p \cdot x_1, \tag{12}$$

$$a_0 = f_1(p,k) x_2, \tag{13}$$

$$\begin{aligned} x_1 = &- \operatorname{dn} p \Big(2 \operatorname{sn} p \operatorname{dn} p \operatorname{E}^3(p) + \big((4k^2-5)p \operatorname{sn} p \operatorname{dn} p + \operatorname{cn} p\,(3 - 6k^2 \operatorname{sn}^2 p)\big) \operatorname{E}^2(p) \\ &+ \big((4k^2-5)\operatorname{cn} p\,(1-2k^2 \operatorname{sn}^2 p)p + \operatorname{sn} p \operatorname{dn} p\,\big(4p^2 - 1 + k^2(6\operatorname{sn}^2 p - 4 - 4p^2)\big)\big) \operatorname{E}(p) \\ &+ p \operatorname{sn} p \operatorname{dn} p\,\big(1 - (1-k^2)p^2 + k^2(4k^2-5)\operatorname{sn}^2 p\big) \\ &+ 2 \operatorname{cn} p\,\big(k^2 \operatorname{sn}^2 p \operatorname{dn}^2 p + (1-k^2)(1-2k^2\operatorname{sn}^2 p)p^2\big)\Big), \end{aligned} \tag{14}$$

$$x_2 = \operatorname{cn} p\,\big(2(1-k^2)\operatorname{E}(p) - \operatorname{E}^2(p) - (1-k^2)p^2\big) + \operatorname{sn} p \operatorname{dn} p\,(\operatorname{E}(p) - (1-k^2)p), \tag{15}$$

$$f_1(p,k) = \operatorname{sn} p \operatorname{dn} p - (2\operatorname{E}(p) - p)\operatorname{cn} p, \tag{16}$$

$$p = \sqrt{r}t/2, \qquad \tau = \sqrt{r}(\varphi + t/2), \qquad \Delta = 1 - k^2 \operatorname{sn}^2 p \operatorname{sn}^2 \tau.$$

Lemma 3.1 [10]. *The equation*

$$2E(k) - K(k) = 0, \qquad k \in [0,1),$$

has a unique root $k_0 \in (0,1)$.

Numerical computations using the Mathematica system [43] yield the approximate value $k_0 = 0.9089085575485414782361189087 44\ldots$. The value $k = k_0$ of the modulus corresponds to the closed eight-shaped elastica shown by a dashed curve in Fig. 2 (see also Fig. 8a below).

Proposition 3.1 [10]. *For any* $k \in [0,1)$, *the function* $f_1(p,k)$ (16) *has a countable number of roots* $p = p_n^1$, $n \in \mathbb{Z}$. *These roots are odd in* n; *in particular,* $p_0^1 = 0$. *The roots* p_n^1 *are localized as follows*:

$$p_n^1 \in (-K + 2Kn, K + 2Kn), \qquad n \in \mathbb{Z}.$$

In particular, the roots p_n^1 *are monotone in* n. *Moreover, the following relations hold for* $n \in \mathbb{N}$:

$$\begin{aligned} k \in [0,k_0) \quad &\Rightarrow \quad p_n^1 \in (2Kn, K + 2Kn), \\ k = k_0 \quad &\Rightarrow \quad p_n^1 = 2Kn, \\ k \in (k_0,1) \quad &\Rightarrow \quad p_n^1 \in (-K + 2Kn, 2Kn), \end{aligned}$$

where k_0 *is the unique root of the equation* $2E(k) - K(k) = 0$ (*see Lemma* 3.1).

The following lemma describes the roots of the function x_2 (15) that enters decomposition (13) of the function a_0.

Lemma 3.2 [38]. *For any* $k \in (0,1)$, *the function* $x_2(p)$ (15) *has a countable number of roots* $p = p_n^{x_2} \geq 0$. *Further,* $p_0^{x_2} = 0$ *and* $p_n^{x_2} \in (2Kn, K + 2Kn)$ *for* $n \in \mathbb{N}$; *moreover,*

$$k < k_0 \quad \Rightarrow \quad p_n^{x_2} \in (p_n^1, K + 2Kn). \tag{17}$$

The following theorem gives bounds on the first conjugate time along inflectional elasticae:

$$t_1^{\mathrm{conj}}(\lambda) = \min\bigl\{t > 0 \bigm| t \text{ is a conjugate time along the trajectory } q(s) = \mathrm{Exp}_s(\lambda)\bigr\}.$$

Theorem 3.1 [38]**.** *Let $\lambda = (\varphi, k, r) \in N_1$. Then the number $t_1^{\mathrm{conj}}(\lambda)$ belongs to the segment bounded by the points $\frac{4K(k)}{\sqrt{r}}$ and $\frac{2p_1^1(k)}{\sqrt{r}}$; namely,*

$$k \in (0, k_0) \quad \Rightarrow \quad t_1^{\mathrm{conj}} \in \left[\frac{4K(k)}{\sqrt{r}}, \frac{2p_1^1(k)}{\sqrt{r}}\right],$$

$$k = k_0 \quad \Rightarrow \quad t_1^{\mathrm{conj}} = \frac{4K(k)}{\sqrt{r}} = \frac{2p_1^1(k)}{\sqrt{r}},$$

$$k \in (k_0, 1) \quad \Rightarrow \quad t_1^{\mathrm{conj}} \in \left[\frac{2p_1^1(k)}{\sqrt{r}}, \frac{4K(k)}{\sqrt{r}}\right].$$

A natural measure of time on extremal trajectories in Euler's problem is provided by the period of elasticae $T(k) = 4K(k)/\sqrt{r}$ (the period of oscillations of a pendulum, Kirchhoff's kinetic analogue of elasticae). In terms of the period T, the bounds of Theorem 3.1 are written as follows.

