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MAXWELL STRATA IN THE EULER ELASTIC PROBLEM

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Abstract. The classical Euler problem on stationary configurations of elastic rod in the plane is studied in detail by geometric control techniques as a left-invariant optimal control problem on the group of motions of a two-dimensional plane E(2).

The attainable set is described, the existence and boundedness of optimal controls are proved. Extremals are parametrized by the Jacobi elliptic functions of natural coordinates induced by the flow of the mathematical pendulum on fibers of the cotangent bundle of E(2).

The group of discrete symmetries of the Euler problem generated by reflections in the phase space of the pendulum is studied. The corresponding Maxwell points are completely described via the study of fixed points of this group. As a consequence, an upper bound on cut points in the Euler problem is obtained.

1. Introduction

In 1744, L. Euler considered the following problem on stationary configurations of an elastic rod. Given a rod in the plane with fixed endpoints and tangents at the endpoints, one should determine possible profiles of the rod under the given boundary conditions. Euler obtained ODEs for stationary configurations of the elastic rod and described their possible qualitative types. These configurations are called *Euler elasticae*.

An Euler elastica is a critical point of the functional of elastic energy on the space of smooth planar curves that satisfy the boundary conditions specified. In this paper, we address the issue of *optimality* of an elastica: whether a critical point is a minimum of the energy functional? That is, which elasticae provide the minimum of the energy functional among all curves satisfying the boundary conditions (the *global optimality*), or the

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minimum compared with sufficiently close curves satisfying the boundary conditions (the *local optimality*). These questions remained open despite their obvious importance.

For the elasticity theory, the problem of local optimality is essential since it corresponds to the *stability* of Euler elasticae under small perturbations that preserve the boundary conditions. In the calculus of variations and optimal control, the point where an extremal trajectory loses its local optimality is called a *conjugate point*. We will give an exact description of conjugate points in the problem on Euler elasticae, which were previously known only numerically.

From the mathematical point of view, the problem of global optimality is fundamental. We will study *cut points* in the Euler elastic problem — the points where elasticae lose their global optimality.

This is the first of three planned works on the Euler elastic problem. The aim of this paper is to give a complete description of *Maxwell points*, i.e., points where distinct extremal trajectories with the same value of the cost functional meet one another. Such points provide an upper bound on cut points: an extremal trajectory cannot be globally optimal after a Maxwell point. In the second work [35], we prove that conjugate points in the Euler elastic problem are bounded by Maxwell points. Moreover, we pursue the study of the global optimal problem: in [36], we describe the global diffeomorphic properties of the *exponential mapping*.

This paper is organized as follows. In Sec. 2, we review the history of the problem on elasticae. In Sec. 3, we state the Euler problem as a left-invariant optimal control problem on the group of motions of a two-dimensional plane E(2) and discuss the continuous symmetries of the problem. In Sec. 4, we describe the attainable set of the control system in question. In Sec. 5, we prove the existence and boundedness of optimal controls in the Euler problem. In Sec. 6, we apply the Pontryagin maximum principle to the problem, describe abnormal extremals, and deduce the Hamiltonian system for normal extremals.

Since the problem is left-invariant, the normal Hamiltonian system of the PMP becomes triangular after an appropriate choice of the parametrization of fibers of the cotangent bundle of E(2): the vertical subsystem is independent of the horizontal coordinates. Moreover, this vertical subsystem is essentially the equation of the *mathematical pendulum*. For the detailed subsequent analysis of the extremals, it is crucial to choose convenient coordinates. In Sec. 7, we construct such natural coordinates in the fiber of the cotangent bundle over the initial point. First, we consider the "angle-action" coordinates in the phase cylinder of the standard pendulum, and then extend them to the whole fiber via continuous symmetries of the problem. One of the coordinates is the time of motion of the pendulum, and the other two are integrals of motion of the pendulum. In Sec. 8, we apply the *elliptic coordinates* constructed in such way for the integration of the normal Hamiltonian system. In particular, we recover the different classes of elasticae discovered by Euler.

The flow of the pendulum plays the key role not only in the parametrization of extremal trajectories, but also in the study of their optimality. In Sec. 9, we describe the *discrete symmetries* of the Euler problem generated by reflections in the phase cylinder of the standard pendulum. Further, we study the action of the group of reflections in the preimage and image of the exponential mapping of the problem.

In Sec. 10, we consider Maxwell points of the Euler problem. The Maxwell strata corresponding to reflections are described by certain equations in elliptic functions. In Sec. 11, we study solvability of these equations, give sharp estimates of their roots, and describe their mutual disposition via the analysis of the elliptic functions involved.

A complete description of the Maxwell strata obtained is important both for global and local optimality of extremal trajectories. In Sec. 12, we derive an upper bound on the cut time in the Euler problem due to the fact that such a trajectory cannot be globally optimal after a Maxwell point. In our subsequent work [35] we will show that conjugate points in the Euler problem are bounded by Maxwell points and give a complete solution to the problem of local optimality of extremal trajectories.

We used the system "Mathematica" [41] to carry out complicated calculations and to produce illustrations in this paper.

This paper is an essentially abridged version of the initial preprint [34]. The preprint contains additional information, illustrations, and complete details of some proofs that appear here in a shortened form due to the limitation of space.

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2. History of the Euler elastic problem

In addition to the original works of the scholars who contributed to the theory, in this section we follow also the sources on history of the subject by C. Truesdell [39], A. E. H. Love [24], and S. Timoshenko [38].

In 1691, J. Bernoulli considered the problem on the form of a uniform planar elastic bar bent by an external force. His hypothesis was that the bending moment of the rod is equal to $\mathcal{B}/R$, where $\mathcal{B}$ is the "flexural rigidity" and R is the curvature radius of the bent bar. For an elastic bar of unit excursion built vertically into a horizontal wall and bent by a load sufficient to make its top horizontal (*rectangular elastica*), J. Bernoulli obtained the ODEs

$$dy = \frac{x^2\,dx}{\sqrt{1-x^4}}, \quad ds = \frac{dx}{\sqrt{1-x^4}}, \quad x \in [0,1]$$

(where (x,y) is the elastic bar and s is its length parameter), integrated them in series, and calculated precise upper and lower bounds for their value at the endpoint $x = 1$ (see [7]).

In 1742, D. Bernoulli in his letter [6] to Euler wrote that the elastic energy of the bent rod is proportional to the magnitude $E = \int ds/R^2$ and suggested to find the elastic curves from the variational principle $E \to \min$. At that time, Euler was writing his treatise on the calculus of variations *Methodus inveniendi ...* [13] published in 1744, and he adjoined to his book an appendix *De curvis elasticis*, where he applied the newly developed techniques to the problem on elasticae. Euler considered a thin homogeneous elastic plate, rectilinear in the natural (unstressed) state. For the profile of the plate, Euler stated the following problem:

> *"That among all curves of the same length which not only pass through the points A and B, but are also tangent to given straight lines at these points, that curve be determined in which the value of $\int_A^B \frac{ds}{R^2}$ be a minimum."* (2.1)

Euler wrote down the ODE known now as the Euler–Lagrange equation for the corresponding problem of calculus of variations and reduced it to the equations

$$dy = \frac{(\alpha + \beta x + \gamma x^2)\,dx}{\sqrt{a^4 - (\alpha + \beta x + \gamma x^2)^2}}, \quad ds = \frac{a^2\,dx}{\sqrt{a^4 - (\alpha + \beta x + \gamma x^2)^2}}, \tag{2.2}$$

where α/a^2, β/a, and γ are real parameters expressed in terms of $\mathcal{B}$, the load of the elastic rod, and its length. Euler studied the quadrature defined by the first of equations (2.2). In the modern terminology, he investigated the qualitative behavior of the elliptic functions that parametrize the elastic curves via the qualitative analysis of the determining ODEs. Euler described all possible types of elasticae and indicated the values of parameters for which these types are realized (see a copy of Euler's original sketches in [34]).

Euler divided all elastic curves into nine classes, they are plotted respectively as follows:

1. straight line, Fig. 4;
2. Fig. 5;
3. rectangular elastica, Fig. 6;
4. Fig. 7;
5. periodic elastica in the form of Fig. 8;
6. Fig. 9;

7. elastica with one loop, Fig. 10;
8. Fig. 11;
9. circle, Fig. 12.

Following the tradition introduced by A. E. H.Love [24], the elastic curves with inflection points (classes 2–6) are said to be *inflectional*, the elastica of class 7 is said to be *critical*, and elasticae without inflection points of class 8 are said to be *noninflectional*.

Further, Euler established the magnitude of the force applied to the elastic plate that results in each type of elasticae. He indicated the experimental method for evaluation of the flexural rigidity of the elastic plate by its form under bending. Finally, he studied the problem of stability of a column modelled by the loaded rod whose lower end is constrained to remain vertical, by presenting it as an elastica of the class 2 close to the straight line (a sinusoid). After the work of Leonhard Euler, the elastic curves are called *Euler elasticae*.

The first explicit parametrization of Euler elasticae was performed by L. Saalchütz in 1880 [29].

In 1906, the future Nobel prize-winner Max Born defended a Ph. D. thesis called *Stability of elastic lines in the plane and the space* [9]. Born considered the problem on elasticae as a problem of calculus of variations and derived from the Euler–Lagrange equation that its solutions $(x(t), y(t))$ satisfy the ODEs of the form

$$\begin{gathered} \dot{x} = \cos\theta, \quad \dot{y} = \sin\theta, \\ A\ddot{\theta} + R\sin(\theta - \gamma) = 0, \quad A, R, \gamma = \text{const}, \end{gathered} \tag{2.3}$$

and, therefore, the angle θ defining the slope of elasticae satisfies the equation of the *mathematical pendulum* (2.3). Further, Born studied stability of elasticae with fixed endpoints and fixed tangents at the endpoints. Born proved that an elastic arc without inflection points is stable (in this case, the angle θ is monotone and, therefore, it can be taken as a parameter along the elastica; Born showed that the second variation of the functional of the elastic energy $E = \dfrac{1}{2}\displaystyle\int \dot{\theta}^2\,dt$ is positive). In the general case, Born wrote down the Jacobian that vanishes at conjugate points. Since the functions entering this Jacobian were too complicated, Born restricted himself to numerical investigation of conjugate points. He was the first to plot elasticae numerically and verify the theory against experiments on elastic rods, see the photos from Born's thesis in [34]. Moreover, Born studied stability of Euler elasticae with various other boundary conditions and obtained some results for elastic curves in $\mathbb{R}^3$.

In 1993, V. Jurdjevic [19] discovered that Euler elasticae appear in the ball-plate problem stated as follows. Consider a ball rolling on a horizontal plane without slipping or twisting. The problem is to roll the ball from an initial contact configuration (defined by the contact point of the ball with the plane and orientation of the ball in the 3-space) to a terminal contact configuration, so that the curve traced by the contact point in the plane was the shortest possible. Jurdjevic showed that such optimal curves are Euler elasticae. Moreover, Jurdjevic also extensively studied the elastic problem in $\mathbb{R}^3$ and its analogs in the sphere S^3 and in the Lorentz space H^3 [20, 21].

In 1993, R. Brockett and L. Dai [11] discovered that Euler elasticae are projections of optimal trajectories in the nilpotent sub-Riemannian problem with the growth vector (2,3,5) known also as the generalized Dido problem [30–33].

Elasticae were considered in the approximation theory as nonlinear splines [8, 14, 17, 18, 23], in computer vision as a maximum likelihood reconstruction of occluded edges [27], their 3-dimensional analogues are used in the modelling of DNA minicircles [25, 26], etc.

Euler elasticae and their various generalizations play an important role in the modern mathematics, mechanics, and their applications. However, the initial variational problem as it was stated by Euler (2.1) is far from the complete solution: neither local nor global optimality of Euler elasticae is studied. This is the first of three planned works that will give a complete description of the local optimality and present an essential progress in the study of the global optimality of elasticae. In this paper, we give an upper bound on the cut points along Euler elasticae, i.e., points where they lose their global optimality. In the next work [35] we obtain a complete characterization of conjugate points, i.e., points where elasticae lose their local optimality. The global diffeomorphic properties of the exponential mapping in the Euler problem will be described in [36].

3. Statement of the problem

3.1. **Optimal control problem.** First, we state the elastic problem mathematically. Let a homogeneous elastic rod in the two-dimensional Euclidean plane $\mathbb{R}^2$ have a fixed length $l > 0$. Take any points $a_0,\ a_1 \in \mathbb{R}^2$ and arbitrary unit tangent vectors at these points $v_i \in T_{a_i}\mathbb{R}^2$, $|v_i| = 1$, $i = 0, 1$. The problem is to find the profile of a rod $\gamma : [0, t_1] \to \mathbb{R}^2$ starting from the point a_0 and coming to the point a_1 with the corresponding tangent vectors v_0 and v_1:

$$\gamma(0) = a_0, \quad \gamma(t_1) = a_1, \tag{3.1}$$

$$\dot\gamma(0) = v_0, \quad \dot\gamma(t_1) = v_1, \tag{3.2}$$

with the minimum elastic energy. The curve $\gamma(t)$ is assumed to be absolutely continuous with the Lebesgue square-integrable curvature $k(t)$. We suppose that $\gamma(t)$ is arc-length parametrized, i.e., $|\dot\gamma(t)| \equiv 1$ and, therefore, the time

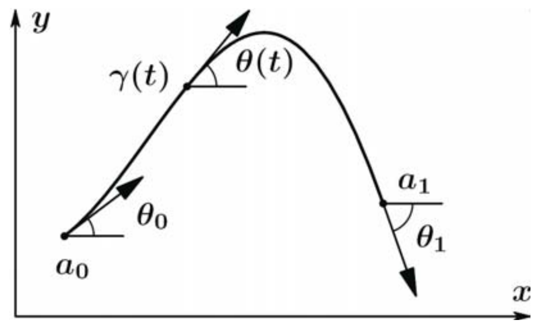

Fig. 1. Statement of the Euler problem.

of motion along the curve γ coincides with its length:

$$t_1 = l. \tag{3.3}$$

The elastic energy of the rod is measured by the integral

$$J = \frac{1}{2}\int_0^{t_1} k^2(t)\,dt.$$

We choose the Cartesian coordinates (x, y) in the two-dimensional plane $\mathbb{R}^2$. Let the required curve be parametrized as $\gamma(t) = (x(t), y(t))$, $t \in [0, t_1]$, and let its endpoints have coordinates $a_i = (x_i, y_i)$, $i = 0, 1$. Denote by θ the angle between the tangent vector to the curve γ and the positive direction of the axis x. Further, let the tangent vectors at the endpoints of γ have the coordinates $v_i = (\cos\theta_i, \sin\theta_i)$, $i = 0, 1$ (see Fig. 1).

Then the required curve $\gamma(t) = (x(t), y(t))$ is defined by a trajectory of the following control system:

$$\dot{x} = \cos\theta, \tag{3.4}$$

$$\dot{y} = \sin\theta, \tag{3.5}$$

$$\dot{\theta} = u, \tag{3.6}$$

$$q = (x, y, \theta) \in M = \mathbb{R}^2_{x,y} \times S^1_\theta, \quad u \in \mathbb{R}, \tag{3.7}$$

$$q(0) = q_0 = (x_0, y_0, \theta_0), \quad q(t_1) = q_1 = (x_1, y_1, \theta_1), \quad t_1 \text{ fixed}. \tag{3.8}$$

For an arc-length parametrized curve, the curvature is, up to the sign, equal to the angular velocity: $k^2 = \dot{\theta}^2 = u^2$, whence we obtain the cost functional

$$J = \frac{1}{2}\int_0^{t_1} u^2(t)\,dt \to \min. \tag{3.9}$$

We study the optimal control problem (3.4)–(3.9). Following V. Jurdjevic [21], this problem is called the *Euler elastic problem*. Admissible controls are $u(t) \in L_2[0, t_1]$, and admissible trajectories are absolutely continuous curves $q(t) \in AC([0, t_1]; M)$.

In the vector notation, the problem has the following form:

$$\dot q = X_1(q) + uX_2(q), \quad q \in M = \mathbb{R}^2 \times S^1, \quad u \in \mathbb{R}, \tag{Σ}$$
$$q(0) = q_0, \quad q(t_1) = q_1, \quad t_1 \text{ fixed},$$
$$J = \frac{1}{2}\int_0^{t_1} u^2 dt \to \min, \quad u \in L_2[0, t_1],$$

where the vector fields in the right-hand side of system (Σ) are

$$X_1 = \cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y}, \quad X_2 = \frac{\partial}{\partial \theta}.$$

Note the multiplication table in the Lie algebra of vector fields generated by X_1, X_2:

$$[X_1, X_2] = X_3 = \sin\theta \frac{\partial}{\partial x} - \cos\theta \frac{\partial}{\partial y}, \tag{3.10}$$
$$[X_2, X_3] = X_1, \qquad [X_1, X_2] = 0. \tag{3.11}$$

3.2. **Left-invariant problem on the group of motions of a plane.** The Euler elastic problem has obvious symmetries — parallel translations and rotations of the two-dimensional plane $\mathbb{R}^2$. Therefore, it can naturally be stated as an invariant problem on the group of proper motions of the two-dimensional plane

$$\mathrm{E}(2) = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; (x, y) \in \mathbb{R}^2,\ \theta \in S^1 \right\}.$$

Indeed, the state space of the problem $M = \mathbb{R}^2_{x,y} \times S^1_\theta$ is parametrized by matrices of the form

$$q = \begin{pmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{E}(2),$$

and dynamics (3.4)–(3.6) is left-invariant on the Lie group E(2):

$$\dot q = \begin{pmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -u & 1 \\ u & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The Lie algebra of the Lie group E(2) has the form

$$\mathrm{e}(2) = \mathrm{span}(E_{21} - E_{12}, E_{13}, E_{23}),$$

where E_{ij} denotes the (3×3)-matrix with the only identity entry in the ith row and jth column, and zeros otherwise. In the basis

$$e_1 = E_{13}, \quad e_2 = E_{21} - E_{12}, \quad e_3 = -E_{23},$$

the multiplication table in the Lie algebra $\mathrm{e}(2)$ takes the form

$$[e_1, e_2] = e_3, \quad [e_2, e_3] = e_1, \quad [e_1, e_3] = 0.$$

Then the Euler elastic problem becomes the following left-invariant problem on the Lie group $\mathrm{E}(2)$:

$$\dot{q} = X_1(q) + uX_2(q), \quad q \in \mathrm{E}(2), \quad u \in \mathbb{R},$$
$$q(0) = q_0, \quad q(t_1) = q_1, \quad t_1 \text{ fixed},$$
$$J = \frac{1}{2}\int_0^{t_1} u^2 dt \to \min,$$

where

$$X_i(q) = qe_i, \quad i = 1, 2, \quad q \in \mathrm{E}(2),$$

are basis left-invariant vector fields on $\mathrm{E}(2)$ (here qe_i denotes the product of (3×3)-matrices).