Corollary 3.1 [38]**.** *Let $\lambda \in N_1$. Then*

$$k \in (0, k_0) \quad \Rightarrow \quad t_1^{\mathrm{conj}} \in [T, t_1^1] \subset [T, 3T/2), \qquad t_1^1 = 2p_1^1/\sqrt{r} \in (T, 3T/2),$$

$$k = k_0 \quad \Rightarrow \quad t_1^{\mathrm{conj}} = T,$$

$$k \in (k_0, 1) \quad \Rightarrow \quad t_1^{\mathrm{conj}} \in [t_1^1, T] \subset (T/2, T], \qquad t_1^1 = 2p_1^1/\sqrt{r} \in (T/2, T).$$

It is instructive to express the local optimality conditions for elasticae in terms of their inflection points.

Corollary 3.2 [38]**.** *Let $\lambda \in N_1$, $q(s) = (x_s, y_s, \theta_s) = \mathrm{Exp}(\lambda_s)$, and let $\Gamma = \{\gamma_s = (x_s, y_s) \mid s \in [0, t]\}$ be the corresponding inflectional elastica.*

1. *If the arc Γ does not contain inflection points, then it is locally optimal.*
2. *If $k \in (0, k_0]$ and the arc Γ contains exactly one inflection point, then it is locally optimal.*
3. *If the arc Γ contains no less than three inflection points inside itself, then it is not locally optimal.*

The following statement describes conjugate points for elasticae centered at an inflection point or a vertex.

Corollary 3.3 [38]**.** *Let $\lambda \in N_1$, $q(s) = (x_s, y_s, \theta_s) = \mathrm{Exp}(\lambda_s)$, and let $\Gamma = \{\gamma_s = (x_s, y_s) \mid s \in [0, t]\}$ be the corresponding inflectional elastica.*

1. *If the elastica Γ is centered at a vertex, then the terminal instant t is a conjugate time if and only if*
$$p = \frac{\sqrt{r}t}{2} \in \{p_n^1 \mid n \in \mathbb{N}\} \cup \{p_m^{x_2} \mid m \in \mathbb{N}\}.$$
2. *If the elastica Γ is centered at an inflection point, then the terminal instant t is a conjugate time if and only if*
$$p = \frac{\sqrt{r}t}{2} \in \{2Kn \mid n \in \mathbb{N}\} \cup \{p_m^{x_1} \mid m \in \mathbb{N}\}.$$

Here the roots of the function $x_1(p)$ (14) are denoted by $p_m^{x_1}$, and those of the function $x_2(p)$ (15), by $p_m^{x_2}$ (see Lemma 3.2).

4. STABILITY OF ELASTICAE CENTERED AT A VERTEX

Here we obtain stability conditions for inflectional elasticae whose midpoint is a vertex:

$$\Gamma = \{(x_s, y_s) \mid s \in [0,t]\}, \qquad q(s) = (x_s, y_s, \theta_s) = \operatorname{Exp}_s(\lambda), \qquad \lambda = (\varphi, k, r) \in N_1, \tag{18}$$

$$\text{the point } (x_{t/2}, y_{t/2}) \text{ is a vertex of the elastica } \Gamma. \tag{19}$$

It is well known [37, 38] that condition (19) is equivalent to the equality

$$\operatorname{sn}(\tau, k) = 0, \qquad \tau = \sqrt{r}(\varphi + t/2).$$

Denote

$$t_1^1 = t_1^1(k,r) = \frac{2}{\sqrt{r}} p_1^1(k),$$

where p_1^1 is the number defined in Proposition 3.1.

Theorem 4.1. *Let an inflectional elastica* Γ (18) *be centered at a vertex.*

1. *If* $t < t_1^1$, *then the elastica* Γ *is stable.*
2. *If* $t = t_1^1$, *then the elastica* Γ *is critical*; *i.e., its endpoint is the first conjugate point.*
3. *If* $t > t_1^1$, *then the elastica* Γ *is unstable.*

Proof. Consider the comparison elastica

$$\widetilde{\Gamma} = \{(\widetilde{x}_s, \widetilde{y}_s) \mid s \in [0, \widetilde{t}\,]\}, \qquad \widetilde{q}(s) = (\widetilde{x}_s, \widetilde{y}_s, \widetilde{\theta}_s) = \operatorname{Exp}_s(\widetilde{\lambda}), \qquad \widetilde{\lambda} = (\widetilde{\varphi}, k, r) \in N_1,$$

where

$$\widetilde{t} = t_1^1, \qquad \widetilde{\varphi} = \varphi + (t - \widetilde{t}\,)/2.$$

Since the parameters (k,r) have the same values for the elasticae Γ and $\widetilde{\Gamma}$, these elasticae are finite arcs of the same infinite elastica, up to a motion of the plane.[1] The equality $\varphi + t/2 = \widetilde{\varphi} + \widetilde{t}/2$ means that the elastica $\widetilde{\Gamma}$, as well as Γ, is centered at a vertex. The elasticae $\widetilde{\Gamma}$ and Γ have the same vertex and are embedded one in the other: if $t \le \widetilde{t}$, then $\Gamma \subset \widetilde{\Gamma}$, and if $t \ge \widetilde{t}$, then $\Gamma \supset \widetilde{\Gamma}$.

By virtue of the equality $\widetilde{t} = t_1^1 = (2/\sqrt{r}) p_1^1$ and item 1 of Corollary 3.3, the instant $\widetilde{t}$ is a conjugate time for $\widetilde{q}(s)$.

1. Let $t < t_1^1$.

1.1. Let $k \in [k_0, 1)$. Then by Theorem 3.1 the inequality $t_1^{\mathrm{conj}}(\widetilde{\lambda}) \ge \widetilde{t}$ holds. Thus $t_1^{\mathrm{conj}}(\widetilde{\lambda}) = \widetilde{t}$. Since the trajectory $\widetilde{q}(s)$, $s \in [0, (\widetilde{t} + t)/2]$, does not contain conjugate points, it is locally optimal. In other words, the elastica $\{(\widetilde{x}_s, \widetilde{y}_s) \mid s \in [0, (\widetilde{t} + t)/2]\}$ is stable. Then the arc Γ contained in this elastica is stable as well.

1.2. Let $k \in (0, k_0)$. Consider an auxiliary continuous family of elasticae centered at the same vertex:

$$\begin{gathered}\Gamma^\alpha = \{(x_s^\alpha, y_s^\alpha) \mid s \in [0, t^\alpha]\}, \qquad q^\alpha(s) = (x_s^\alpha, y_s^\alpha, \theta_s^\alpha) = \operatorname{Exp}_s(\lambda^\alpha), \qquad \lambda^\alpha = (\varphi^\alpha, k, r) \in N_1, \\ t^\alpha = \alpha, \qquad \varphi^\alpha = \widetilde{\varphi} + (\widetilde{t} - t^\alpha)/2, \qquad \alpha \in (0, \widetilde{t}\,).\end{gathered} \tag{20}$$

By virtue of the regularity of normal extremals in Euler's elastic problem [38], their sufficiently short arcs are locally optimal, and hence there exists $\alpha_0 \in (0, \widetilde{t}\,)$ such that the trajectory $q^{\alpha_0}(s)$, $s \in [0, t^{\alpha_0}]$, does not contain conjugate points. Consider the continuous family of extremal trajectories $q^\alpha(s)$, $s \in [0, t^\alpha]$, $\alpha \in [\alpha_0, \widetilde{t}\,)$.

[1] Hereafter we omit this phrase for brevity.

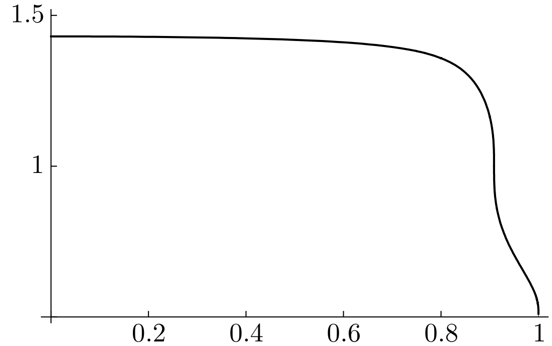


Fig. 3. Graph of the function $k \mapsto \frac{p_1^1(k)}{2K(k)}$.

1.2a. We show that, for any $\alpha \in [\alpha_0, \widetilde{t}\,)$, the point $q^\alpha(t^\alpha)$ is not conjugate for the trajectory $q^\alpha(s)$, $s \in [0, t^\alpha]$. By contradiction, let for some $\alpha \in [\alpha_0, \widetilde{t}\,)$ the point $q^\alpha(t^\alpha)$ be conjugate for the trajectory $q^\alpha(s)$. Since the elastica Γ^α is centered at a vertex, by Corollary 3.3 we obtain

$$\frac{\sqrt{r}}{2} t^\alpha \in \{p_n^1 \mid n \in \mathbb{N}\} \cup \{p_m^{x_2} \mid m \in \mathbb{N}\}. \tag{21}$$

But $t^\alpha = \alpha < \widetilde{t} = \frac{2}{\sqrt{r}} p_1^1$, so $\frac{\sqrt{r}}{2} t^\alpha < p_1^1$. On the other hand, the inequality $k < k_0$ and Lemma 3.2 imply that $p_m^{x_2} > p_1^1$ for all $m \in \mathbb{N}$. It follows that inclusion (21) is impossible. The statement of item 1.2a is proved.

1.2b. By virtue of the homotopic invariance of the index of the second variation [38, 13], any trajectory $q^\alpha(s)$, $s \in [0, t^\alpha]$, $\alpha \in [\alpha_0, \widetilde{t}\,)$, does not contain conjugate points; thus it is locally optimal. In particular, the elastica $\Gamma = \Gamma^\alpha$, $\alpha = t$, is locally optimal.

2. Now we make use of the family of elasticae Γ^α (20). By virtue of stability of these elasticae for $\alpha < t_1^1$ and by the homotopic invariance of the index of the second variation, it follows that for $\alpha = t_1^1$ the elastica Γ^α does not contain conjugate points. But t_1^1 is not a conjugate time, so $t_1^{\mathrm{conj}}(\widetilde{\lambda}) = t_1^1$.