3.3. **Continuous symmetries and normalization of conditions of the problem.** Left translations on the Lie group $\mathrm{E}(2)$ are symmetries of the Euler elastic problem. By virtue of these symmetries, we can assume that initial point of trajectories is the identity element of the group $\mathrm{Id} = E_{11} + E_{22} + E_{33}$, i.e.,

$$q_0 = (x_0, y_0, \theta_0) = (0, 0, 0). \tag{3.12}$$

In other words, parallel translations in the plane $\mathbb{R}^2_{x,y}$ shift the initial point of the elastic rod γ to the origin $(0,0) \in \mathbb{R}^2_{x,y}$, and rotations of this plane combine the initial tangent vector $\dot{\gamma}(0)$ with the positive direction of the axis x.

Moreover, one can easily see one more continuous family of symmetries of the problem — dilations in the plane $\mathbb{R}^2_{x,y}$. Consider the following one-parameter group of transformations of variables of the problem:

$$\begin{aligned}(x, y, \theta, t, u, t_1, J) &\mapsto (\tilde{x}, \tilde{y}, \tilde{\theta}, \tilde{t}, \tilde{u}, \tilde{t}_1, \tilde{J}) \\ &= (e^s x, e^s y, \theta, e^s t, e^{-s} u, e^s t_1, e^{-s} J).\end{aligned} \tag{3.13}$$

One immediately verifies that the Euler problem is preserved by this group of transformations. Thus, choosing $s = -\ln t_1$, we can assume that $t_1 = 1$. In other words, we obtain an elastic rod of the unit length by virtue of dilations in the plane $\mathbb{R}^2_{x,y}$.

In the sequel, we usually fix the initial point q_0 as in (3.12). However, the terminal time t_1 will remain a parameter, not necessarily equal to 1.

4. Attainable set

Consider a smooth control system of the form

$$\dot{q} = f(q, u), \quad q \in M, \quad u \in U. \tag{4.1}$$

Let $u = u(t)$ be an admissible control, and let $q_0 \in M$. Denote by $q(t; u, q_0)$ the trajectory of the system corresponding to the control $u(t)$ and satisfying the initial condition $q(0; u, q_0) = q_0$. The *attainable set* of system (4.1) from the point q_0 for the time t_1 is defined as follows:

$$\mathcal{A}_{q_0}(t_1) = \{q(t_1; u, q_0) \mid u = u(t) \text{ admissible control, } t \in [0, t_1]\}.$$

Moreover, one can consider the attainable set for a time not greater than t_1: $\mathcal{A}_{q_0}^{t_1} = \bigcup\limits_{0 \le t \le t_1} \mathcal{A}_{q_0}(t)$, and the attainable set for an arbitrary nonnegative time: $\mathcal{A}_{q_0} = \bigcup\limits_{0 \le t < \infty} \mathcal{A}_{q_0}(t)$. The *orbit* of system (4.1) is defined as

$$\mathcal{O}_{q_0} = \left\{ e^{\tau_N f_N} \circ \cdots \circ e^{\tau_1 f_1}(q_0) \mid \tau_i \in \mathbb{R},\ f_i = f(\cdot, u_i),\ u_i \in U,\ N \in \mathbb{N} \right\},$$

where $e^{\tau_i f_i}$ is the flow of the vector field f_i (see [2, 21] for basic properties of attainable sets and orbits). In this section we describe the orbit and attainable sets for the Euler elastic problem.

Multiplication rules (3.10), (3.11) imply that the control system (Σ) has the full rank:

$$\mathrm{Lie}_q(X_1, X_2) = \mathrm{span}(X_1(q), X_2(q), X_3(q)) = T_q M \quad \forall q \in M.$$

By the Nagano–Sussmann Orbit Theorem [2, 21], the whole state space is a single orbit: $\mathcal{O}_{q_0} = M$ for any $q_0 \in M$. Moreover, the system is completely controllable: $\mathcal{A}_{q_0} = M$ for any $q_0 \in M$. This can be shown either by applying a general controllability condition for control-affine systems with recurrent drift (see [21, Sec. 4, Theorem 5]), or via the controllability test for left-invariant systems on semidirect products of Lie groups (see [21, Sec. 6, Theorem 10]). On the other hand, it is obvious that system (Σ) is not completely controllable on a compact time segment $[0, t_1]$: $\mathcal{A}_{q_0}^{t_1} \neq M$ in view of the bound

$$(x(t) - x_0)^2 + (y(t) - y_0)^2 \le t_1^2, \tag{4.2}$$

the distance between the endpoints of the elastic rod should not exceed the length of the rod. We have the following description of the exact-time attainable sets for the Euler problem.

Theorem 4.1. *Let $q_0 = (x_0, y_0, \theta_0) \in M = \mathbb{R}^2 \times S^1$ and $t_1 > 0$. Then the attainable set of system (Σ) is*

$$\begin{aligned}\mathcal{A}_{q_0}(t_1) = \{(x, y, \theta) \in M \mid {} & (x - x_0)^2 + (y - y_0)^2 < t_1^2 \\ & \text{or } (x, y, \theta) = (x_0 + t_1 \cos\theta_0, y_0 + t_1 \sin\theta_0, \theta_0)\}.\end{aligned}$$

Proof. The proof is straightforward. First, in view of continuous symmetries of the problem (see Sec. 3.3), it suffices to prove this theorem in the case $q_0 = (0, 0, 0)$, $t_1 = 1$. Second, one should apply bound (4.2). Third, it suffices to consider trajectories of the system corresponding to piecewise-constant controls — concatenations of straight lines and circles in the plane (x, y). All details are given in [34]. □

The following properties of attainable sets of system (Σ) follow immediately from Theorem 4.1.

Corollary 4.1. *Let q_0 be an arbitrary point of M. Then*:

(1) $\mathcal{A}_{q_0}(t_1) \subset \mathcal{A}_{q_0}(t_2)$ *for any* $0 < t_1 < t_2$;
(2) $\mathcal{A}^t_{q_0} = \mathcal{A}_{q_0}(t)$ *for any* $t > 0$;
(3) $q_0 \in \operatorname{int} \mathcal{A}^t_{q_0}$ *for any* $t > 0$.

Item (3) means that system (Σ) is small-time locally controllable. However, the restriction of (Σ) to a small neighborhood of a point $q_0 \in M$ is not controllable since some points in the neighborhood of q_0 are reachable from q_0 by trajectories of Σ far from q_0.

Topologically, the attainable set $\mathcal{A}_{q_0}(t)$ is an open solid torus united with a single point at its boundary. In particular, the attainable set is neither open nor closed.

In the sequel, we study the Euler problem under the natural condition

$$q_1 \in \mathcal{A}_{q_0}(t_1). \tag{4.3}$$

5. Existence and regularity of optimal solutions

We apply known results of the optimal control theory in order to show that, in the Euler elastic problem, optimal controls exist and are essentially bounded.

5.1. Embedding the problem into $\mathbb{R}^3$.

The state space and attainable sets of the Euler problem have a nontrivial topology, and we start from the embedding of the problem into the Euclidean space. By Theorem 4.1, the attainable set $\mathcal{A} = \mathcal{A}_{q_0}(1)$, $q_0 = (0, 0, 0)$, is contained in the set

$$\widetilde{M} = \operatorname{cl} \mathcal{A} = \{(x, y, \theta) \in M \mid x^2 + y^2 \leq 1\}.$$

Moreover, by item (2) of Corollary 4.1, any trajectory of system (Σ) starting from q_0 does not leave the set $\widetilde{M}$ in the time segment $t \in [0, 1]$. Therefore, this set can be considered as a new state space of the problem. The set $\widetilde{M}$ is embedded into the Euclidean space $\mathbb{R}^3_{x_1x_2x_3}$ by the diffeomorphism

$$\begin{gathered} \Phi : \widetilde{M} \to \mathbb{R}^3_{x_1x_2x_3}, \\ \Phi(x, y, \theta) = (x_1, x_2, x_3) = ((2 + x)\cos\theta,\ (2 + x)\sin\theta, y). \end{gathered} \tag{5.1}$$

The image

$$\Phi(\widetilde{M}) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid (2-\rho)^2 + x_3^2 \le 1\}, \quad \rho = \sqrt{x_1^2 + x_2^2},$$

is the closed solid torus.

In the coordinates (x_1, x_2, x_3), the Euler problem has the form

$$\dot{x}_1 = \frac{x_1^2}{x_1^2 + x_2^2} - ux_2, \tag{5.2}$$

$$\dot{x}_2 = \frac{x_1 x_2}{x_1^2 + x_2^2} + ux_1, \tag{5.3}$$

$$\dot{x}_3 = \frac{x_2}{\sqrt{x_1^2 + x_2^2}}, \tag{5.4}$$

$$x = (x_1, x_2, x_3) \in \Phi(\widetilde{M}), \qquad u \in \mathbb{R}, \tag{5.5}$$

$$x(0) = x^0 = (2, 0, 0), \qquad x(1) = x^1 = (x_1^1, x_2^1, x_3^1), \tag{5.6}$$

$$J = \frac{1}{2} \int_0^1 u^2 dt \to \min, \tag{5.7}$$

$$u(\cdot) \in L_2[0, 1], \qquad x(\cdot) \in AC[0, 1]. \tag{5.8}$$

5.2. **Existence of optimal controls.** First, we recall an appropriate general existence result for control-affine systems from Sec. 11.4.C of the textbook of L. Cesari [12]. Consider an optimal control problem of the form

$$\dot{x} = f(t, x) + \sum_{i=1}^{m} u_i g_i(t, x), \quad x \in X \subset \mathbb{R}^n, \quad u = (u_1, \dots, u_m) \in \mathbb{R}^m, \tag{5.9}$$

$$J = \int_0^{t_1} f_0(t, x, u)\, dt \to \min, \tag{5.10}$$

$$x(\cdot) \in AC([0, t_1], X), \quad u(\cdot) \in L_2([0, t_1], \mathbb{R}^m), \tag{5.11}$$

$$x(0) = x^0, \quad x(t_1) = x^1, \quad t_1 \text{ fixed}. \tag{5.12}$$

For such a problem, a general existence theorem is formulated as follows.

Theorem 5.1 (see [12, Theorem 11.4.VI])**.** *Assume that the following conditions hold*:

(C') *the set X is closed, and the function f_0 is continuous on $[0, t_1] \times X \times \mathbb{R}^m$*;

(L_1) *there exists a real-valued function $\psi(t) \ge 0$, $t \in [0, t_1]$, $\psi \in L_1[0, t_1]$, such that $f_0(t, x, u) \ge -\psi(t)$ for $(t, x, u) \in [0, t_1] \times X \times \mathbb{R}^m$ and almost all t*;

(CL) *the vector fields* $f(t,x)$, $g_1(t,x)$, $\dots$, $g_m(t,x)$ *are continuous on* $[0,t_1] \times X$,
- *the vector fields* $f(t,x)$, $g_1(t,x)$, $\dots$, $g_m(t,x)$ *have bounded components on* $[0,t_1] \times X$,
- *the function* $f_0(t,x,u)$ *is convex in* u *for all* $(t,x) \in [0,t_1] \times X$,
- $x^1 \in \mathcal{A}_{x^0}(t_1)$.

Then there exists an optimal control $u \in L_2([0,t_1], \mathbb{R}^m)$ *for problem* (5.9)–(5.12).

For the Euler problem embedded into $\mathbb{R}^3$ (5.2)–(5.8), we have:
- $m = 1$,
- the set $X = \Phi(\widetilde{M})$ is compact,
- the function $f_0 = u^2$ is continuous, nonnegative, and convex,
- the vector fields $f(x) = \dfrac{x_1^2}{x_1^2+x_2^2}\dfrac{\partial}{\partial x_1} + \dfrac{x_1 x_2}{x_1^2+x_2^2}\dfrac{\partial}{\partial x_2} + \dfrac{x_2}{\sqrt{x_1^2+x_2^2}}\dfrac{\partial}{\partial x_3}$ and $g_1(x) = -x_2\dfrac{\partial}{\partial x_1} + x_1\dfrac{\partial}{\partial x_2}$ are continuous and have bounded components on X,
- $x^1 \in \mathcal{A}_{x^0}(t_1)$ as is supposed in (4.3).

Thus, all assumptions of Theorem 5.1 are satisfied, and there exists an optimal control $u \in L_2[0,t_1]$ for the Euler problem.

5.3. **Boundedness of optimal controls.** One can prove the essential boundedness of optimal control in the Euler elastic problem by virtue of the following general result due to A. Sarychev and D. Torres.

Theorem 5.2 (see [37, Theorem 1 and Corollary 1])**.** *Consider an optimal control problem of the form* (5.9)–(5.12). *Let* $f_0 \in C^1([0,t_1] \times X \times \mathbb{R}^m, \mathbb{R})$, $f, g_i \in C^1([0,t_1] \times X; \mathbb{R}^n)$, $i = 1,\dots,m$, *and* $\varphi(t,x,u) = f(t,x) + \sum_{i=1}^{m} u_i g_i(t,x)$.

Under the assumtions:

$(H1)$ *full rank condition*: $\dim \operatorname{span}(g_1(t,x),\dots,g_m(t,x)) = m$ *for all* $t \in [0,t_1]$ *and* $x \in X$;

$(H2)$ *coercivity*: *there exists a function* $\theta : \mathbb{R} \to \mathbb{R}$ *such that*

$$f_0(t,x,u) \geq \theta(\|u\|) > \zeta \quad \forall (t,x,u) \in [0,t_1] \times X \times \mathbb{R}^m$$

and $\lim\limits_{r \to +\infty} \dfrac{r}{\theta(r)} = 0$;

$(H3)$ *growth condition*: *there exist constants* γ, β, η, *and* μ, *where* $\gamma > 0$, $\beta < 2$, *and* $\mu \geq \max\{\beta - 2, \ -2\}$ *such that for all* $t \in [0,t_1]$, $x \in X$, *and* $u \in \mathbb{R}^m$, *the relation*

$$(|f_{0t}| + |f_{0x_i}| + \|f_0 \varphi_t - f_{0t}\varphi\| + \|f_0 \varphi_{x_i} - f_{0x_i}\varphi\|)\|u\|^\mu \leq \gamma f_0^\beta + \eta, \quad i = 1,\dots,n,$$

holds,

all optimal controls $u(\cdot)$ *of the problem* (5.9)–(5.12) *satisfy the Pontryagin maximum principle, and the normal ones are essentially bounded on* $[0, t_1]$.

It is easy to see that all assumptions of Theorem 5.2 hold:

$(H1)$ $g(x) = -x_2 \dfrac{\partial}{\partial x_1} + x_1 \dfrac{\partial}{\partial x_2} \neq 0$ on X;

$(H2)$ $\theta(r) = r^2$;

$(H3)$ $f_{0t} = f_{0x_i} = \varphi_t = 0$, $\|\varphi_{x_i}\| \leq C$ on X. The required bound $\|f_0 \varphi_{x_i}\| \cdot \|u\|^\mu \leq \gamma f_0^\beta + \eta$ is satisfied for $\beta = 1$, $\mu = 1$, $\gamma = C$, $\eta = 0$.

Thus, in the Euler elastic problem, all optimal controls satisfy the PMP, and the normal ones are in $L_\infty[0, t_1]$. In Sec. 6.2, we show that the abnormal optimal controls are not strictly abnormal (i.e., are simultaneously normal) and, therefore, they are also essentially bounded.

We summarize our results for the Euler elastic problem given in this section. Obviously, we can return back from problem (5.2)–(5.8) in $\mathbb{R}^3_{x_1 x_2 x_3}$ to initial problem (3.4)–(3.9) in $\mathbb{R}^2_{x,y} \times S^1_\theta$.

Theorem 5.3. *Let* $q_1 \in \mathcal{A}_{q_0}(t_1)$.

(1) *Then there exists an optimal control for the Euler problem* (3.4)–(3.9) *in the class* $u(\cdot) \in L_2[0, t_1]$.
(2) *All optimal solutions of the Euler problem satisfy the Pontryagin maximum principle.*
(3) *If an optimal control* $u(\cdot)$ *is normal, then* $u(\cdot) \in L_\infty[0, t_1]$. *The corresponding optimal trajectory* $q(\cdot)$ *is Lipschitzian.*

Certainly, Theorem 5.3 is not the best possible statement on regularity of solutions of the Euler problem. We will deduce from the Pontryagin maximum principle that optimal controls and optimal trajectories are analytic (see Theorem 6.3).

6. Extremals

6.1. **Pontryagin maximum principle.** In order to apply the Pontryagin maximum principle (PMP) in the invariant form, we recall the basic notions of the Hamiltonian formalism [2, 21]. Note that the approach and conclusions of this section have much intersection with the book [21] of V. Jurdjevic.