3. Let $t > t_1^1$. For any $\varepsilon \in (0, t - t_1^1)$, the elastica Γ contains an unstable elastica $\{(\widetilde{x}_s, \widetilde{y}_s) \mid s \in [0, \widetilde{t} + \varepsilon]\}$; thus Γ is unstable as well. $\square$

For a fixed infinite inflectional elastica

$$\{(x_s, y_s) \mid s \in [0, t]\}, \qquad q(s) = (x_s, y_s, \theta_s) = \mathrm{Exp}_s(\lambda), \qquad \lambda = (\varphi, k, r) \in N_1,$$

i.e., for fixed parameters (k, r), Theorem 4.1 provides a supremum of lengths of stable elasticae centered at a vertex: this supremum is equal to $t_1^1 = \frac{2}{\sqrt{r}} p_1^1(k)$. It is natural to compare this value with the length of the period of the elastica, $T = \frac{4}{\sqrt{r}} K(k)$. A graph of the ratio $\frac{t_1^1}{T} = \frac{p_1^1(k)}{2K(k)}$ is shown in Fig. 3. This graph has a vertical tangent at the point $(k_0, 1)$.

All stable inflectional elasticae centered at a vertex contain two inflection points inside since $t_1^1 \in \left(\frac{1}{2}T, \frac{3}{2}T\right)$.

An inflectional elastica centered at a vertex, with the length equal to the period T, is

- stable for $k < k_0$ since in this case $t_1^1 > T$ (case 1 of Theorem 4.1);
- critical for $k = k_0$ since in this case $t_1^1 = T$ (case 2);
- unstable for $k > k_0$ since in this case $t_1^1 < T$ (case 3).

These elasticae are shown below (see Fig. 6; experimentally obtained elasticae are presented in Fig. 7 for case 1, in Fig. 8a for case 2, and in Figs. 8b and 8c for case 3). In the critical and unstable cases the elasticae are stabilized by finger.

5. STABILITY OF ELASTICAE CENTERED AT AN INFLECTION POINT

Here we obtain stability conditions for elasticae whose midpoint is an inflection point:

$$\Gamma = \{(x_s, y_s) \mid s \in [0, t]\}, \qquad q(s) = (x_s, y_s, \theta_s) = \mathrm{Exp}_s(\lambda), \qquad \lambda = (\varphi, k, r) \in N_1, \tag{22}$$

$$\text{the point } (x_{t/2}, y_{t/2}) \text{ is an inflection point of the elastica } \Gamma. \tag{23}$$

It is known [37, 38] that condition (23) is equivalent to the equality

$$\mathrm{cn}(\tau, k) = 0, \qquad \tau = \sqrt{r}(\varphi + t/2).$$

Recall that the length of a period of the elastica is given by $T = \frac{4K(k)}{\sqrt{r}}$.

Theorem 5.1. *Let an elastica* Γ (22) *be centered at an inflection point. Let also* $k \in (0, k_0]$.

1. *If* $t < T$, *then the elastica* Γ *is stable.*
2. *If* $t = T$, *then the elastica* Γ *is critical*; *i.e., its endpoint is the first conjugate point.*
3. *If* $t > T$, *then the elastica* Γ *is unstable.*

Proof. 1. Let $t < T$. By Theorem 3.1, we obtain $t_1^{\mathrm{conj}}(\lambda) > T > t$. Thus, the extremal trajectory $q(s)$, $s \in [0, t]$, does not contain conjugate points and is locally optimal. In other words, the elastica Γ is stable.

2. In the case $t = T$, the proof is similar to the proof of item 2 of Theorem 4.1.

3. Let $t > T$. We construct a comparison elastica

$$\widetilde{\Gamma} = \{(\widetilde{x}_s, \widetilde{y}_s) \mid s \in [0, T]\}, \qquad \widetilde{q}(s) = (\widetilde{x}_s, \widetilde{y}_s, \widetilde{\theta}_s) = \mathrm{Exp}_s(\widetilde{\lambda}), \qquad \widetilde{\lambda} = (\widetilde{\varphi}, k, r) \in N_1,$$

$$\widetilde{t} = T, \qquad \widetilde{\varphi} = \varphi + (t - \widetilde{t})/2.$$

By Corollary 3.3, the point $\widetilde{q}(\widetilde{t})$ is conjugate for the trajectory $\widetilde{q}(s)$. Thus, the elastica $\widetilde{\Gamma}$ is unstable. Consequently, the arc Γ containing this elastica is unstable as well. $\square$

For $k > k_0$, statement 1 of Theorem 5.1 is, generally speaking, not valid. An elastica is unstable for $t > T$, but for k sufficiently close[2] to 1 it becomes unstable for certain $t < T$ since the first conjugate time occurs before the period T.

Theorem 5.1 provides the supremum T of lengths of stable elasticae centered at an inflection point. Elasticae of length T centered at an inflection point contain three inflection points: one at the center and two at the boundary.

Elasticae centered at an inflection point are shown below (see Figs. 9 and 10; in Fig. 10a, the photo of an elastic film is superimposed on the plot of the corresponding elastica constructed in the Mathematica system: the physical and mathematical elasticae are in excellent agreement, which confirms the high accuracy of the mathematical model of elastic rods).

6. EXPERIMENTAL STUDY OF STABILITY OF ELASTICAE

Here we consider several one-parameter families of inflectional elasticae centered at a vertex or an inflection point:

$$\begin{gathered} \Gamma^\alpha = \{(x_t^\alpha, y_t^\alpha) \mid t \in [0, t^\alpha]\}, \qquad q^\alpha(t) = (x_t^\alpha, y_t^\alpha, \theta_t^\alpha) = \mathrm{Exp}_t(\lambda^\alpha), \\ \lambda^\alpha = (\varphi^\alpha, k^\alpha, r^\alpha) \in N_1, \qquad \alpha \in [\alpha_0, \alpha_1]. \end{gathered} \tag{24}$$

For each of these families, one can observe the loss of stability; i.e., there exists a critical value of the family parameter $\alpha_* \in (\alpha_0, \alpha_1)$ such that the elastica Γ^α is stable for $\alpha < \alpha_*$ and unstable for $\alpha > \alpha_*$ (or vice versa). We compute this critical parameter for each of these families.