Let M be a smooth n-dimensional manifold, then its cotangent bundle T^*M is a smooth $2n$-dimensional manifold. The canonical projection $\pi : T^*M \to M$ maps a covector $\lambda \in T^*_q M$ to the base point $q \in M$. The tautological 1-form $s \in \Lambda^1(T^*M)$ on the cotangent bundle is defined as follows. Let $\lambda \in T^*M$ and $v \in T_\lambda(T^*M)$, then $\langle s_\lambda, v \rangle = \langle \lambda, \pi_* v \rangle$ (in coordinates $s = p\,dq$). The canonical symplectic structure on the cotangent bundle $\sigma \in \Lambda^2(T^*M)$ is defined as $\sigma = ds$ (in coordinates $\sigma = dp \wedge dq$). To any Hamiltonian $h \in C^\infty(T^*M)$, there corresponds a Hamiltonian vector field on the cotangent bundle $\vec{h} \in \operatorname{Vec}(T^*M)$ by the rule $\sigma_\lambda(\cdot, \vec{h}) = d_\lambda h$.

Now let $M = \mathbb{R}^2_{x,y} \times S^1_\theta$ be the state space of the Euler problem. Recall that the vector fields

$$X_1 = \cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y}, \quad X_2 = \frac{\partial}{\partial \theta}, \quad X_3 = \sin\theta \frac{\partial}{\partial x} - \cos\theta \frac{\partial}{\partial y}$$

form a basis in the tangent spaces to M. The Lie brackets of these vector fields are given in (3.10), (3.11). Introduce the Hamiltonians linear on fibers of T^*M and corresponding to these basis vector fields:

$$h_i(\lambda) = \langle \lambda, X_i \rangle, \quad \lambda \in T^*M, \quad i = 1, 2, 3,$$

and the family of Hamiltonian functions

$$h_u^\nu(\lambda) = \langle \lambda, X_1 + uX_2 \rangle + \frac{\nu}{2}u^2 = h_1(\lambda) + uh_2(\lambda) + \frac{\nu}{2}u^2,$$
$$\lambda \in T^*M, \quad u \in \mathbb{R}, \quad \nu \in \mathbb{R},$$

the control-dependent Hamiltonian of the PMP for the Euler problem (3.4)–(3.9).

By Theorem 5.3, all optimal solutions of the Euler problem satisfy the Pontryagin maximum principle. We write it in the following invariant form.

Theorem 6.1 (see [2, Theorem 12.3])**.** *Let $u(t)$ and $q(t)$, $t \in [0, t_1]$, be an optimal control and the corresponding optimal trajectory in the Euler problem* (3.4)–(3.9). *Then there exist a curve $\lambda_t \in T^*M$, $\pi(\lambda_t) = q(t)$, $t \in [0, t_1]$, and a number $\nu \leq 0$ such that the following conditions hold for almost all $t \in [0, t_1]$*:

$$\dot\lambda_t = \vec{h}^\nu_{u(t)}(\lambda_t) = \vec{h}_1(\lambda_t) + u(t)\vec{h}_2(\lambda_t), \tag{6.1}$$
$$h^\nu_{u(t)}(\lambda_t) = \max_{u \in \mathbb{R}} h^\nu_u(\lambda_t), \tag{6.2}$$
$$(\nu, \lambda_t) \neq 0. \tag{6.3}$$

Using the coordinates $(h_1, h_2, h_3, x, y, \theta)$, we can write the Hamiltonian system of the PMP (6.1) as follows:

$$\dot h_1 = -uh_3, \tag{6.4}$$
$$\dot h_2 = h_3, \tag{6.5}$$
$$\dot h_3 = uh_1, \tag{6.6}$$
$$\dot x = \cos\theta, \tag{6.7}$$
$$\dot y = \sin\theta, \tag{6.8}$$
$$\dot\theta = u. \tag{6.9}$$

Note that the subsystem for the vertical coordinates (h_1, h_2, h_3) (6.4)–(6.6) is independent of the horizontal coordinates (x, y, θ); this is a consequence of the left-invariant symmetry of system (Σ) and of an appropriate choice of the coordinates (h_1, h_2, h_3) (see [2]).

As usual, the constant parameter ν can be either zero (the abnormal case), or negative (the normal case, then one can normalize $\nu = -1$).

6.2. **Abnormal extremals.** First, consider the *abnormal case*: let $\nu = 0$. The maximality condition of PMP (6.2) has the form:

$$h_u^\nu(\lambda) = h_1(\lambda) + u h_2(\lambda) \to \max_{u \in \mathbb{R}}, \tag{6.10}$$

and, therefore, $h_2(\lambda_t) \equiv 0$ along an abnormal extremal λ_t. Then Eq. (6.5) yields $h_3(\lambda_t) \equiv 0$, and Eq. (6.6) gives $u(t)h_1(\lambda_t) \equiv 0$. But in view of the nontriviality condition of the PMP (6.3), we have $h_1(\lambda_t) \neq 0$ and, therefore, $u(t) \equiv 0$. Thus, abnormal extremal controls in the Euler problem are identically zero. Note that these controls are singular since they are not uniquely defined by the maximality condition of the PMP (6.10).

Now we find the abnormal extremal trajectories. For $u \equiv 0$, the horizontal equations (6.7)–(6.9) have the form

$$\dot{q} = X_1(q) \quad \Leftrightarrow \quad \begin{cases} \dot{x} = \cos\theta, \\ \dot{y} = \sin\theta, \\ \dot{\theta} = 0, \end{cases}$$

and the initial condition $(x, y, \theta)(0) = (0, 0, 0)$ gives

$$x(t) = t, \quad y(t) \equiv 0, \quad \theta(t) \equiv 0.$$

The abnormal extremal trajectory through $q_0 = \mathrm{Id}$ is the one-parameter subgroup of the Lie group $\mathrm{E}(2)$ corresponding to the left-invariant field X_1. It is projected on the straight line $(x, y) = (t, 0)$ in the plane (x, y). The corresponding elastica is a straight line segment — the elastic rod without any external forces applied. This is the trajectory connecting q_0 with the only attainable point q_1 at the boundary of the attainable set $\mathcal{A}_{q_0}(t_1)$.

For $u \equiv 0$, the elastic energy is $J = 0$, the absolute minimum. Therefore, the abnormal extremal trajectory $q(t)$, $t \in [0, t_1]$, is optimal; it gives an optimal solution for the boundary conditions $q_0 = (0, 0, 0)$, $q_1 = (t_1, 0, 0)$.

Combining the description of abnormal controls just obtained with Theorem 5.3, we obtain the following statement.

Theorem 6.2. *For any $q_1 \in \mathcal{A}_{q_0}(t_1)$, the corresponding optimal control for the Euler problem* (3.4)–(3.9) *is essentially bounded.*

6.3. **Normal case.** Now let $\nu = -1$. The maximality condition of the PMP (6.2) has the form

$$h_u^{-1} = h_1 + u h_2 - \frac{1}{2}u^2 \to \max_{u \in \mathbb{R}},$$

whence $\dfrac{\partial h_u^{-1}}{\partial u} = h_2 - u = 0$ and

$$u = h_2. \tag{6.11}$$

The corresponding normal Hamiltonian of the PMP is $H = h_1 + \dfrac{1}{2}h_2^2$, and the *normal Hamiltonian system of the PMP* has the form

$$\dot{\lambda} = \vec{H}(\lambda) \quad \Leftrightarrow \quad \begin{cases} \dot{h}_1 = -h_2 h_3, \\ \dot{h}_2 = h_3, \\ \dot{h}_3 = h_1 h_2, \\ \dot{q} = X_1 + h_2 X_2. \end{cases} \tag{6.12}$$

This system is analytic and, therefore, we obtain the following statement (taking into account the analyticity in the abnormal case, see Sec. 6.2).

Theorem 6.3. *All extremal* (*in particular, optimal*) *controls and trajectories of the Euler problem are real-analytic.*

Note that the vertical subsystem of the Hamiltonian system (6.12) admits a particular solution $(h_1, h_2, h_3) \equiv (0, 0, 0)$ with the corresponding normal control $u = h_2 \equiv 0$. Thus, abnormal extremal trajectories are simultaneously normal, i.e., they are not strictly abnormal.

We define the *exponential mapping* for the problem

$$\operatorname{Exp}_{t_1} : T^*_{q_0} M \to M, \quad \operatorname{Exp}_{t_1}(\lambda_0) = \pi \circ e^{t_1 \vec{H}}(\lambda_0) = q(t_1).$$

The vertical subsystem of system (6.12) has an obvious integral:

$$h_1^2 + h_3^2 \equiv r^2 = \text{const} \geq 0,$$

and it is natural to introduce the polar coordinates

$$h_1 = r\cos\alpha, \qquad h_3 = r\sin\alpha.$$

Then the normal Hamiltonian system (6.12) takes the following form:

$$\begin{aligned} &\dot{\alpha} = h_2, \quad \dot{h}_2 = r\sin\alpha, \quad \dot{r} = 0, \\ &\dot{x} = \cos\theta, \quad \dot{y} = \sin\theta, \quad \dot{\theta} = h_2. \end{aligned} \tag{6.13}$$

The vertical subsystem of the Hamiltonian system (6.13) is reduced to the equation

$$\ddot{\alpha} = r\sin\alpha.$$

In the coordinates

$$c = h_2, \quad \beta = \alpha + \pi$$

we obtain the *equation of the pendulum*

$$\dot{\beta} = c, \quad \dot{c} = -r\sin\beta$$

known as the *Kirchhoff kinetic analog* of elasticae. Note the physical meaning of the constant:

$$r = \frac{g}{L}, \tag{6.14}$$

where g is the gravitational acceleration and L is the length of the suspension of the pendulum.

The total energy of the pendulum is

$$E = \frac{c^2}{2} - r\cos\beta \in [-r, +\infty), \tag{6.15}$$

this is just the Hamiltonian H.

The equation of the mathematical pendulum is integrable in elliptic functions. Consequently, the whole Hamiltonian system (6.13) is integrable in quadratures: one can integrate first the vertical subsystem

$$\begin{cases} \dot h_1 = -h_2 h_3, \\ \dot h_2 = h_3, \\ \dot h_3 = h_1 h_2 \end{cases} \qquad \Leftrightarrow \qquad \begin{cases} \dot\beta = c, \\ \dot c = -r\sin\beta, \\ \dot r = 0 \end{cases} \tag{6.16}$$

then the equation for θ, and then the equations for x, y. In Sec. 8, we find an explicit parametrization of the normal extremals by the Jacobi elliptic functions in terms of natural coordinates in the phase space of pendulum (6.16).

First, we apply continuous symmetries of the problem. The normal Hamiltonian vector field has the form

$$\begin{aligned} \vec H &= -h_2 h_3 \frac{\partial}{\partial h_1} + h_3 \frac{\partial}{\partial h_2} + h_1 h_2 \frac{\partial}{\partial h_3} + \cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y} + h_2 \frac{\partial}{\partial \theta} \\ &= h_2 \frac{\partial}{\partial \alpha} + r\sin\alpha \frac{\partial}{\partial h_2} + \cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y} + h_2 \frac{\partial}{\partial \theta}. \end{aligned}$$

The Hamiltonian system (6.13) is preserved by the one-parameter group of transformations

$$(\alpha, r, h_2, x, y, \theta, t) \mapsto (\alpha, re^{-2s}, h_2 e^{-s}, xe^s, ye^s, \theta, te^s) \tag{6.17}$$

obtained by the continuation of the group of dilations of the plane $\mathbb{R}^2_{x,y}$ (3.13) to the vertical coordinates.

The one-parameter group (6.17) is generated by the vector field

$$Z = -2r\frac{\partial}{\partial r} - h_2 \frac{\partial}{\partial h_2} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}.$$

We have the Lie bracket and Lie derivatives

$$[Z, \vec H] = -\vec H, \tag{6.18}$$

$$Zr = -2r, \quad Zh_2 = -h_2, \quad \vec H r = 0, \quad \vec H h_2 = r\sin\alpha. \tag{6.19}$$

The infinitesimal symmetry Z of the Hamiltonian field $\vec{H}$ is integrated to the symmetry at the level of flows:

$$e^{t'\vec{H}} \circ e^{sZ}(\lambda) = e^{sZ} \circ e^{t\vec{H}}(\lambda), \quad t' = e^s t, \quad \lambda \in T^*M.$$

The following decomposition of the preimage of the exponential mapping N into invariant subsets of the fields $\vec{H}$ and Z will be very important in the sequel:

$$T^*_{q_0}M = N = \bigcup_{i=1}^{7} N_i, \tag{6.20}$$

$$N_1 = \{\lambda \in N \mid r \neq 0,\ E \in (-r, r)\}, \tag{6.21}$$

$$N_2 = \{\lambda \in N \mid r \neq 0,\ E \in (r, +\infty)\} = N_2^+ \cup N_2^-, \tag{6.22}$$

$$N_3 = \{\lambda \in N \mid r \neq 0,\ E = r,\ \beta \neq \pi\} = N_3^+ \cup N_3^-, \tag{6.23}$$

$$N_4 = \{\lambda \in N \mid r \neq 0,\ E = -r\}, \tag{6.24}$$

$$N_5 = \{\lambda \in N \mid r \neq 0,\ E = r,\ \beta = \pi\}, \tag{6.25}$$

$$N_6 = \{\lambda \in N \mid r = 0,\ c \neq 0\} = N_6^+ \cup N_6^-, \tag{6.26}$$

$$N_7 = \{\lambda \in N \mid r = c = 0\}, \tag{6.27}$$

$$N_i^{\pm} = N_i \cup \{\lambda \in N \mid \operatorname{sgn} c = \pm 1\}, \quad i = 2, 3, 6. \tag{6.28}$$

Any cylinder $\{\lambda \in N \mid r = \text{const} \neq 0\}$ can be transformed to the cylinder $C = \{\lambda \in N \mid r = 1\}$ by the dilation Z; the corresponding decomposition of the phase space of the *standard pendulum*

$$\dot{\beta} = c, \quad \dot{c} = -\sin\beta, \quad (\beta, c) \in C = S^1_\beta \times \mathbb{R}_c,$$

is shown in Fig. 3, where

$$C_i = N_i \cap \{r = 1\}, \quad i = 1, \dots, 5.$$

In order to integrate the normal Hamiltonian system

$$\dot{\lambda} = \vec{H}(\lambda),$$

i.e.,

$$\begin{aligned} &\dot{\beta} = c, \quad \dot{c} = -r\sin\beta, \quad \dot{r} = 0, \\ &\dot{x} = \cos\theta, \quad \dot{y} = \sin\theta, \quad \dot{\theta} = c, \end{aligned} \tag{6.29}$$

we consider natural coordinates in the phase space of the pendulum.

7. Elliptic coordinates

7.1. Time of motion of the pendulum. *Elliptic coordinates* in the phase cylinder of the standard pendulum

$$\dot{\beta} = c, \quad \dot{c} = -\sin\beta, \quad (\beta, c) \in C = S^1_\beta \times \mathbb{R}_c, \tag{7.1}$$

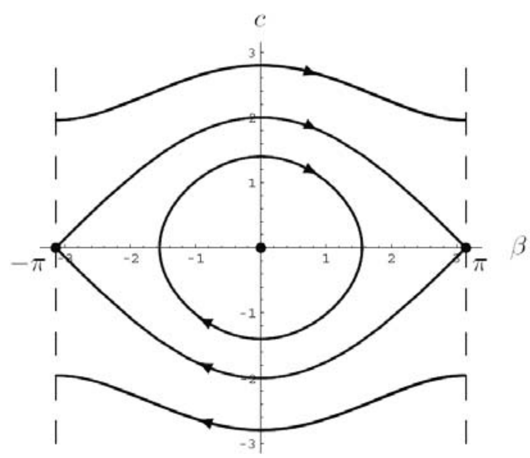

Fig. 2. Phase portrait of the pendulum.

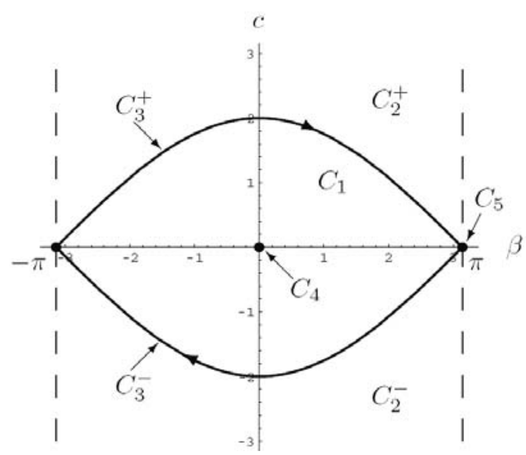

Fig. 3. Decomposition of the phase cylinder of the pendulum.

were introduced in [30] for the integration and study of the nilpotent sub-Riemannian problem with the growth vector $(2, 3, 5)$. Here we propose a more natural and efficient construction of these coordinates.

Denote

$$P = \mathbb{R}_{+c} \times \mathbb{R}_t, \quad \widehat{C} = C_1 \cup C_2^+ \cup C_3^+,$$

and consider the mapping

$$\Phi : P \to \widehat{C}, \quad \Phi : (c, t) \mapsto (\beta_t, c_t),$$

where (β_t, c_t) is the solution of the equation of pendulum (7.1) with the initial condition

$$\beta_0 = 0, \quad c_0 = c. \tag{7.2}$$

The mapping $\Phi : P \to \widehat{C}$ is real-analytic since the equation of pendulum (7.1) is a real-analytic ODE.

It is easy to show that Φ is a local diffeomorphism, see details in [34].

Denote by

$$k_1 = \sqrt{\frac{E+1}{2}} = \sqrt{\sin^2\frac{\beta}{2} + \frac{c^2}{4}} \in (0,1), \quad (\beta, c) \in C_1,$$
$$k_2 = \sqrt{\frac{2}{E+1}} = \frac{1}{\sqrt{\sin^2\frac{\beta}{2} + \frac{c^2}{4}}} \in (0,1), \quad (\beta, c) \in C_2,$$

a reparametrized energy ($E = \frac{c^2}{2} - \cos\beta$) of the standard pendulum; below k_1, k_2 play the role of the modulus for the Jacobi elliptic functions, and

$$K(k) = \int_0^{\pi/2} \frac{dt}{\sqrt{1 - k^2 \sin^2 t}}, \quad k \in (0,1),$$

is the complete elliptic integral of the first kind, see [22, 34]. It is well known [22] that the standard pendulum (7.1) has the following period of motion T depending on its energy E:

$$-1 < E < 1 \quad \Leftrightarrow \quad (\beta, c) \in C_1 \quad \Rightarrow \quad T = 4K(k_1), \tag{7.3}$$
$$E = 1,\ \beta \neq \pi \quad \Leftrightarrow \quad (\beta, c) \in C_3 \quad \Rightarrow \quad T = \infty, \tag{7.4}$$
$$E > 1 \quad \Leftrightarrow \quad (\beta, c) \in C_2 \quad \Rightarrow \quad T = 2K(k_2)k_2. \tag{7.5}$$

Introduce the equivalence relation $\sim$ in the domain P as follows. For $(c_1, t_1) \in P$ and $(c_2, t_2) \in P$, we set $(c_1, t_1) \sim (c_2, t_2)$ iff $c_1 = c_2 = c$ and

$$\begin{aligned}
&t_2 = t_1 \pmod{4K(k_1)}, \quad k_1 = \frac{c}{2} && \text{for } c \in (0,2),\\
&t_2 = t_1 && \text{for } c = 2,\\
&t_2 = t_1 \pmod{2K(k_2)k_2}, \quad k_2 = \frac{2}{c} && \text{for } c \in (2, +\infty).
\end{aligned}$$

That is, we identify the points (c, t_1), (c, t_2) iff the corresponding solutions of the equation of pendulum (7.1) with the initial condition (7.2) give the same point (β_t, c_t) in the phase cylinder of the pendulum $S^1_\beta \times \mathbb{R}_c$ at the instants t_1 and t_2.

Denote the quotient $\widetilde{P} = P/\sim$. In view of the periodicity properties (7.3)–(7.5) of pendulum (7.1), the mapping

$$\Phi : \widetilde{P} \to \widehat{C}, \quad \Phi(c, t) = (\beta_t, c_t),$$

is a global analytic diffeomorphism. Thus, there exists the inverse mapping, an analytic diffeomorphism

$$F : \widehat{C} \to \widetilde{P}, \quad F(\beta, c) = (c_0, \varphi), \tag{7.6}$$

where φ is the time of motion of the pendulum in the reverse time from a point $(\beta, c) \in \widehat{C}$ to the semi-axis $\{\beta = 0,\ c = 0\}$. In the domains C_1 and C_2^+, the time φ is defined modulo the period of the pendulum $4K(k_1)$ and $2K(k_2)k_2$, respectively. We summarize the above construction in the following proposition.

Theorem 7.1. *There exists an analytic multivalued function*

$$\varphi : \widehat{C} = C_1 \cup C_2^+ \cup C_3^+ \to \mathbb{R}$$

such that for $\beta_0 = 0$*,* $c_0 > 0$*, and the corresponding solution* (β_t, c_t) *of the Cauchy problem* (7.1)*,* (7.2)*, the equality*

$$\varphi(\beta_t, c_t) = \begin{cases} t \pmod{4K(k_1)} & \textit{for } (\beta_t, c_t) \in C_1, \\ t & \textit{for } (\beta_t, c_t) \in C_2^+, \\ t \pmod{2K(k_2)k_2} & \textit{for } (\beta_t, c_t) \in C_3^+ \end{cases}$$

holds.

In other words, $\varphi(\beta_t, c_t)$ is the time of motion of the pendulum in the reverse time from the point $(\beta_t, c_t) \in \widehat{C}$ to the semi-axis $\{\beta = 0,\ c > 0\}$.

7.2. **Elliptic coordinates in the phase space of the pendulum.** In the domain $C_1 \cup C_2 \cup C_3$, we introduce the *elliptic coordinates* (φ, k), where φ is the time of motion of the pendulum from the semi-axis $\{\beta = 0,\ c > 0\}$ (in the domain $\widehat{C} = C_1 \cup C_2^+ \cup C_3^+$) or from the semi-axis $\{\beta = 0,\ c < 0\}$ (in the domain $\widetilde{C} = C_2^- \cup C_3^-$), and $k \in (0, 1)$ is a reparametrized energy of the pendulum — the modulus of the Jacobi elliptic functions.

7.2.1. *Elliptic coordinates in* C_1. If $(\beta, c) \in C_1$, then we set

$$\sin\frac{\beta}{2} = k_1 \operatorname{sn}(\varphi, k_1), \quad \frac{c}{2} = k_1 \operatorname{cn}(\varphi, k_1), \quad \cos\frac{\beta}{2} = \operatorname{dn}(\varphi, k_1), \tag{7.7}$$

$$k_1 = \sqrt{\frac{E+1}{2}} = \sqrt{\sin^2\frac{\beta}{2} + \frac{c^2}{4}} \in (0, 1), \tag{7.8}$$

$$\varphi \pmod{4K(k_1)} \in [0, 4K(k_1)].$$

Here and below, cn, sn, and dn are the Jacobi elliptic functions (see [22,34]).

The function φ defined in such way is indeed the time of motion of the pendulum from the semi-axis $\{\beta = 0,\ c > 0\}$ in view of the following:

$$(\beta = 0,\ c > 0) \quad \Rightarrow \quad \varphi = 0, \tag{7.9}$$

$$\left.\frac{d\varphi}{dt}\right|_{\text{Eq. (7.1)}} = 1, \tag{7.10}$$

the total derivative with respect to the equation of pendulum (7.1).

The mapping $(\beta, c) \mapsto (k, \varphi)$ is an analytic diffeomorphism since it is decomposed into the chain of analytic diffeomorphisms:

$$(\beta, c) \overset{(a)}{\mapsto} (c_0, \varphi) \overset{(b)}{\mapsto} (k_1, \varphi),$$

where (a) is defined by F (7.6), while (b) is given by

$$k_1 = \sqrt{\frac{E+1}{2}} = \frac{c_0}{2}$$

(cf. (7.8)).

7.2.2. *Elliptic coordinates in C_2^+.* Let $(\beta, c) \in C_2^+$. The elliptic coordinates (φ, k_1) in the domain C_2^+ are analytic functions (φ, k_1) defined as follows: φ is the time of motion of the pendulum from the semi-axis $\{\beta = 0,\ c > 0\}$, and $k_1 = \dfrac{E+1}{2}$. By the uniqueness theorem for analytic functions, in the domain C_2^+ we have the same formulas as in C_1:

$$\sin\frac{\beta}{2} = k_1 \operatorname{sn}(\varphi, k_1), \tag{7.11}$$

$$\frac{c}{2} = k_1 \operatorname{cn}(\varphi, k_1), \tag{7.12}$$

$$\cos\frac{\beta}{2} = \operatorname{dn}(\varphi, k_1), \tag{7.13}$$

$$k_1 = \sqrt{\frac{E+1}{2}} \in (1, +\infty).$$

Here the Jacobi elliptic functions $\operatorname{sn}(u, k_1)$, $\operatorname{cn}(u, k_1)$, and $\operatorname{dn}(u, k_1)$ for the modulus $k_1 > 1$ are obtained from the corresponding functions in the normal case $k_1 \in (0, 1)$ by the analytic continuation along the complex modulus $k_1 \in \mathbb{C}$ through the complex plane around the singularity $k_1 = 1$, see Sec. 3.9 and Sec. 8.14 [22]. In order to obtain the Jacobi functions with the modulus in the interval $(0, 1)$, we apply the transformation of modulus $k \mapsto \dfrac{1}{k}$ by the formulas

$$\operatorname{sn}\left(u, \frac{1}{k}\right) = k \operatorname{sn}\left(\frac{u}{k}, k\right), \qquad \operatorname{cn}\left(u, \frac{1}{k}\right) = \operatorname{dn}\left(\frac{u}{k}, k\right), \tag{7.14}$$

$$\operatorname{dn}\left(u, \frac{1}{k}\right) = \operatorname{cn}\left(\frac{u}{k}, k\right), \qquad \operatorname{E}\left(u, \frac{1}{k}\right) = \frac{1}{k} \operatorname{E}\left(\frac{u}{k}, k\right) - \frac{1-k^2}{k^2} u \tag{7.15}$$

(see [22]). Transforming Eqs. (7.11)–(7.13) via formulas (7.14) and (7.15), we obtain the following expressions for elliptic coordinates (φ, k_2):

$$\sin\frac{\beta}{2} = \operatorname{sn}\left(\frac{\varphi}{k_2}, k_2\right), \quad \frac{c}{2} = \frac{1}{k_2} \operatorname{dn}\left(\frac{\varphi}{k_2}, k_2\right), \quad \cos\frac{\beta}{2} = \operatorname{cn}\left(\frac{\varphi}{k_2}, k_2\right), \tag{7.16}$$

$$k_2 = \frac{1}{k_1} = \sqrt{\frac{2}{E+1}} \in (0,1).$$

Certainly, one can directly verify that φ is indeed the time of motion of the standard pendulum from the point (β, c) to the semi-axis $\{\beta = 0,\ c > 0\}$ in the reverse time by verifying conditions (7.9) and (7.10) in the domain C_2^+, but our idea is to obtain “for free” equalities in C_2^+ from equalities in C_1 via the transformation of the modulus $k \mapsto 1/k$.

7.2.3. *Elliptic coordinates in C_3^+.* Let $(\beta, c) \in C_3^+$. Elliptic coordinates on the set C_3^+ are given by $(\varphi, k = 1)$, where φ is the time of motion of the pendulum from the semi-axis $\{\beta = 0,\ c > 0\}$, and $k = \sqrt{\frac{E+1}{2}} = 1$. The analytic expressions for φ are obtained by passing to the limit $k_1 \to 1 - 0$ in formulas (7.7) or to the limit $k_2 \to 1 - 0$ in formulas (7.16), with the use of formulas of degeneration of elliptic functions [22, 34]. As a result of the both limit passages, we obtain the following expression for the elliptic coordinate φ on the set C_3^+:

$$\sin\frac{\beta}{2} = \tanh\varphi, \quad \frac{c}{2} = \frac{1}{\cosh\varphi}, \quad \cos\frac{\beta}{2} = \frac{1}{\cosh\varphi}.$$

7.2.4. *Elliptic coordinates in $C_2^- \cup C_3^-$.* For a point $(\beta, c) \in \widetilde{C} = C_2^- \cup C_3^-$, the elliptic coordinates (φ, k) cannot be defined in the same way as in $\widehat{C} = C_1 \cup C_2^+ \cup C_3^+$ since such a point is not attainable along the flow of pendulum (7.1) from the semi-axis $\{\beta = 0,\ c > 0\}$, see the phase portrait in Fig. 2. Now we take the initial semi-axis $\{\beta = 0,\ c < 0\}$, and define φ in $\widetilde{C}$ equal to the time of motion of the pendulum from this semi-axis to the current point. That is, for points $(\beta, c) \in \widetilde{C}$ we consider the mapping

$$F(c,t) = (\beta_t, c_t), \quad c < -2,$$
$$\beta_0 = 0, \quad c_0 = c,$$

and construct the inverse mapping

$$\Phi(\beta, c) = (c_0, \varphi).$$

Pendulum (7.1) has an obvious symmetry — reflection with respect to the origin $(\beta = 0,\ c = 0)$:

$$i : (\beta, c) \mapsto (-\beta, -c). \tag{7.17}$$

In view of this symmetry, we obtain:

$$\Phi(\beta, c) = (c_0, \varphi), \qquad (\beta, c) \in C_2^- \cup C_3^-,$$
$$\Phi(-\beta, -c) = (-c_0, \varphi), \qquad (-\beta, -c) \in C_2^+ \cup C_3^+,$$

and, therefore,

$$\varphi(\beta, c) = \varphi(-\beta, -c), \quad (\beta, c) \in C_2^- \cup C_3^-.$$

On the other hand, the energy of the pendulum E and the modulus of elliptic functions k_2 are preserved by reflection (7.17). Therefore, we have the following formulas for elliptic functions in $\widetilde{C}$.

$$(\beta, c) \in C_2^- \quad \Rightarrow \quad \begin{cases} \sin \dfrac{\beta}{2} = -\operatorname{sn}\left(\dfrac{\varphi}{k_2}, k_2\right), \\ \dfrac{c}{2} = -\dfrac{1}{k_2} \operatorname{dn}\left(\dfrac{\varphi}{k_2}, k_2\right), \\ \cos \dfrac{\beta}{2} = \operatorname{cn}\left(\dfrac{\varphi}{k_2}, k_2\right), \end{cases}$$

$$(\beta, c) \in C_3^- \quad \Rightarrow \quad \begin{cases} \sin \dfrac{\beta}{2} = -\tanh \varphi, \\ \dfrac{c}{2} = -\dfrac{1}{\cosh \varphi}, \\ \cos \dfrac{\beta}{2} = \dfrac{1}{\cosh \varphi}. \end{cases}$$

Summing up, in the domain $C_1 \cup C_2 \cup C_3$ the elliptic coordinates (φ, k) are defined as follows:

$$(\beta, c) \in C_1 \quad \Rightarrow \quad \begin{cases} \sin \dfrac{\beta}{2} = k_1 \operatorname{sn}(\varphi, k_1), \\ \dfrac{c}{2} = k_1 \operatorname{cn}(\varphi, k_1), \\ \cos \dfrac{\beta}{2} = \operatorname{dn}(\varphi, k_1), \end{cases}$$

$$k_1 = \sqrt{\frac{E+1}{2}} \in (0,1), \quad \varphi \pmod{4K(k_1)} \in [0, 4K(k_1)],$$

$$(\beta, c) \in C_2^\pm \quad \Rightarrow \quad \begin{cases} \sin \dfrac{\beta}{2} = \pm \operatorname{sn}\left(\dfrac{\varphi}{k_2}, k_2\right), \\ \dfrac{c}{2} = \pm\dfrac{1}{k_2} \operatorname{dn}\left(\dfrac{\varphi}{k_2}, k_2\right), \\ \cos \dfrac{\beta}{2} = \operatorname{cn}\left(\dfrac{\varphi}{k_2}, k_2\right), \end{cases}$$

$$k_2 = \sqrt{\frac{2}{E+1}} \in (0,1), \quad \varphi \pmod{2K(k_2)k_2} \in [0, 2K(k_2)k_2], \quad \pm = \operatorname{sgn} c,$$

$$(\beta, c) \in C_3^\pm \quad \Rightarrow \quad \begin{cases} \sin \dfrac{\beta}{2} = \pm \tanh \varphi, \\ \dfrac{c}{2} = \pm\dfrac{1}{\cosh \varphi}, \\ \cos \dfrac{\beta}{2} = \dfrac{1}{\cosh \varphi}, \end{cases}$$

$$k = 1, \quad \varphi \in \mathbb{R}, \quad \pm = \operatorname{sgn} c.$$

A grid of elliptic coordinates in the phase cylinder of the standard pendulum $(\mathbb{R}_{+c} \times S^1_\beta)$ is plotted in [34].

7.3. **Elliptic coordinates in the preimage of the exponential mapping.** In the domain $\widehat{N} = N_1 \cup N_2 \cup N_3$ (recall decomposition (6.20)–(6.28)), the vertical subsystem of the Hamiltonian system (6.12) has the form of the *generalized pendulum*

$$\dot\beta = c, \quad \dot c = -r\sin\beta, \quad \dot r = 0. \tag{7.18}$$

Elliptic coordinates in the domain $\widehat{N}$ have the form (φ, k, r). On the set $N_1 \cup N_2^+ \cup N_3^+$, the coordinate φ is equal to the time of motion of the generalized pendulum (7.18) from a point $(\beta = 0, c = c_0 > 0, r)$ to a point (β, c, r), while on the set $N_2^- \cup N_3^-$ the time of motion is taken from a point $(\beta = 0, c = c_0 < 0, r)$.

The one-parameter group of symmetries $(\beta, c, r, t) \mapsto (\beta, ce^{-s}, re^{-2s}, te^s)$ of the generalized pendulum (7.18) is a restriction of the action of group (6.17). We apply this group to transform the generalized pendulum (7.18) in the domain $\{r > 0\}$ to the standard pendulum (7.1) for $r = 1$. This transformation preserves the integral of the generalized pendulum

$$k_1 = \sqrt{\frac{E+r}{2r}} = \sqrt{\sin^2\frac{\beta}{2} + \frac{c^2}{4r}}.$$

Thus, we obtain the following expressions for elliptic coordinates in the domain $\widehat{N}$ from similar expressions in the domain $\widehat{C}$ (see Sec. 7.2):

$$\lambda = (\beta, c, r) \in N_1 \quad \Rightarrow \quad \begin{cases} \sin\dfrac{\beta}{2} = k_1 \operatorname{sn}(\sqrt{r}\varphi, k_1), \\ \dfrac{c}{2} = k_1\sqrt{r}\operatorname{cn}(\sqrt{r}\varphi, k_1), \\ \cos\dfrac{\beta}{2} = \operatorname{dn}(\sqrt{r}\varphi, k_1), \end{cases}$$

$$k_1 = \sqrt{\frac{E+r}{2r}} \in (0,1), \quad \sqrt{r}\varphi \pmod{4K(k_1)} \in [0, 4K(k_1)],$$

$$\lambda = (\beta, c, r) \in N_2^\pm \quad \Rightarrow \quad \begin{cases} \sin\dfrac{\beta}{2} = \pm\operatorname{sn}\left(\dfrac{\sqrt{r}\varphi}{k_2}, k_2\right), \\ \dfrac{c}{2} = \pm\dfrac{\sqrt{r}}{k_2}\operatorname{dn}\left(\dfrac{\sqrt{r}\varphi}{k_2}, k_2\right), \\ \cos\dfrac{\beta}{2} = \operatorname{cn}\left(\dfrac{\sqrt{r}\varphi}{k_2}, k_2\right), \end{cases}$$

$$k_2 = \sqrt{\frac{2r}{E+r}} \in (0,1), \ \sqrt{r}\varphi \pmod{2K(k_2)k_2} \in [0, 2K(k_2)k_2], \ \pm = \operatorname{sgn} c,$$

$$\lambda = (\beta, c, r) \in N_3^\pm \quad \Rightarrow \quad \begin{cases} \sin\dfrac{\beta}{2} = \pm\tanh(\sqrt{r}\varphi), \\ \dfrac{c}{2} = \pm\dfrac{\sqrt{r}}{\cosh(\sqrt{r}\varphi)}, \\ \cos\dfrac{\beta}{2} = \dfrac{1}{\cosh(\sqrt{r}\varphi)}, \end{cases}$$

$$k = 1, \quad \varphi \in \mathbb{R}, \quad \pm = \operatorname{sgn} c.$$

In the domain N_2, it will also be convenient to use the coordinates (k_2, ψ, r), where

$$\psi = \frac{\varphi}{k_2}, \quad \sqrt{r}\psi \pmod{2K(k_2)} \in [0, 2K(k_2)].$$

In computations, if this does not lead to an ambiguity, we denote the both moduli of the Jacobi functions k_1 and k_2 by k, noting that $k \in (0, 1)$; this is the normal case in the theory of the Jacobi elliptic functions (see [22]).