[2] For $k \in (\bar{k}, 1)$, where $\bar{k} \approx 0.998$.

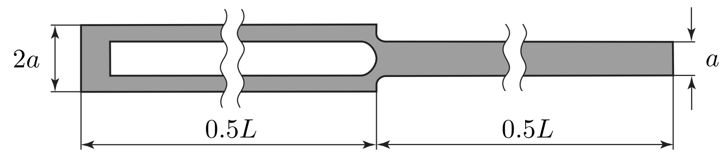


Fig. 4. Specimen for modeling self-intersecting elasticae.

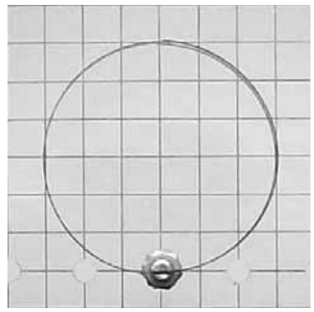


Fig. 5. Non-inflectional elastica-circle.

6.1. Experimental setup. Theoretical findings concerning stability of elasticae were supported experimentally using specimens made of a thin celluloid film which can sustain substantial curvature changes without residual strains. The experimental setup is a thick organic-glass sheet on which a quadratic grid with a step of 10 mm is drawn. In the grid nodes, holes are made to install bolts with longitudinal cuts in which the specimens are clamped. Since the stability of elastica is investigated under various boundary conditions, it is necessary to model highly bent configurations with allowance for self-intersection. For a specimen of uniform rectangular cross section, these states are difficult to model since the line that passes through the cross-sectional centroids of the specimen axis twists out of the bending plane and becomes a spatial curve. To satisfy the condition of a planar elastic curve, special specimens shaped like a strip of variable cross section with a slit (see Fig. 4) are used. One part of the specimen is narrow and continuous, whereas the other part has a slit. The slit is made slightly wider than the width of the narrow part to ensure free penetration of the narrow part into the wide part of the specimen so that the elastic line remains a planar curve. The cross-sectional dimensions are such that the flexural rigidity $Eah^3/12$ (E is Young's modulus; a and h are the width and height of the cross section, respectively) remains constant along the specimen except for a very narrow part in the middle. The condition of constant flexural rigidity is confirmed experimentally by pure bending (non-inflectional elastica): when the ends of the slit specimen are rotated through an angle of π radians and clamped at one point, the specimen becomes a circle (Fig. 5). The photographs below show the specimen of length $L = 198$ mm and width $a = 4.5$ mm.

6.2. Elasticae of series 1. Consider the following family of elasticae (24) centered at a vertex:

$$\varphi^\alpha \equiv -t_1/2, \qquad k^\alpha = \alpha, \qquad \sqrt{r^\alpha} = 4K(\alpha)/t_1, \qquad t^\alpha \equiv t_1, \qquad \alpha \in (0,1).$$

For this family, we have

$$y_{t_1} = 0, \qquad \theta_{t_1} = 0$$

(see Fig. 6 and photos in Figs. 7 and 8). The critical value of the parameter is given by

$$\alpha_* = k_0,$$

which agrees with the stable elasticae in Fig. 7 and unstable elasticae in Figs. 8b and 8c. The critical curve is the closed elastica shown in Fig. 8a.

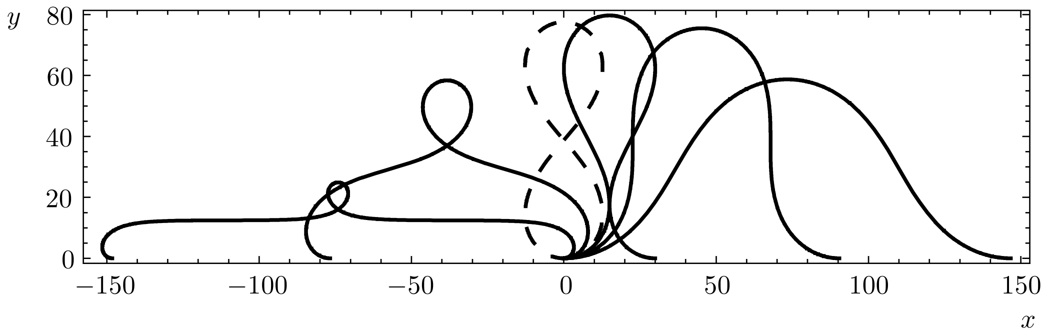


Fig. 6. Elasticae of series 1, centered at a vertex.

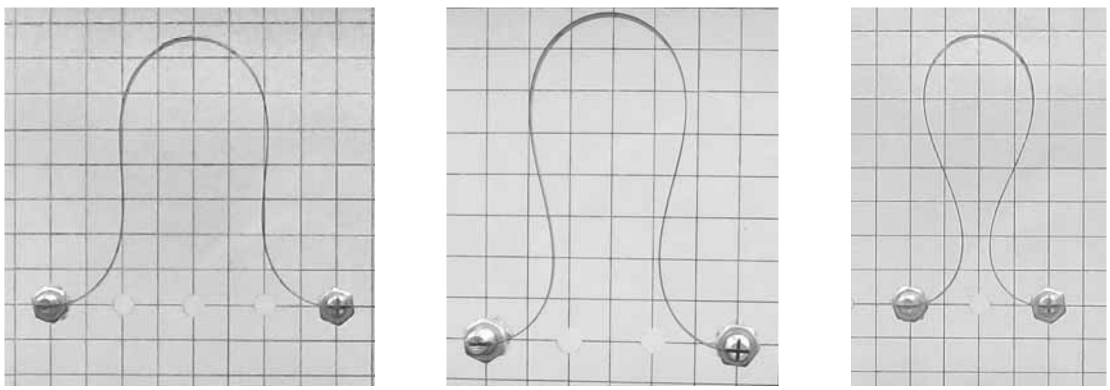
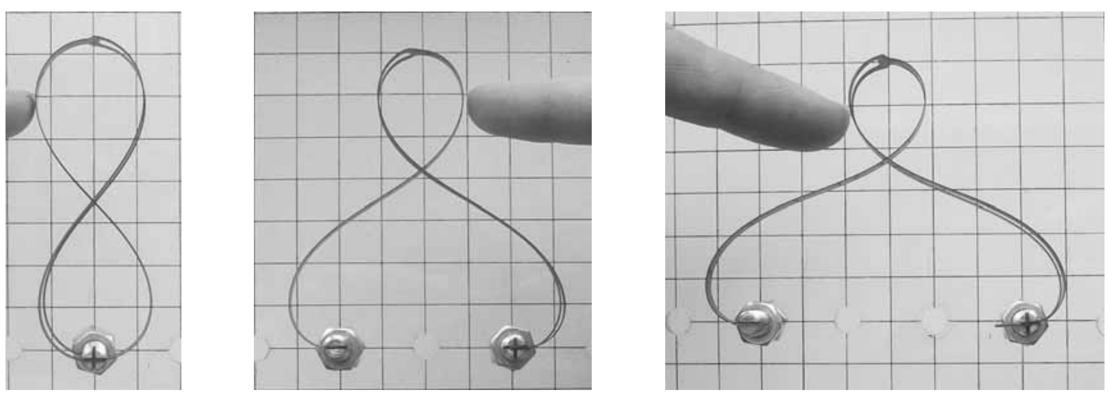


Fig. 7. Series 1: stable elasticae.



(a) (b) (c)

Fig. 8. Series 1: (a) critical 8-shaped elastica and (b, c) unstable elasticae.

6.3. Elasticae of series 2. Consider the following family of elasticae (24) centered at an inflection point:

$$\varphi^\alpha = t_1(3K(\alpha) - p_1^1(\alpha))/(2p_1^1(\alpha)), \qquad k^\alpha = \alpha, \qquad \sqrt{r^\alpha} = 2p_1^1(\alpha)/t_1, \qquad t^\alpha \equiv t_1, \qquad \alpha \in (0,1).$$

For this family, we have

$$y_{t_1} = 0, \qquad \theta_{t_1} = 0$$

(see Fig. 9 and photos in Figs. 10 and 11). The critical value of the parameter is given by

$$\alpha_* = k_0,$$

which agrees with the stable elasticae in Fig. 10. The critical curve is the closed elastica.

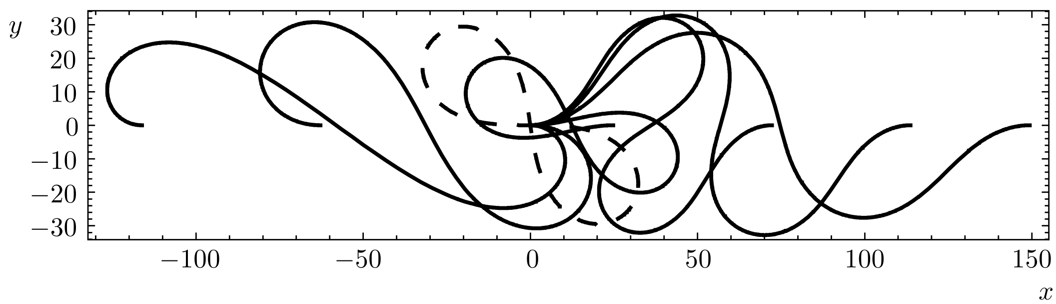
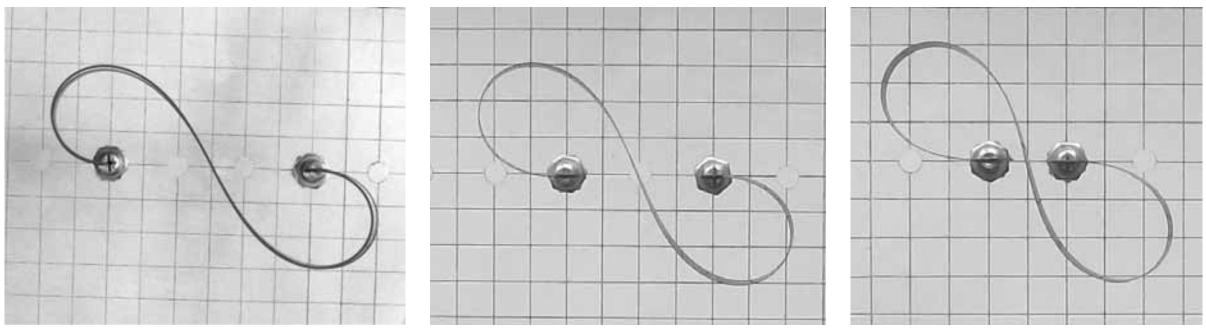
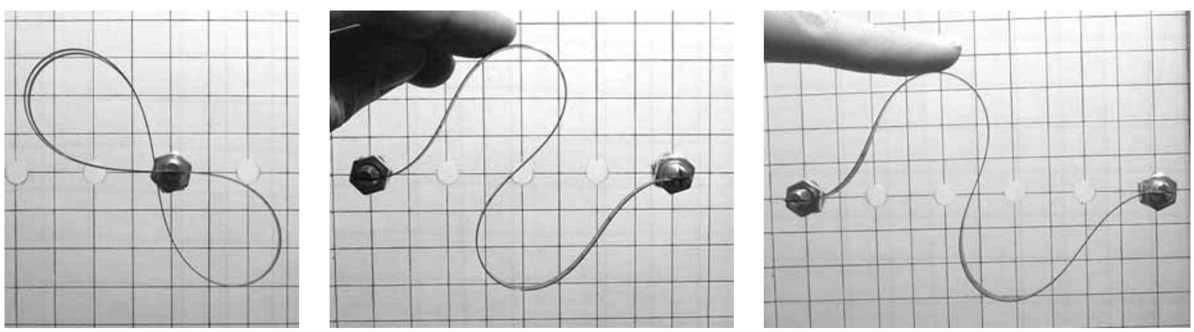