8. Integration of the normal Hamiltonian system

8.1. **Integration of the vertical subsystem.** In the elliptic coordinates (φ, k, r) in the domain $\widehat{N}$, the vertical subsystem (7.18) of the normal Hamiltonian system $\dot{\lambda} = \vec{H}(\lambda)$ is rectified:

$$\dot{\varphi} = 1, \quad \dot{k} = 0, \quad \dot{r} = 0,$$

and, therefore, has the solutions

$$\varphi_t = \varphi + t, \quad k = \text{const}, \quad r = \text{const}.$$

Then expressions for the vertical coordinates (β, c, r) are immediately given by the formulas for elliptic coordinates found in Sec. 7.3. For $\lambda \in N \setminus \widehat{N}$, the vertical subsystem is degenerated and is easily integrated. Thus, we obtain the following description of the solution (β_t, c_t, r) of the vertical subsystem (7.18) with the initial condition $(\beta_t, c_t, r)|_{t=0} = (\beta, c, r)$:

$$\lambda \in N_1 \quad \Rightarrow \quad \begin{cases} \sin\dfrac{\beta_t}{2} = k_1 \operatorname{sn}(\sqrt{r}\varphi_t), \\ \cos\dfrac{\beta_t}{2} = \operatorname{dn}(\sqrt{r}\varphi_t), \\ \dfrac{c_t}{2} = k_1\sqrt{r}\operatorname{cn}(\sqrt{r}\varphi_t), \end{cases}$$

$$\lambda \in N_2^{\pm} \quad \Rightarrow \quad \begin{cases} \sin\dfrac{\beta_t}{2} = \pm\operatorname{sn}\left(\dfrac{\sqrt{r}\varphi_t}{k}\right), \\ \cos\dfrac{\beta_t}{2} = \operatorname{cn}\left(\dfrac{\sqrt{r}\varphi_t}{k}\right), \\ \dfrac{c_t}{2} = \pm\dfrac{\sqrt{r}}{k}\operatorname{dn}\left(\dfrac{\sqrt{r}\varphi_t}{k}\right), \end{cases}$$

$$\lambda \in N_3^{\pm} \quad \Rightarrow \quad \begin{cases} \sin\dfrac{\beta_t}{2} = \pm\tanh(\sqrt{r}\varphi_t), \\ \cos\dfrac{\beta_t}{2} = \dfrac{1}{\cosh(\sqrt{r}\varphi_t)}, \\ \dfrac{c_t}{2} = \pm\dfrac{\sqrt{r}}{\cosh(\sqrt{r}\varphi_t)}, \end{cases}$$

$$\lambda \in N_4 \quad \Rightarrow \quad \beta_t \equiv 0, \quad c_t \equiv 0,$$

$$\lambda \in N_5 \quad \Rightarrow \quad \beta_t \equiv \pi, \quad c_t \equiv 0,$$
$$\lambda \in N_6 \quad \Rightarrow \quad \beta_t = ct + \beta, \quad c_t \equiv c,$$
$$\lambda \in N_7 \quad \Rightarrow \quad c_t \equiv 0, \quad r \equiv 0.$$

8.2. **Integration of the horizontal subsystem.** The Cauchy problem for the horizontal variables (x, y, θ) of the normal Hamiltonian system (6.29) has the form

$$\dot{x} = \cos\theta = 2\cos^2\frac{\theta}{2} - 1, \qquad x_0 = 0,$$
$$\dot{y} = \sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}, \qquad y_0 = 0,$$
$$\dot{\theta} = c = \dot{\beta}, \qquad \theta_0 = 0,$$

and, therefore,

$$\theta_t = \beta_t - \beta. \tag{8.1}$$

We apply known formulas for integrals of the Jacobi elliptic functions (see [22, 34]) and obtain the following parametrization of normal extremal trajectories.

If $\lambda \in N_1$, then

$$\sin\frac{\theta_t}{2} = k\,\mathrm{dn}(\sqrt{r}\varphi)\,\mathrm{sn}(\sqrt{r}\varphi_t) - k\,\mathrm{sn}(\sqrt{r}\varphi)\,\mathrm{dn}(\sqrt{r}\varphi_t),$$
$$\cos\frac{\theta_t}{2} = \mathrm{dn}(\sqrt{r}\varphi)\,\mathrm{dn}(\sqrt{r}\varphi_t) + k^2\,\mathrm{sn}(\sqrt{r}\varphi)\,\mathrm{sn}(\sqrt{r}\varphi_t),$$
$$\begin{aligned} x_t = {} & \frac{2}{\sqrt{r}}\,\mathrm{dn}^2(\sqrt{r}\varphi)(\mathrm{E}(\sqrt{r}\varphi_t) - \mathrm{E}(\sqrt{r}\varphi)) \\ & + \frac{4k^2}{\sqrt{r}}\,\mathrm{dn}(\sqrt{r}\varphi)\,\mathrm{sn}(\sqrt{r}\varphi)(\mathrm{cn}\,\sqrt{r}\varphi) - \mathrm{cn}(\sqrt{r}\varphi_t)) \\ & + \frac{2k^2}{\sqrt{r}}\,\mathrm{sn}^2(\sqrt{r}\varphi)(\sqrt{r}t + \mathrm{E}(\sqrt{r}\varphi) - \mathrm{E}(\sqrt{r}\varphi_t)) - t, \end{aligned}$$
$$\begin{aligned} y_t = {} & \frac{2k}{\sqrt{r}}(2\,\mathrm{dn}^2(\sqrt{r}\varphi) - 1)(\mathrm{cn}(\sqrt{r}\varphi) - \mathrm{cn}(\sqrt{r}\varphi_t)) \\ & - \frac{2k}{\sqrt{r}}\,\mathrm{sn}(\sqrt{r}\varphi)\,\mathrm{dn}(\sqrt{r}\varphi)(2(\mathrm{E}(\sqrt{r}\varphi_t) - \mathrm{E}(\sqrt{r}\varphi)) - \sqrt{r}t). \end{aligned}$$

Here $\mathrm{E}(u, k)$ is the Jacobi epsilon function (see [22, 34]).

The parametrization of trajectories in N_2^+ is obtained from the above parametrization in N_1 via the transformation $k \mapsto \dfrac{1}{k}$ described in Sec. 7.2.4; after this, trajectories in N_2^- are obtained via the reflection i (7.17). In the domain N_2, we will use the coordinate

$$\psi_t = \frac{\varphi_t}{k}.$$

Then we obtain the following.

If $\lambda \in N_2^{\pm}$, then

$$
\begin{aligned}
\sin\frac{\theta_t}{2} &= \pm(\operatorname{cn}(\sqrt{r}\psi)\operatorname{sn}(\sqrt{r}\psi_t) - \operatorname{sn}(\sqrt{r}\psi)\operatorname{cn}(\sqrt{r}\psi_t)),\\
\cos\frac{\theta_t}{2} &= \operatorname{cn}(\sqrt{r}\psi)\operatorname{cn}(\sqrt{r}\psi_t) + \operatorname{sn}(\sqrt{r}\psi)\operatorname{sn}(\sqrt{r}\psi_t),\\
x_t &= \frac{1}{\sqrt{r}}(1 - 2\operatorname{sn}^2(\sqrt{r}\psi))\left(\frac{2}{k}(\mathrm{E}(\sqrt{r}\psi_t) - \mathrm{E}(\sqrt{r}\psi)) - \frac{2-k^2}{k^2}\sqrt{r}t\right)\\
&\quad + \frac{4}{k\sqrt{r}}\operatorname{cn}(\sqrt{r}\psi)\operatorname{sn}(\sqrt{r}\psi)(\operatorname{dn}(\sqrt{r}\psi) - \operatorname{dn}(\sqrt{r}\psi_t)),\\
y_t &= \pm\left(\frac{2}{k\sqrt{r}}(2\operatorname{cn}^2(\sqrt{r}\psi) - 1)(\operatorname{dn}(\sqrt{r}\psi) - \operatorname{dn}(\sqrt{r}\psi_t))\right.\\
&\quad \left. -\frac{2}{\sqrt{r}}\operatorname{sn}(\sqrt{r}\psi)\operatorname{cn}(\sqrt{r}\psi)\left(\frac{2}{k}(\mathrm{E}(\sqrt{r}\psi_t) - \mathrm{E}(\sqrt{r}\psi)) - \frac{2-k^2}{k^2}\sqrt{r}t\right)\right).
\end{aligned}
$$

The formulas in $N_3^{\pm}$ are obtained from the above formulas in $N_2^{\pm}$ via the limit $k \to 1 - 0$.

Consequently, if $\lambda \in N_3^{\pm}$, then

$$
\begin{aligned}
\sin\frac{\theta_t}{2} &= \pm\left(\frac{\tanh(\sqrt{r}\varphi_t)}{\cosh(\sqrt{r}\varphi)} - \frac{\tanh\sqrt{r}\varphi)}{\cosh(\sqrt{r}\varphi_t)}\right),\\
\cos\frac{\theta_t}{2} &= \frac{1}{\cosh(\sqrt{r}\varphi)\cosh(\sqrt{r}\varphi_t)} + \tanh(\sqrt{r}\varphi)\tanh(\sqrt{r}\varphi_t),\\
x_t &= (1 - 2\tanh^2(\sqrt{r}\varphi))t\\
&\quad + \frac{4\tanh(\sqrt{r}\varphi)}{\sqrt{r}\cosh(\sqrt{r}\varphi)}\left(\frac{1}{\cosh(\sqrt{r}\varphi)} - \frac{1}{\cosh(\sqrt{r}\varphi_t)}\right),\\
y_t &= \pm\left(\frac{2}{\sqrt{r}}\left(\frac{2}{\cosh^2\sqrt{r}\varphi)} - 1\right)\left(\frac{1}{\cosh(\sqrt{r}\varphi)} - \frac{1}{\cosh(\sqrt{r}\varphi_t)}\right)\right.\\
&\qquad\qquad \left. -2\frac{\tanh(\sqrt{r}\varphi)}{\cosh(\sqrt{r}\varphi)}t\right).
\end{aligned}
$$

Now we consider the special cases.

If $\lambda \in N_4 \cup N_5 \cup N_7$, then

$$\theta_t = 0, \quad x_t = t, \quad y_t = 0.$$

If $\lambda \in N_6$, then

$$\theta_t = ct, \quad x_t = \frac{\sin ct}{c}, \quad y_t = \frac{1 - \cos ct}{c}.$$

Thus, we parametrized the exponential mapping of the Euler elastic problem

$$\mathrm{Exp}_t : \lambda = (\beta, c, r) \mapsto q_t = (\theta_t, x_t, y_t), \quad \lambda \in N = T_{q_0}^* M, \quad q_t \in M,$$

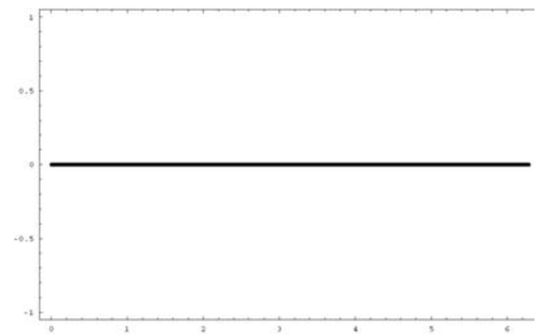

Fig. 4. $E = \pm r$, $r > 0$, $c = 0$.

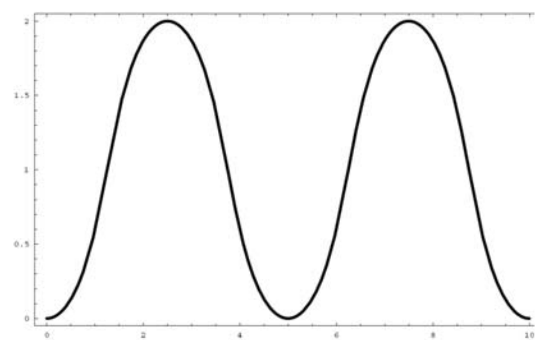

Fig. 5. $E \in (-r, r)$, $r > 0$, $k \in (0, \dfrac{1}{\sqrt{2}})$.

by the Jacobi elliptic functions.

8.3. **Euler elasticae.** Projections of extremal trajectories on the plane (x, y) are stationary configurations of the elastic rod in the plane — Euler elasticae. These curves satisfy the system of ODEs

$$\dot{x} = \cos\theta, \quad \dot{y} = \sin\theta, \quad \ddot{\theta} = -r\sin(\theta - \beta), \quad r, \beta = \text{const}\,. \tag{8.2}$$

Depending on the value of energy $E = \dfrac{\dot{\theta}^2}{2} - r\cos(\theta - \beta) \in [-r, +\infty)$ and constants of motion $r \in [0, +\infty)$, $\beta \in S^1$, of the generalized pendulum (8.2), elasticae have different forms discovered by Euler.

If the energy E takes the absolute minimum $-r \neq 0$ and, therefore, $\lambda \in N_4$, then the corresponding elastica (x_t, y_t) is a straight line (Fig. 4). The corresponding motion of the generalized pendulum (the Kirchhoff kinetic analog) is the stable equilibrium.

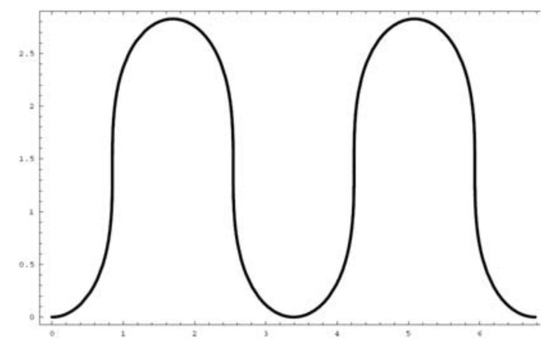

Fig. 6. $E \in (-r, r)$, $r > 0$, $k = \dfrac{1}{\sqrt{2}}$.

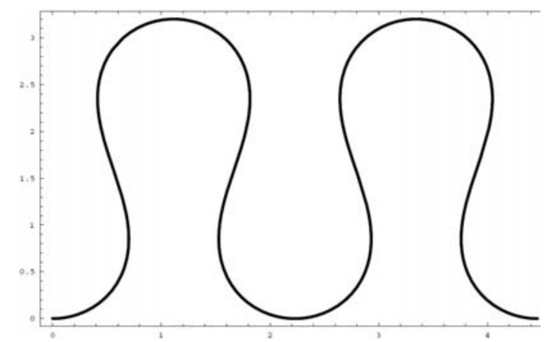

Fig. 7. $E \in (-r, r)$, $r > 0$, $k \in (\dfrac{1}{\sqrt{2}}, k_0)$.

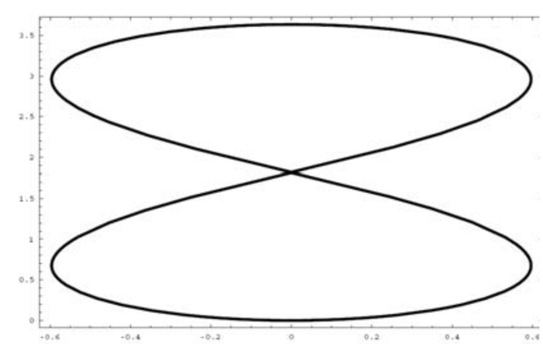

Fig. 8. $E \in (-r, r)$, $r > 0$, $k = k_0$.

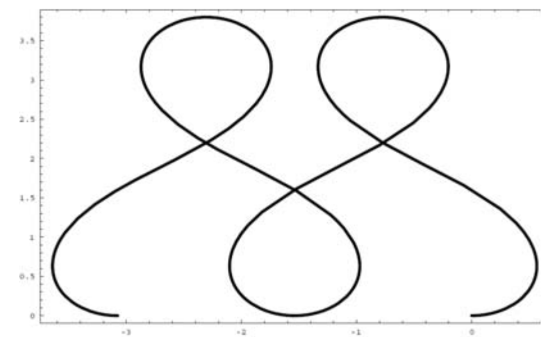

Fig. 9. $E \in (-r, r)$, $r > 0$, $k \in (k_0, 1)$.

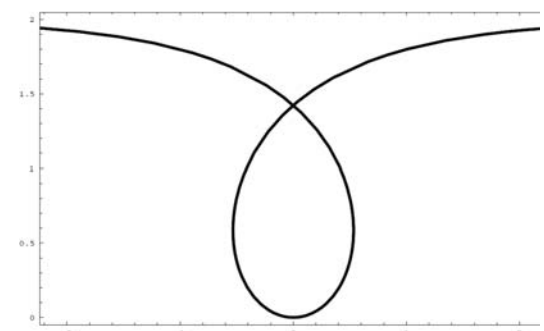

Fig. 10. $E = r > 0$, $\beta \neq \pi$.

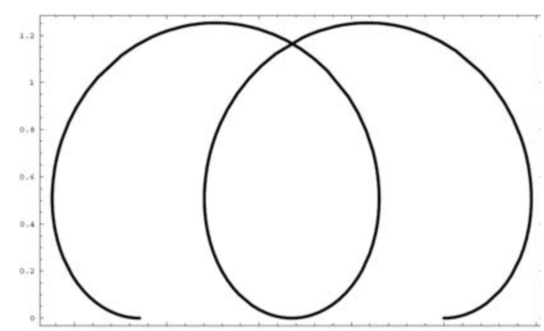

Fig. 11. $E > r > 0$.

If $E \in (-r, r)$, $r \neq 0$, and, therefore, $\lambda \in N_1$, then the pendulum oscillates between extremal values of the angle, and the angular velocity

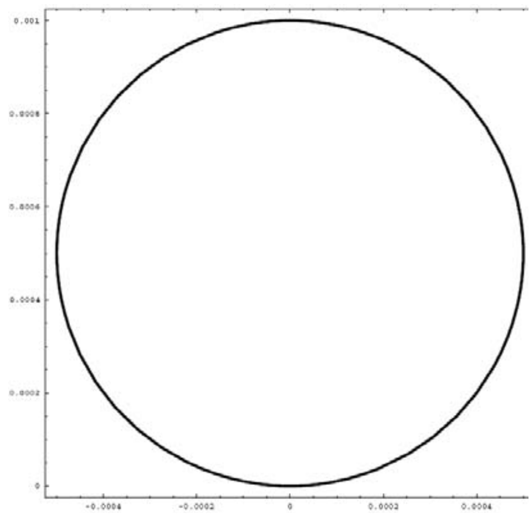

Fig. 12. $r = 0$, $c \neq 0$.

$\dot{\theta}$ changes its sign. The corresponding elasticae have inflections at the points where $\dot{\theta} = 0$ and vertices at the points where $|\dot{\theta}| = \max$, since $\dot{\theta}$ is the curvature of an elastica (x_t, y_t). Such elasticae are called inflectional. See the plots of different classes of inflectional elasticae in Figs. 5–9. The correspondence between the values of the modulus of elliptic functions $k = \sqrt{\dfrac{E+r}{2r}} \in (0,1)$ and these figures is as follows:

$$
\begin{aligned}
&k \in \left(0, \frac{1}{\sqrt{2}}\right) &&\Rightarrow \quad \text{Fig. 5,}\\
&k = \frac{1}{\sqrt{2}} &&\Rightarrow \quad \text{Fig. 6,}\\
&k \in \left(\frac{1}{\sqrt{2}}, k_0\right) &&\Rightarrow \quad \text{Fig. 7,}\\
&k = k_0 &&\Rightarrow \quad \text{Fig. 8,}\\
&k \in (k_0, 1) &&\Rightarrow \quad \text{Fig. 9.}
\end{aligned}
$$

The value $k = 1/\sqrt{2}$ corresponds to the rectangular elastica studied by James Bernoulli (see Sec. 2). The value $k_0 \approx 0.909$ corresponds to the periodic elastica in the form of figure-8 and is described below in Proposition 11.5. As was mentioned by Euler, as $k \to 0$, the inflectional elasticae tend to sinusoids. The corresponding Kirchhoff kinetic analog is provided by the harmonic oscillator $\ddot{\theta} = -r(\theta - \beta)$.

If $E = r \neq 0$ and $\theta - \beta \neq \pi$ and, therefore, $\lambda \in N_3$, then the pendulum approaches its unstable equilibrium ($\theta - \beta = \pi, \dot{\theta} = 0$) along the saddle separatrix, and the corresponding critical elastica has one loop (see Fig. 10).

If $E = r \neq 0$ and $\theta - \beta = \pi$ and, therefore, $\lambda \in N_5$, then the pendulum stays at its unstable equilibrium ($\theta - \beta = \pi, \dot{\theta} = 0$), and the elastica is a straight line (see Fig. 4).

If $E > r \neq 0$ and, therefore, $\lambda \in N_2$, then the Kirchhoff kinetic analog is the pendulum rotating counterclockwise ($\dot{\theta} > 0 \Leftrightarrow \lambda \in N_2^+$) or clockwise ($\dot{\theta} < 0 \Leftrightarrow \lambda \in N_2^-$). The corresponding elasticae have nonvanishing curvature $\dot{\theta}$ and, therefore, they have no inflection points and are called non-inflectional (see Fig. 11). The points where $|\dot{\theta}|$ has local maxima or minima are vertices of inflectional elasticae.

If $r = 0$ and $\dot{\theta} \neq 0$ and, therefore, $\lambda \in N_6$, then the pendulum rotates uniformly: one may think that the gravitational acceleration is $g = 0$ (see the physical meaning of the constant r (6.14)), while the angular velocity $\dot{\theta}$ is nonzero. The corresponding elastica is a circle (see Fig. 12).

Finally, if $r = 0$ and $\dot{\theta} = 0$ and, therefore, $\lambda \in N_7$, then the pendulum is stationary (no gravity with zero angular velocity $\dot{\theta}$), and the elastica is a straight line (see Fig. 4).

Note that the plots of elasticae in Figs. 5–11 do not preserve the real ratio y/x for the sake of saving space.

9. Discrete symmetries of the Euler problem

In this section, we lift discrete symmetries of the standard pendulum (7.1) to discrete symmetries of the normal Hamiltonian system

$$\begin{aligned} &\dot{\beta} = c, \quad \dot{c} = -r\sin\beta, \quad \dot{r} = 0, \\ &\dot{\theta} = c, \quad \dot{x} = \cos\theta, \quad \dot{y} = \sin\theta. \end{aligned} \tag{9.1}$$

9.1. **Reflections in the phase cylinder of the standard pendulum.** It is obvious that the following reflections of the phase cylinder of the standard pendulum $C = S^1_\beta \times \mathbb{R}_c$ preserve the field of directions (although, not the vector field) defined by the ODE of the standard pendulum (7.1):

$$\begin{aligned} &\varepsilon^1 : (\beta, c) \mapsto (\beta, -c), \\ &\varepsilon^2 : (\beta, c) \mapsto (-\beta, c), \\ &\varepsilon^3 : (\beta, c) \mapsto (-\beta, -c) \end{aligned}$$

(see Fig. 13).

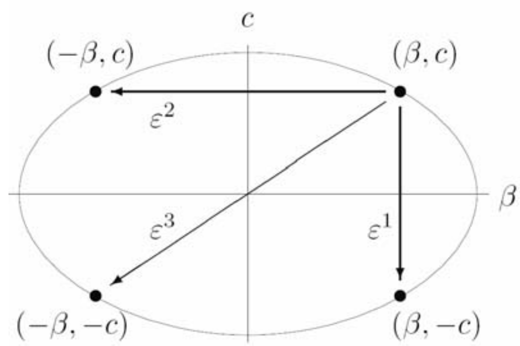

Fig. 13. Reflections in the phase cylinder of the pendulum.

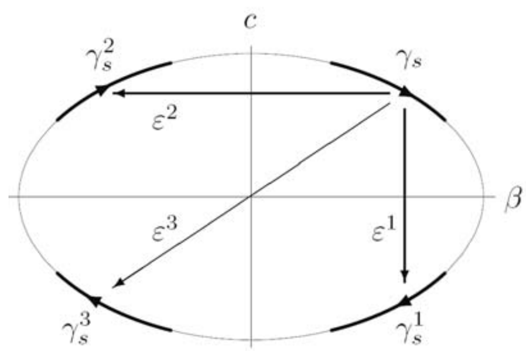

Fig. 14. Reflections of trajectories of the pendulum.

These reflections generate the dihedral group — the group of symmetries of the rectangle $D_2 = \{\mathrm{Id}, \varepsilon^1, \varepsilon^2, \varepsilon^3\}$ with the multiplication table

	ε^1	ε^2	ε^3
ε^1	Id	ε^3	ε^2
ε^2	ε^3	Id	ε^1
ε^3	ε^2	ε^1	Id

Note that the reflections ε^1 and ε^2 reverse the direction of time on trajectories of the pendulum, while ε^3 preserves the direction of time (in fact, ε^3 is the inversion i defined in (7.17)). All reflections ε^i preserve the energy of the pendulum $E = \dfrac{c^2}{2} - \cos\beta$.

9.2. **Reflections of trajectories of the standard pendulum.** We can define the action of reflections on trajectories of the standard pendulum as follows:

$$\varepsilon^i : \gamma_s = \{(\beta_s, c_s) \mid s \in [0,t]\} \mapsto \gamma_s^i = \{(\beta_s^i, c_s^i) \mid s \in [0,t]\},$$

where

$$(\beta_s^1, c_s^1) = (\beta_{t-s}, -c_{t-s}), \tag{9.2}$$
$$(\beta_s^2, c_s^2) = (-\beta_{t-s}, c_{t-s}), \tag{9.3}$$
$$(\beta_s^3, c_s^3) = (-\beta_s, -c_s) \tag{9.4}$$

(see Fig. 14).

All reflections ε^i map trajectories γ_s to trajectories γ_s^i; they preserve both the total time of motion t and the energy $E = \dfrac{c^2}{2} - \cos\beta$.

9.3. **Reflections of trajectories of the generalized pendulum.** The action of reflections is obviously continued to trajectories of the generalized pendulum (7.18) — the vertical subsystem of the normal Hamiltonian system (9.1) as follows:

$$\varepsilon^i : \{(\beta_s, c_s, r) \mid s \in [0,t]\} \mapsto \{(\beta_s^i, c_s^i, r) \mid s \in [0,t]\}, \quad i = 1,2,3, \tag{9.5}$$

where the functions β_s^i, c_s^i are given by (9.2)–(9.4). Then the reflections ε^i preserve both the total time of motion t, the energy of the generalized pendulum $E = \dfrac{c^2}{2} - r\cos\beta$, and the elastic energy of the rod

$$J = \frac{1}{2}\int_0^t \dot\theta_s^2\,ds = \frac{1}{2}\int_0^t c_s^2\,ds.$$

9.4. **Reflections of normal extremals.** Now we define action of the reflections ε^i on the normal extremals

$$\lambda_s = e^{s\vec H}(\lambda_0) \in T^*M, \quad s \in [0,t],$$

i.e., solutions of the Hamiltonian system

$$\begin{gathered} \dot\beta_s = c_s, \quad \dot c_s = -r\sin\beta_s, \\ \dot r = 0, \quad \dot q_s = X_1(q_s) + c_s X_2(q_s), \end{gathered} \tag{9.6}$$

as follows:

$$\varepsilon^i : \{\lambda_s \mid s \in [0,t]\} \mapsto \{\lambda_s^i \mid s \in [0,t]\}, \quad i = 1,2,3, \tag{9.7}$$
$$\lambda_s = (\nu_s, q_s) = (\beta_s, c_s, r, q_s), \quad \lambda_s^i = (\nu_s^i, q_s^i) = (\beta_s^i, c_s^i, r, q_s^i). \tag{9.8}$$

Here λ_s^i is a solution of the Hamiltonian system (9.6), and the action on the vertical coordinates

$$\varepsilon^i : \{\nu_s = (\beta_s, c_s, r)\} \mapsto \{\nu_s^i = (\beta_s^i, c_s^i, r)\}$$

was defined in Sec. 9.3. The action of reflections on horizontal coordinates (θ, x, y) is described in the next section.

9.5. **Reflections of Euler elasticae.** Here we describe the action of reflections on the normal extremal trajectories

$$\varepsilon^i : \{q_s = (\theta_s, x_s, y_s) \mid s \in [0, t]\} \mapsto \{q^i_s = (\theta^i_s, x^i_s, y^i_s) \mid s \in [0, t]\}.$$

Proposition 9.1. *Let*

$$\lambda_s = (\beta_s, c_s, r, q_s), \quad \lambda^i_s = \varepsilon^i(\lambda_s) = (\beta^i_s, c^i_s, r, q^i_s), \quad s \in [0, t],$$

be normal extremals defined in (9.7), (9.8). *Then the following equalities hold*:

(1) $\theta^1_s = \theta_{t-s} - \theta_t$, $\begin{pmatrix} x^1_s \\ y^1_s \end{pmatrix} = \begin{pmatrix} \cos\theta_t & \sin\theta_t \\ -\sin\theta_t & \cos\theta_t \end{pmatrix} \begin{pmatrix} x_t - x_{t-s} \\ y_t - y_{t-s} \end{pmatrix}$,

(2) $\theta^2_s = \theta_t - \theta_{t-s}$, $\begin{pmatrix} x^2_s \\ y^2_s \end{pmatrix} = \begin{pmatrix} \cos\theta_t & -\sin\theta_t \\ \sin\theta_t & \cos\theta_t \end{pmatrix} \begin{pmatrix} x_t - x_{t-s} \\ y_{t-s} - y_t \end{pmatrix}$,

(3) $\theta^3_s = -\theta_s$, $\begin{pmatrix} x^3_s \\ y^3_s \end{pmatrix} = \begin{pmatrix} x_s \\ -y_s \end{pmatrix}$.

Proof. We prove only the formulas in item (1), the next two items are studied similarly. By virtue of (8.1) and (9.2), we have

$$\theta^1_s = \beta^1_s - \beta^1_0 = \beta_{t-s} - \beta_t = \theta_{t-s} - \theta_t.$$

Further,

$$x^1_s = \int_0^s \cos\theta^1_r \, dr = \int_0^s \cos(\theta_{t-r} - \theta_t) \, dr = \cos\theta_t(x_t - x_{t-s}) + \sin\theta_t(y_t - y_{t-s})$$

and similarly

$$y^1_s = \int_0^s \sin\theta^1_r \, dr = \int_0^s \sin(\theta_{t-r} - \theta_t) \, dr = \cos\theta_t(y_t - y_{t-s}) - \sin\theta_t(x_t - x_{t-s}).$$

The proposition is proved. □

Remark. Note the visual meaning of the action of the reflections ε^i on the elastica $\{(x_s, y_s) \mid s \in [0, t]\}$ in the case $(x_t, y_t) \neq (x_0, y_0)$.

One can show that modulo inversion of time on elasticae and rotations of the plane (x, y), we have:

- ε^1 is the reflection of elastica in the center of its chord;
- ε^2 is the reflection of elastica in the middle perpendicular to its chord;
- ε^3 is the reflection of elastica in its chord

(see details in [34]).

9.6. **Reflections of endpoints of extremal trajectories.** Now we can define the action of reflections in the state space $M = \mathbb{R}^2_{x,y} \times S^1_\theta$ as the action on endpoints of extremal trajectories:

$$\varepsilon^i : M \to M, \quad \varepsilon^i : q_t \mapsto q_t^i, \tag{9.9}$$

as follows:

$$\varepsilon^1 : \begin{pmatrix} \theta_t \\ x_t \\ y_t \end{pmatrix} \mapsto \begin{pmatrix} -\theta_t \\ x_t \cos\theta_t + y_t \sin\theta_t \\ -x_t \sin\theta_t + y_t \cos\theta_t \end{pmatrix}, \tag{9.10}$$

$$\varepsilon^2 : \begin{pmatrix} \theta_t \\ x_t \\ y_t \end{pmatrix} \mapsto \begin{pmatrix} \theta_t \\ x_t \cos\theta_t + y_t \sin\theta_t \\ x_t \sin\theta_t - y_t \cos\theta_t \end{pmatrix}, \tag{9.11}$$

$$\varepsilon^3 : \begin{pmatrix} \theta_t \\ x_t \\ y_t \end{pmatrix} \mapsto \begin{pmatrix} -\theta_t \\ x_t \\ -y_t \end{pmatrix}. \tag{9.12}$$

These formulas directly follow from Proposition 9.1. Note that the action of reflections $\varepsilon^i : M \to M$ is well-defined in the sense that the image $\varepsilon^i(q_t)$ depends only on the point q_t, but not on the whole trajectory $\{q_s \mid s \in [0,t]\}$.

9.7. **Reflections as symmetries of the exponential mapping.** The action of reflections ε^i on the vertical subsystem of the normal Hamiltonian system (9.5) defines the action of ε^i in the preimage of the exponential mapping by restriction to the initial instant $s = 0$:

$$\varepsilon^i : \nu = (\beta, c, r) \mapsto \nu^i = (\beta^i, c^i, r),$$

where $(\beta, c, r) = (\beta_0, c_0, r)$, $(\beta^i, c^i, r) = (\beta_0^i, c_0^i, r)$ are the initial points of the curves $\nu_s = (\beta_s, c_s, r)$ and $\nu_s^i = (\beta_s^i, c_s^i, r)$. The explicit formulas for (β^i, c^i) are found from formulas (9.2)–(9.4):

$$(\beta^1, c^1) = (\beta_t, -c_t),$$
$$(\beta^2, c^2) = (-\beta_t, c_t),$$
$$(\beta^3, c^3) = (-\beta_0, -c_0).$$

Thus, we have the action of reflections in the preimage of the exponential mapping:

$$\varepsilon^i : N \to N, \quad \varepsilon^i(\nu) = \nu^i, \quad \nu, \nu^i \in N = T^*_{q_0} M.$$

Since the both actions of ε^i in N and M are induced by the action of ε^i on extremals λ_s (9.7), we obtain the following statement.