Fig. 9. Elasticae of series 2 centered at an inflection point.



(a) (b) (c)

Fig. 10. Series 2: stable elasticae; in (a) the predicted curve is superimposed on the photo of an experimentally obtained elastica.



(a) (b) (c)

Fig. 11. Series 2: (a) critical 8-shaped elastica and (b, c) unstable elasticae.

6.4. Elasticae of series 3. Consider the following family of elasticae (24) centered at a vertex:

$$\varphi^\alpha \equiv -t_1/2, \qquad k^\alpha = \alpha, \qquad \sqrt{r^\alpha} = \frac{2}{t_1} F\left(\pi - \arcsin\frac{1}{\sqrt{2}\alpha}, \alpha\right), \qquad t_1^\alpha \equiv t_1, \qquad \alpha \in \left[\frac{1}{\sqrt{2}}, 1\right).$$

For this family, we have

$$x_{t_1} = 0, \qquad \theta_{t_1} = \pi,$$

and the critical value α_* is determined from the equation

$$\operatorname{sn}(p_1^1(\alpha), \alpha) = -1/(\sqrt{2}\alpha),$$

which gives

$$\alpha_* = 0.924902\ldots.$$

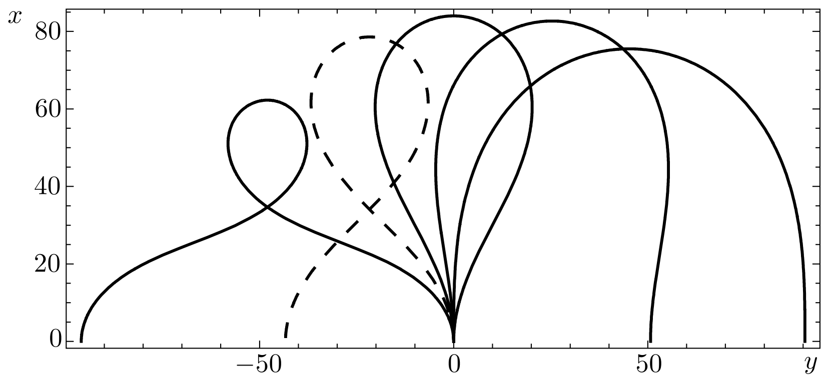


Fig. 12. Elasticae of series 3 centered at a vertex.

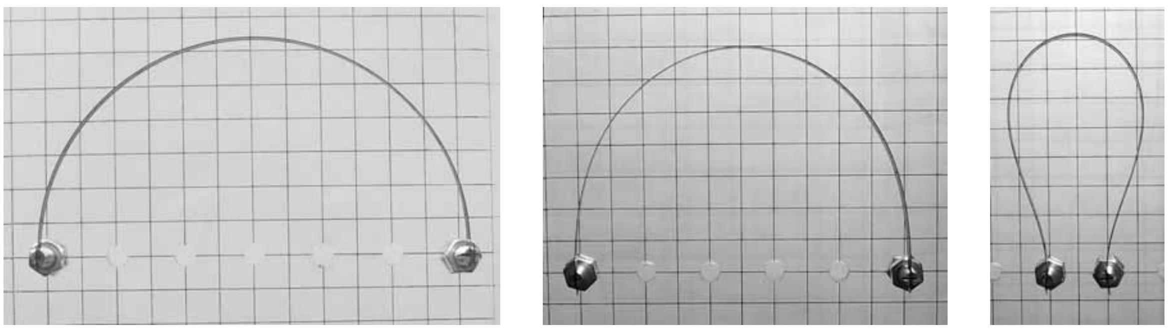
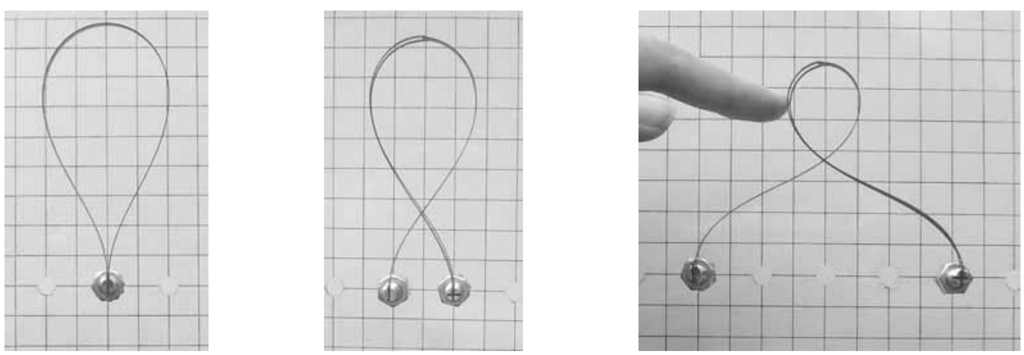


Fig. 13. Series 3: stable elasticae.