Proposition 9.2. *Reflections ε^i are symmetries of the exponential mapping $\mathrm{Exp}_t : N \to M$, i.e., the following diagram is commutative*:

$$\begin{array}{ccc} N & \xrightarrow{\mathrm{Exp}_t} & M \\ \downarrow{\scriptstyle \varepsilon^i} & & \downarrow{\scriptstyle \varepsilon^i} \\ N & \xrightarrow{\mathrm{Exp}_t} & M \end{array} \qquad\qquad \begin{array}{ccc} \nu & \xmapsto{\mathrm{Exp}_t} & q_t \\ \downarrow{\scriptstyle \varepsilon^i} & & \downarrow{\scriptstyle \varepsilon^i} \\ \nu^i & \xmapsto{\mathrm{Exp}_t} & q_t^i \end{array}$$

9.8. **Action of reflections in the preimage of the exponential mapping.** In this section, we describe the action of reflections

$$\varepsilon^i : N \to N, \quad \varepsilon^i(\nu) = \nu^i,$$

in the elliptic coordinates (Sec. 7.3) in the preimage of the exponential mapping N.

Proposition 9.3. (1) *If $\nu = (k, \varphi, r) \in N_1$, then $\nu^i = (k, \varphi^i, r) \in N_1$ and*

$$\varphi^1 + \varphi_t = \frac{2K}{\sqrt{r}} \left(\mathrm{mod} \frac{4K}{\sqrt{r}} \right),$$
$$\varphi^2 + \varphi_t = 0 \left(\mathrm{mod} \frac{4K}{\sqrt{r}} \right),$$
$$\varphi^3 - \varphi = \frac{2K}{\sqrt{r}} \left(\mathrm{mod} \frac{4K}{\sqrt{r}} \right).$$

(2) *If $\nu = (k, \psi, r) \in N_2$, then $\nu^i = (k, \psi^i, r) \in N_2$ and, moreover,*

$$\nu \in N_2^{\pm} \quad \Rightarrow \quad \nu^1 \in N_2^{\mp}, \quad \nu^2 \in N_2^{\pm}, \quad \nu^3 \in N_2^{\mp}, \tag{9.13}$$

and

$$\psi^1 + \psi_t = 0 \left(\mathrm{mod} \frac{2K}{\sqrt{r}} \right),$$
$$\psi^2 + \psi_t = 0 \left(\mathrm{mod} \frac{2K}{\sqrt{r}} \right),$$
$$\psi^3 - \psi = 0 \left(\mathrm{mod} \frac{2K}{\sqrt{r}} \right).$$

(3) *If $\nu = (\varphi, r) \in N_3$, then $\nu^i = (\varphi^i, r) \in N_3$ and, moreover,*

$$\nu \in N_3^{\pm} \quad \Rightarrow \quad \nu^1 \in N_3^{\mp}, \quad \nu^2 \in N_3^{\pm}, \quad \nu^3 \in N_3^{\mp},$$

and

$$\varphi^1 + \varphi_t = 0, \quad \varphi^2 + \varphi_t = 0, \quad \varphi^3 - \varphi = 0.$$

(4) *If $\nu = (\beta, c, r) \in N_6$, then $\nu^i = (\beta^i, c^i, r) \in N_6$ and, moreover,*

$$\nu \in N_6^{\pm} \quad \Rightarrow \quad \nu^1 \in N_6^{\mp}, \quad \nu^2 \in N_6^{\pm}, \quad \nu^3 \in N_6^{\mp}, \tag{9.14}$$

and

$$(\beta^1, c^1) = (\beta_t, -c), \quad (\beta^2, c^2) = (-\beta_t, c), \quad (\beta^3, c^3) = (-\beta, -c). \qquad (9.15)$$

Proof. We prove only item (1) since the other items are proved similarly.

The reflections ε^i preserve the domain N_1 since

$$\varepsilon^i : E \mapsto E, \quad \varepsilon^i : r \mapsto r,$$
$$\varepsilon^1,\ \varepsilon^3 : c \mapsto -c, \quad \varepsilon^2 : c \mapsto c.$$

This follows from equalities (9.2)–(9.3). Further, we obtain from (9.2) that

$$\theta^1 = \theta_t, \quad c^1 = -c_t,$$

whence by virtue of the construction of elliptic coordinates (Sec. 7.3) it follows that

$$\operatorname{sn}(\sqrt{r}\varphi^1) = \operatorname{sn}(\sqrt{r}\varphi_t), \quad \operatorname{cn}(\sqrt{r}\varphi^1) = -\operatorname{cn}(\sqrt{r}\varphi_t)$$

and, therefore, $\varphi^1 + \varphi_t = \dfrac{2K}{\sqrt{r}} \left(\operatorname{mod} \dfrac{4K}{\sqrt{r}}\right)$. The expressions for the action of the rest reflections in the elliptic coordinates are obtained similarly. □

10. Maxwell strata

10.1. **Optimality of normal extremal trajectories.** Consider an analytic optimal control problem of the form:

$$\dot{q} = f(q, u), \qquad q \in M, \quad u \in U, \qquad (10.1)$$

$$q(0) = q_0, \qquad q(t_1) = q_1, \qquad t_1 \text{ fixed}, \qquad (10.2)$$

$$J_{t_1}[q, u] = \int_0^{t_1} \varphi(q(t), u(t))\, dt \to \min. \qquad (10.3)$$

Here M and U are finite-dimensional analytic manifolds and $f(q, u)$ and $\varphi(q, u)$ are an analytic vector field and a function depending on the control parameter u, respectively. Let

$$h_u(\lambda) = \langle \lambda, f(q, u) \rangle - \varphi(q, u), \quad \lambda \in T^*M, \quad q = \pi(\lambda) \in M, \quad u \in U,$$

be the normal Hamiltonian of the Pontryagin maximum principle for this problem (see Sec. 6.1 and [2]). Suppose that all normal extremals λ_t of the problem are regular, i.e., the strong Legendre condition is satisfied:

$$\left.\frac{\partial^2}{\partial u^2}\right|_{u(t)} h_u(\lambda_t) < -\delta, \quad \delta > 0, \qquad (10.4)$$

for the corresponding extremal control $u(t)$. Then the maximized Hamiltonian $H(\lambda) = \max_{u \in U} h_u(\lambda)$ is analytic, and the exponential mapping

$$\operatorname{Exp}_t : N = T^*_{q_0} M \to M, \quad \operatorname{Exp}_t(\lambda) = \pi \circ e^{t\vec{H}}(\lambda) = q(t)$$

is defined for the time t.

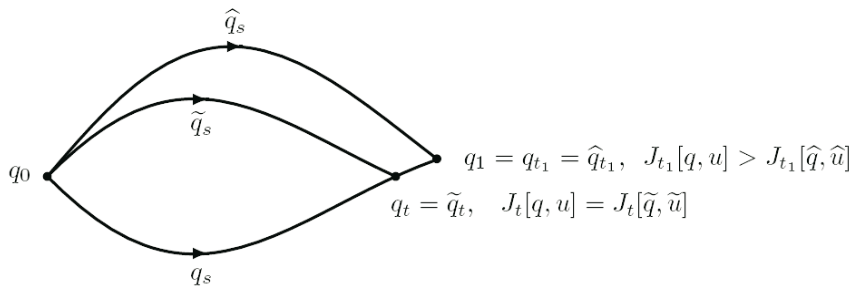

Fig. 15. Maxwell point q_t.

Suppose that the control u maximizing the Hamiltonian $h_u(\lambda)$ is an analytic function $u = u(\lambda)$, $\lambda \in T^*M$.

For covectors $\lambda, \widetilde{\lambda} \in T^*_{q_0}M$, we denote the corresponding extremal trajectories as follows:

$$q_s = \operatorname{Exp}_s(\lambda), \quad \widetilde{q}_s = \operatorname{Exp}_s(\widetilde{\lambda})$$

and the extremal controls as follows:

$$u(s) = u(\lambda_s), \qquad \lambda_s = e^{s\vec{H}}(\lambda),$$
$$\widetilde{u}(s) = u(\widetilde{\lambda}_s), \qquad \widetilde{\lambda}_s = e^{s\vec{H}}(\widetilde{\lambda}).$$

The time t *Maxwell set* in the preimage of the exponential mapping $N = T^*_{q_0}M$ is defined as follows:

$$\mathrm{MAX}_t = \Big\{\lambda \in N \mid \exists\ \widetilde{\lambda} \in N : \\ \widetilde{q}_s \not\equiv q_s,\ s \in [0,t],\ \widetilde{q}_t = q_t,\ J_t[q,u] = J_t[\widetilde{q},\widetilde{u}]\Big\}. \quad (10.5)$$

The inclusion $\lambda \in \mathrm{MAX}_t$ means that two distinct extremal trajectories $\widetilde{q}_s \not\equiv q_s$ with the same value of the cost functional $J_t[q,u] = J_t[\widetilde{q},\widetilde{u}]$ intersect one another at the point $\widetilde{q}_t = q_t$ (see Fig. 15).

The point q_t is called a *Maxwell point* of the trajectory q_s, $s \in [0,t_1]$, and the instant t is called a *Maxwell time*.

The Maxwell set is closely connected with the optimality of extremal trajectories: such a trajectory cannot be optimal after a Maxwell point. The following statement is a modification of a similar proposition proved by S. Jacquet [15] in the context of sub-Riemannian problems.

Proposition 10.1. *If a normal extremal trajectory q_s, $s \in [0,t_1]$, admits a Maxwell point q_t, $t \in (0,t_1)$, then q_s is not optimal in problem* (10.1)–(10.3).

Proof. By contradiction, assume that the trajectory q_s, $s \in [0, t_1]$, is optimal. Then the broken curve

$$q'_s = \begin{cases} \widetilde{q}_s, & s \in [0, t], \\ q_s, & s \in [t, t_1] \end{cases}$$

is an admissible trajectory of system (10.1) with the control

$$u'_s = \begin{cases} \widetilde{u}(s), & s \in [0, t], \\ u(s), & s \in [t, t_1]. \end{cases}$$

Moreover, the trajectory q'_s is optimal in problem (10.1)–(10.3) since

$$\begin{aligned} J_{t_1}[q', u'] &= \int_0^{t_1} \varphi(q'_s, u'(s))\, ds = \int_0^{t} \varphi(q'_s, u'(s))\, ds + \int_t^{t_1} \varphi(q'_s, u'(s))\, ds \\ &= \int_0^{t} \varphi(\widetilde{q}_s, \widetilde{u}(s))\, ds + \int_t^{t_1} \varphi(q_s, u(s))\, ds \\ &= J_t[\widetilde{q}, \widetilde{u}] + \int_t^{t_1} \varphi(q_s, u(s))\, ds = J_t[q, u] + \int_t^{t_1} \varphi(q_s, u(s))\, ds \\ &= J_{t_1}[q, u], \end{aligned}$$

which is minimal since q_s is optimal.

Therefore, the trajectory q'_s is extremal, in particular, it is analytic. Thus, the analytic curves q_s and q'_s coincide one with another on the segment $s \in [t, t_1]$. By the uniqueness theorem for analytic functions, these curves must coincide everywhere: $q_s \equiv q'_s$, $s \in [0, t_1]$, and, therefore, $q_s \equiv \widetilde{q}_s$, $s \in [0, t_1]$, which contradicts the definition of the Maxwell point q_t. □

Maxwell points were successfully applied for the study of the optimality of geodesics in several sub-Riemannian problems [1, 10, 28]. We will apply this notion in order to obtain an upper bound on the cut time, i.e., the time, where the normal extremals lose optimality (see [31–33] for a similar result for the nilpotent sub-Riemannian problem with the growth vector $(2, 3, 5)$).

As was noted in the book of V. I. Arnold [5], the term *Maxwell point* is originated "in connection with the Maxwell rule of the van der Waals theory, according to which the phase transition takes place at a value of the parameter for which two maxima of a certain smooth function are equal to each other."

10.2. **Maxwell strata generated by reflections.** We return to the Euler elastic problem (3.4)–(3.9). It is easy to see that this problem has the form (10.1)–(10.3) and satisfies all assumptions stated in the previous subsection, and therefore, Proposition 10.1 holds for the Euler problem.

Consider the action of reflections in the preimage of the exponential mapping:

$$\varepsilon^i : N \to N, \quad \varepsilon^i(\lambda) = \lambda^i,$$

and denote the corresponding extremal trajectories

$$q_s = \operatorname{Exp}_s(\lambda), \quad q_s^i = \operatorname{Exp}_s(\lambda^i)$$

and extremal controls (6.11)

$$u(s) = c_s, \quad u^i(s) = c_s^i.$$

The *Maxwell strata corresponding to reflections* ε^i are defined as follows:

$$\mathrm{MAX}_t^i = \{\lambda \in N \mid q_s^i \not\equiv q_s,\ q_t^i = q_t,\ J_t[q,u] = J_t[q^i,u^i]\}, \quad i = 1,2,3,\ t > 0. \tag{10.6}$$

It is obvious that

$$\mathrm{MAX}_t^i \subset \mathrm{MAX}_t, \quad i = 1,2,3.$$

Remark. Along normal extremals we have

$$J_t[q,u] = \frac{1}{2}\int_0^t c_s^2\, ds.$$

In view of the expression for the action of reflections ε^i on trajectories of pendulum (9.2)–(9.4), we have

$$J_t[q^i,u^i] = J_t[q,u], \quad i = 1,2,3,$$

i.e., the last condition in the definition of the Maxwell stratum MAX_t is always satisfied.

10.3. **Extremal trajectories preserved by reflections.** In this section, we describe the normal extremal trajectories q_s such that $q_s^i \equiv q_s$. This identity appears in the definition of Maxwell strata MAX_t^i (10.6).

Proposition 10.2. (1) $q_s^1 \equiv q_s \quad \Leftrightarrow \quad \lambda^1 = \lambda$.
(2) $q_s^2 \equiv q_s \quad \Leftrightarrow \quad \lambda^2 = \lambda$ *or* $\lambda \in N_6$.
(3) $q_s^3 \equiv q_s \quad \Leftrightarrow \quad \lambda^3 = \lambda$.

Proof. First, consider the chain

$$q_s^i \equiv q_s \quad \Rightarrow \quad \theta_s^i \equiv \theta_s \quad \Rightarrow \quad \beta_s^i - \beta_0^i \equiv \beta_s - \beta_0, \quad i = 1,2,3. \tag{10.7}$$

(1) Let $q_s^1 \equiv q_s$. By equality (9.2), $\beta_s^1 = \beta_{t-s}$ and, therefore, we obtain from (10.7) that

$$\beta_{t-s} - \beta_t \equiv \beta_s - \beta_0.$$

For $s = t$, we have $\beta_t = \beta_0$ and, therefore,

$$\beta_{t-s} \equiv \beta_s.$$

Differentiating with respect to s and taking into account the equation of generalized pendulum (7.18), we obtain

$$c_{t-s} \equiv -c_s.$$

In view of equality (9.2),

$$(\beta^1_s, c^1_s) \equiv (\beta_s, c_s) \quad \Rightarrow \quad (\beta^1, c^1) = (\beta, c) \quad \Rightarrow \quad \lambda = \lambda^1.$$

Conversely, if $\lambda^1 = \lambda$, then $q^1_s \equiv q_s$.

(2) Let $q^2_s \equiv q_s$. In view of (9.3), $\beta^2_s = -\beta_{t-s}$ and, therefore, (10.7) gives the identity

$$-\beta_{t-s} + \beta_t \equiv \beta_s - \beta_0.$$

Differentiating twice with respect to the equation of generalized pendulum (7.18), we obtain

$$\begin{aligned} c_{t-s} \equiv c_s \quad &\Rightarrow \quad -r \sin \beta_{t-s} \equiv r \sin \beta_r \\ &\Rightarrow \quad (\beta_s \equiv \beta_0 \text{ or } \beta_{t-s} \equiv -\beta_s \text{ or } r = 0). \end{aligned}$$

If $\beta_s \equiv \beta_0$, then $c_s \equiv 0$, which means that $\lambda \in N_4 \cup N_5 \cup N_7$. If $\beta_{t-s} \equiv -\beta_s$, then $(\beta^2_s, c^2_s) \equiv (\beta_s, c_s)$ and, therefore, $\lambda^2 = \lambda$. Finally, the equality $r = 0$ means that $\lambda \in N_6 \cup N_7$. Thus, we have proved that

$$q^2_s \equiv q_s \quad \Rightarrow \quad (\lambda^2 = \lambda \text{ or } \lambda \in \bigcup_{i=4}^{7} N_i).$$

But if $\lambda \in N_4 \cup N_5 \cup N_7$, then $\beta_s \equiv 0$ or π, $c_s \equiv 0$ (see Sec. 8.1), and equality (9.3) implies that $(\beta^2_s, c^2_s) = (\beta_s, c_s)$ and, therefore, $\lambda^2 = \lambda$. The implication $\Rightarrow$ in item (2) follows. The reverse implication is directly verified.

(3) Let $q^3_s \equiv q_s$. Equality (9.4) gives $\beta^3_s = -\beta_s$, and condition (10.7) implies that $\beta_s \equiv \beta_0$. Then $c_s \equiv 0$. Consequently, $\lambda \in N_4 \cup N_5 \cup N_7$. But if $\lambda \in N_4 \cup N_5 \cup N_7$, then $\lambda^3 = \lambda$ by the argument used above in the proof of item (2). The implication $\Rightarrow$ in item (3) follows. The reverse implication in item (3) is directly verified. $\square$

Proposition 10.2 means that the identity $q^i_s \equiv q_s$ is satisfied in the following cases:

(a) $\lambda^i = \lambda$, the trivial case, or
(b) $\lambda \in N_6$ for $i = 2$.

10.4. **Multiple points of the exponential mapping.** In this section, we study solutions of the equations $q^i_t = q_t$ related to the Maxwell strata MAX^i_t (10.6).

Recall that in Sec. 9.6, we have defined the action of reflections ε^i in the state space M. We denote $q^i = \varepsilon^i(q)$, $q, q^i \in M$.

The following functions are defined on $M = \mathbb{R}^2_{x,y} \times S^1_\theta$ up to a sign:

$$P = x \sin \frac{\theta}{2} - y \cos \frac{\theta}{2}, \quad Q = x \cos \frac{\theta}{2} + y \sin \frac{\theta}{2},$$

although their zero sets $\{P = 0\}$ and $\{Q = 0\}$ are well defined.