(a) (b) (c)

Fig. 14. Series 3: (a) stable drop-shaped elastica, (b) stable elastica with self-intersection, and (c) unstable elastica.

Elasticae of this series are shown in Fig. 12 (where the x axis is directed upwards and the y axis is directed to the left to agree with the subsequent photos). Figures 13 and 14 show the corresponding equilibrium configurations of the elastic film of length $t_1 \approx 198$ mm. The predicted critical value y_{t_1} is approximately 32 mm, which agrees with the stable elasticae in Figs. 13, 14a, and 14b and unstable elasticae in Fig. 14c. The drop-shaped elastica (with self-contact) shown in Fig. 14a corresponds to the modulus $k = 0.855092407720382690\ldots$.

6.5. Elasticae of series 4. Consider a family of elasticae centered at a vertex whose photos are shown in Figs. 15 and 16: an elastic film of variable length is clamped at angles of $\pi/4$ to the horizontal axis, the distance between the clamps being $b = 60$ mm. When the length of the film between the clamping points decreases, the W-shaped elastica loses stability, but preserves the vertical symmetry axis. The arcs shown in Fig. 15 are stable; the arc in Fig. 16a is stable and close

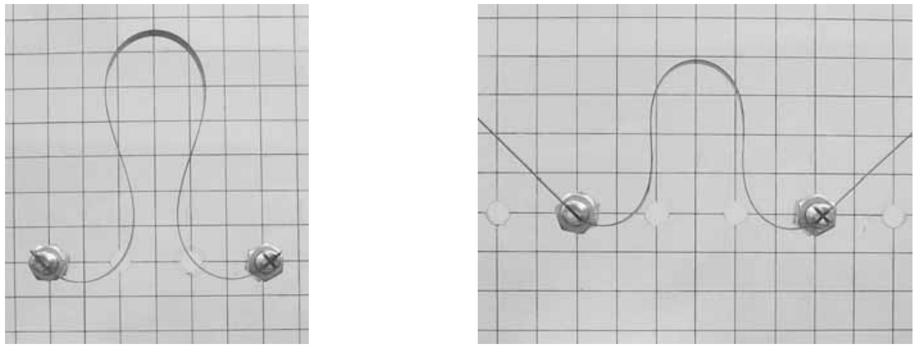
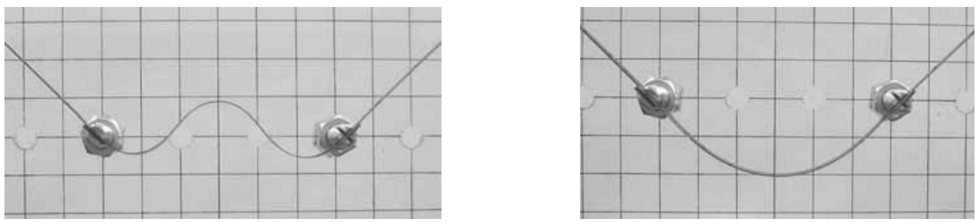


Fig. 15. Series 4: stable elasticae.



(a) (b)

Fig. 16. Series 4: (a) stable subcritical elastica and (b) stable supercritical elastica.

to the critical one: after a small decrease in the arc length between the clamps it snaps through to a stable elastica in Fig. 16b.

Computations for this series yield the following values of the critical parameters:

$$k_* \text{ is the root of the equation } \quad k\,\mathrm{sn}(p_1^1(k), k) = -\sqrt{2-\sqrt{2}}/2, \qquad k_* \in (0.3, 0.5),$$
$$p_* = p_1^1(k_*), \qquad \sqrt{r_*} = 2(2\,\mathrm{E}(p_*, k_*) - p_*)/b, \qquad \varphi_* = -p_*/\sqrt{r_*}, \qquad t_{1*} = -2\varphi_*.$$

According to numerical computations, the critical modulus $k_* \approx 0.39$ and the midspan deflection is approximately 7.3 mm. For the subcritical elastica shown in Fig. 16a, the deflection is found to be approximately 8 mm, which is naturally somewhat higher than the predicted value.

7. CONCLUSIONS

We have considered equilibrium states of a flexible inextensible unshearable rod (elastica) with fixed endpoints and tangents at the endpoints. We have thoroughly studied the stability question of elasticae centered at the vertex or at the inflection point. To this end, we applied the methods of optimal control theory. The theoretical findings obtained have been supported experimentally using specimens made of a thin celluloid film.

It should be noted that the paper deals with kinematic conditions at the rod ends (i.e., elasticae whose coordinates and whose tangents are prescribed at the ends). An open question is the stability of elasticae for the general case where one part of the boundary conditions is formulated in terms of displacements and the other in terms of forces and couples. It is of interest to extend the method considered above to study the stability of curvilinear equilibrium states, for example, of a simply supported rod subject to end couples. Another issue of interest is to investigate the deformation behavior of the rod under continuous kinematic loading (variation of the distance between the ends, rotation of the end tangents, etc.). The mathematical methods for investigating stability can also be used to obtain explicit stability conditions for elastic rods of the general form.

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