Proposition 10.3. (1) $q^1 = q \quad \Leftrightarrow \quad \theta = 0 \pmod{2\pi}$;
(2) $q^2 = q \quad \Leftrightarrow \quad P = 0$;
(3) $q^3 = q \quad \Leftrightarrow \quad (y = 0$ *and* $\theta = 0 \pmod{\pi})$.

Proof. We apply the formulas for the action of reflections ε^i in M obtained in Sec. 9.6.

(1) Formula (9.10) means that

$$\varepsilon^1 : (\theta, x, y) \mapsto (-\theta, x \cos \theta + y \sin \theta, -x \sin \theta + y \cos \theta),$$

which gives statement (1).

(2) Formula (9.11) has the form

$$\varepsilon^2 : q = (\theta, x, y) \mapsto q^2 = (\theta, x \cos \theta + y \sin \theta, x \sin \theta - y \cos \theta).$$

If $(x, y) = (0, 0)$, then $q^2 = (\theta, 0, 0) = q$ and $P = 0$ and, therefore, statement (2) follows.

Suppose that $(x, y) \neq (0, 0)$, then we can introduce polar coordinates: $x = \rho \cos \chi$, $y = \rho \sin \chi$, $\rho > 0$. We have

$$\begin{aligned} q^2 = q \quad &\Leftrightarrow \quad \begin{cases} x \cos \theta + y \sin \theta = x, \\ x \sin \theta - y \cos \theta = y \end{cases} \\ &\Leftrightarrow \quad \begin{cases} \cos \chi \cos \theta + \sin \chi \sin \theta = \cos \chi, \\ \cos \chi \sin \theta - \sin \chi \cos \theta = \sin \chi \end{cases} \\ &\Leftrightarrow \quad \cos \chi \sin \frac{\theta}{2} - \sin \chi \cos \frac{\theta}{2} = 0 \quad \Leftrightarrow \quad P = 0, \end{aligned}$$

and statement (2) is proved also in the case $(x, y) \neq (0, 0)$.

(3) Formula (9.12) has the form

$$\varepsilon^3 : q = (\theta, x, y) \mapsto q^3 = (-\theta, x, -y),$$

and, therefore,

$$q^3 = q \quad \Leftrightarrow \quad \begin{cases} \theta = -\theta, \\ y = -y \end{cases} \quad \Leftrightarrow \quad \begin{cases} \theta = 0 \pmod{\pi}, \\ y = 0. \end{cases}$$

The proposition is proved. □

The visual meaning of the conditions $q^i_t = q_t$ for the corresponding arcs of Euler elasticae (x_s, y_s), $s \in [0, t]$ in the case $x_t^2 + y_t^2 \neq 0$ is described in [34].

10.5. **Fixed points of reflections in the preimage of the exponential mapping.** In order to describe fixed points of the reflections $\varepsilon^i : N \to N$, we use the elliptic coordinates (k, φ, r) in N introduced in Sec. 7.3. Moreover, the following coordinate turns out to be very convenient:

$$\tau = \frac{\sqrt{r}(\varphi_t + \varphi)}{2}.$$

While the values $\sqrt{r}\varphi$ and $\sqrt{r}\varphi_t$ correspond to the initial and terminal points of an elastic arc, their arithmetic mean τ corresponds to the midpoint of the elastic arc.

Proposition 10.4. *Let* $\nu = (k, \varphi, r) \in N_1$*, then* $\nu^i = \varepsilon^i(\nu) = (k, \varphi^i, r) \in N_1$*. Moreover,*

(1) $\nu^1 = \nu \quad \Leftrightarrow \quad \operatorname{cn} \tau = 0,$
(2) $\nu^2 = \nu \quad \Leftrightarrow \quad \operatorname{sn} \tau = 0,$
(3) $\nu^3 = \nu$ *is impossible.*

Proof. We apply Proposition 9.3. The inclusion $\nu^i \in N_1$ holds. Further,

$$\begin{aligned}
\nu^1 = \nu \quad &\Leftrightarrow \quad \varphi^1 = \varphi \quad \Leftrightarrow \quad \varphi + \varphi_t = \frac{2K}{\sqrt{r}} \left(\operatorname{mod} \frac{4K}{\sqrt{r}} \right) \\
&\Leftrightarrow \quad \tau = K \pmod{2K} \quad \Leftrightarrow \quad \operatorname{cn} \tau = 0, \\
\nu^2 = \nu \quad &\Leftrightarrow \quad \varphi^2 = \varphi \quad \Leftrightarrow \quad \varphi + \varphi_t = 0 \left(\operatorname{mod} \frac{4K}{\sqrt{r}} \right) \\
&\Leftrightarrow \quad \tau = 0 \pmod{2K} \quad \Leftrightarrow \quad \operatorname{sn} \tau = 0, \\
\nu^3 = \nu \quad &\Leftrightarrow \quad \varphi^3 = \varphi \quad \Leftrightarrow \quad 0 = \frac{2K}{\sqrt{r}} \left(\operatorname{mod} \frac{4K}{\sqrt{r}} \right),
\end{aligned}$$

which is impossible. $\square$

Note the visual meaning of fixed points of the reflections $\varepsilon^i : N_1 \to N_1$ for standard pendulum (7.1) in the cylinder (β, c), and for the corresponding inflectional elasticae, see the corresponding figures in [34]. The equality $\operatorname{cn} \tau = 0$ is equivalent to $c = 0$, these are inflection points of elasticae (zeros of their curvature c). The equality $\operatorname{sn} \tau = 0$ is equivalent to $\beta = 0$, these are vertices of elasticae (extrema of their curvature c).

In the domain N_2, we use the convenient coordinate

$$\tau = \frac{\sqrt{r}(\psi + \psi_t)}{2}$$

corresponding to the midpoint of a non-inflectional elastic arc.

Proposition 10.5. *Let* $\nu = (k, \psi, r) \in N_2$*, then* $\nu^i = \varepsilon^i(\nu) = (k, \psi^i, r) \in N_2$*. Moreover*:

(1) $\nu^1 = \nu$ *is impossible*;
(2) $\nu^2 = \nu \quad \Leftrightarrow \quad \operatorname{sn} \tau \operatorname{cn} \tau = 0;$

(3) $\nu^3 = \nu$ *is impossible.*

Proof. We apply Proposition 9.3. The inclusion $\nu^i \in N_2$ holds. Implication (9.13) yields items (1) and (3). We prove item (2):

$$\begin{aligned}\nu^2 = \nu \quad &\Leftrightarrow \quad \psi^2 = \psi \quad \Leftrightarrow \quad \psi + \psi_t = 0 \left(\operatorname{mod} \frac{2K}{\sqrt{r}} \right) \\ &\Leftrightarrow \quad \tau = 0 \pmod K \quad \Leftrightarrow \quad \operatorname{sn}\tau \operatorname{cn}\tau = 0.\end{aligned}$$

The proposition is proved. □

Note the visual meaning of the fixed points of the reflections $\varepsilon^i : N_2 \to N_2$. The equality $\operatorname{sn}\tau \operatorname{cn}\tau = 0$ is equivalent to the equalities $\beta = 0 \pmod \pi$, $|c| = \max, \min$, these are vertices of non-inflectional elasticae (local extrema of their curvature c).

Similarly to the previous cases, in the set N_3 we use the parameter

$$\tau = \frac{\sqrt{r}(\varphi_t + \varphi)}{2}.$$

Proposition 10.6. *Let* $\nu = (\varphi, r) \in N_3$, *then* $\nu^i = \varepsilon^i(\nu) = (\varphi^i, r) \in N_3$. *Moreover*:

(1) $\nu^1 = \nu$ *is impossible*;
(2) $\nu^2 = \nu \quad \Leftrightarrow \quad \tau = 0$;
(3) $\nu^3 = \nu$ *is impossible.*

The proof is similar to the proof of Proposition 10.5.

The visual meaning of fixed points of the reflection $\varepsilon^2 : N_3 \to N_3$: the equality $\tau = 0$ means that $\beta = 0$, $|c| = \max$, these are vertices of critical elasticae.

Proposition 10.7. *Let* $\nu = (\beta, c, r) \in N_6$, *then* $\nu^i = \varepsilon^i(\nu) = (\beta^i, c^i, r) \in N_6$. *Moreover*:

(1) $\nu^1 = \nu$ *is impossible*;
(2) $\nu^2 = \nu \quad \Leftrightarrow \quad 2\beta + ct = 0 \pmod{2\pi}$;
(3) $\nu^3 = \nu$ *is impossible.*

Proof. Items (1) and (3) follow from implication (9.14). Item (2) follows from (9.15) and the formula $\beta_t = \beta_0 + ct$ (see Sec. 8.1). □

10.6. **General description of the Maxwell strata generated by reflections.** Now we summarize our computations of Maxwell strata corresponding to reflections.

Theorem 10.1. (1) *Let* $\nu = (k, \varphi, r) \in N_1$. *Then*:

(1.1) $\nu \in \mathrm{MAX}_t^1 \quad \Leftrightarrow \quad \begin{cases} \nu^1 \neq \nu, \\ q_t^1 = q_t \end{cases} \quad \Leftrightarrow \quad \begin{cases} \operatorname{cn}\tau \neq 0, \\ \theta_t = 0; \end{cases}$

(1.2) $\nu \in \mathrm{MAX}_t^2 \quad \Leftrightarrow \quad \begin{cases} \nu^2 \neq \nu, \\ q_t^2 = q_t \end{cases} \quad \Leftrightarrow \quad \begin{cases} \operatorname{sn}\tau \neq 0, \\ P_t = 0; \end{cases}$

(1.3) $\nu \in \mathrm{MAX}_t^3 \quad \Leftrightarrow \quad \begin{cases} \nu^3 \neq \nu, \\ q_t^3 = q_t \end{cases} \quad \Leftrightarrow \quad \begin{cases} y_t = 0, \\ \theta_t = 0 \ \textit{or}\ \pi. \end{cases}$

(2) *Let* $\nu = (k, \psi, r) \in N_2$. *Then*:

(2.1) $\nu \in \mathrm{MAX}_t^1 \quad \Leftrightarrow \quad \begin{cases} \nu^1 \neq \nu, \\ q_t^1 = q_t \end{cases} \quad \Leftrightarrow \quad \theta_t = 0;$

(2.2) $\nu \in \mathrm{MAX}_t^2 \quad \Leftrightarrow \quad \begin{cases} \nu^2 \neq \nu, \\ q_t^2 = q_t \end{cases} \quad \Leftrightarrow \quad \begin{cases} \operatorname{sn}\tau \operatorname{cn}\tau \neq 0, \\ P_t = 0; \end{cases}$

(2.3) $\nu \in \mathrm{MAX}_t^3 \quad \Leftrightarrow \quad \begin{cases} \nu^3 \neq \nu, \\ q_t^3 = q_t \end{cases} \quad \Leftrightarrow \quad \begin{cases} y_t = 0, \\ \theta_t = 0 \ \textit{or}\ \pi. \end{cases}$

(3) *Let* $\nu = (\varphi, r) \in N_3$. *Then*:

(3.1) $\nu \in \mathrm{MAX}_t^1 \quad \Leftrightarrow \quad \begin{cases} \nu^1 \neq \nu, \\ q_t^1 = q_t \end{cases} \quad \Leftrightarrow \quad \theta_t = 0;$

(3.2) $\nu \in \mathrm{MAX}_t^2 \quad \Leftrightarrow \quad \begin{cases} \nu^2 \neq \nu, \\ q_t^2 = q_t \end{cases} \quad \Leftrightarrow \quad \begin{cases} \tau \neq 0, \\ P_t = 0; \end{cases}$

(3.3) $\nu \in \mathrm{MAX}_t^3 \quad \Leftrightarrow \quad \begin{cases} \nu^3 \neq \nu, \\ q_t^3 = q_t \end{cases} \quad \Leftrightarrow \quad \begin{cases} y_t = 0, \\ \theta_t = 0 \ \textit{or}\ \pi. \end{cases}$

(4) $\mathrm{MAX}_t^i \cap N_j = \emptyset$ *for* $i = 1, 2, 3,\ j = 4, 5, 7$.

(6) *Let* $\nu \in N_6$. *Then*:

(6.1) $\nu \in \mathrm{MAX}_t^1 \quad \Leftrightarrow \quad \begin{cases} \nu^1 \neq \nu, \\ q_t^1 = q_t \end{cases} \quad \Leftrightarrow \quad \theta_t = 0;$

(6.2) $\nu \in \mathrm{MAX}_t^2$ *is impossible*;

(6.3) $\nu \in \mathrm{MAX}_t^3 \quad \Leftrightarrow \quad \begin{cases} \nu^3 \neq \nu, \\ q_t^3 = q_t \end{cases} \quad \Leftrightarrow \quad \begin{cases} y_t = 0, \\ \theta_t = 0 \ \textit{or}\ \pi. \end{cases}$

Proof. In view of the remark after the definition of Maxwell strata (10.6) and Proposition 10.2, we have

$$\begin{aligned} &\mathrm{MAX}_t^i = \{\nu \in N \mid \nu^i \neq \nu,\ q_t^i = q_t\}, \quad i = 1, 3, \\ &\mathrm{MAX}_t^2 \cap N_j = \{\nu \in N_j \mid \nu^2 \neq \nu,\ q_t^2 = q_t\}, \quad j \neq 6, \\ &\mathrm{MAX}_t^2 \cap N_6 = \emptyset. \end{aligned}$$

This proves the first implication in items (1.1)–(3.3). The second implication in these items directly follows by combination of Propositions 10.4, 10.5, and 10.6 with Proposition 10.3. Therefore, items (1)–(3) follow.

In the case $\nu \in N_4 \cup N_5 \cup N_7$ the corresponding extremal trajectory is $(x_s, y_s, \theta_s) = (s, 0, 0)$, which is globally optimal since elastic energy of the straight segment is $J = 0$. By Proposition 10.1, there are no Maxwell points in this case.

Finally, let $\nu \in N_6$. Items (6.1) and (6.2) follow by combination of Proposition 10.7 with Proposition 10.3. Item (6.2) was already obtained in (10.8) from item (2) of Proposition 10.2. $\square$

Remark. Items (1.3), (2.3), (3.3.), (4), and (6.3) of Theorem 10.1 show that the Maxwell stratum MAX_t^3 admits a decomposition into two disjoint subsets:

$$\mathrm{MAX}_t^3 = \mathrm{MAX}_t^{3+} \cup \mathrm{MAX}_t^{3-}, \quad \mathrm{MAX}_t^{3+} \cap \mathrm{MAX}_t^{3-} = \emptyset,$$

$$\nu \in \mathrm{MAX}_t^{3+} \quad \Leftrightarrow \quad \begin{cases} y_t = 0, \\ \theta_t = 0, \end{cases}$$

$$\nu \in \mathrm{MAX}_t^{3-} \quad \Leftrightarrow \quad \begin{cases} y_t = 0, \\ \theta_t = \pi. \end{cases}$$

In order to obtain a complete description of the Maxwell strata MAX_t^i, in the next section we solve the equations that determine these strata and appear in Theorem 10.1.

11. Complete description of Maxwell strata

11.1. **Roots of the equation $\theta = 0$.** In this section, we solve the equation $\theta_t = 0$ that determines the Maxwell stratum MAX_t^1 (see Theorem 10.1).

We denote by $\left[\begin{array}{l} A \\ B \end{array} \right.$ the condition $A \vee B$ in contrast to $\begin{cases} A \\ B \end{cases}$, which denotes the condition $A \wedge B$.

Proposition 11.1. *Let $\nu = (k, \varphi, r) \in N_1$, then*

$$\theta_t = 0 \quad \Leftrightarrow \quad \left[\begin{array}{l} p = 2Kn, \ n \in \mathbb{Z}, \\ \operatorname{cn} \tau = 0, \end{array} \right.$$

where $p = \dfrac{\sqrt{r}(\varphi_t - \varphi)}{2}$ and $\tau = \dfrac{\sqrt{r}(\varphi_t + \varphi)}{2}$.

Proof. We have

$$\theta_t = 0 \quad \Leftrightarrow \quad \frac{\beta_t}{2} = \frac{\beta_0}{2} \pmod \pi \quad \Leftrightarrow \quad \begin{cases} \operatorname{sn}(\sqrt{r}\varphi_t) = \operatorname{sn}(\sqrt{r}\varphi), \\ \operatorname{dn}(\sqrt{r}\varphi_t) = \operatorname{dn}(\sqrt{r}\varphi) \end{cases}$$

$$\Leftrightarrow \quad \left[\begin{array}{l} \sqrt{r}\varphi_t = \sqrt{r}\varphi \pmod{4\sqrt{r}K} \\ \sqrt{r}\varphi_t = 2\sqrt{r}K - \sqrt{r}\varphi \pmod{4\sqrt{r}K} \end{array} \right.$$

$$\Leftrightarrow \quad \left[\begin{array}{l} p = 2Kn, \ n \in \mathbb{Z} \\ \operatorname{cn} \tau = 0. \end{array} \right.$$

The proposition is proved. $\square$

Proposition 11.2. *Let $\nu = (k, \psi, r) \in N_2$, then*

$$\theta_t = 0 \quad \Leftrightarrow \quad p = Kn, \ n \in \mathbb{Z},$$

where $p = \dfrac{\sqrt{r}(\psi_t - \psi)}{2}$.

Proof. Let $\nu \in N_2^+$, then

$$\theta_t = 0 \quad \Leftrightarrow \quad \frac{\beta_t}{2} = \frac{\beta_0}{2} \pmod{\pi} \quad \Leftrightarrow \quad \begin{cases} \operatorname{sn}(\sqrt{r}\psi_t) = \pm \operatorname{sn}(\sqrt{r}\psi), \\ \operatorname{cn}(\sqrt{r}\psi_t) = \pm \operatorname{cn}(\sqrt{r}\psi) \end{cases}$$

$$\Leftrightarrow \quad \left[\begin{array}{l} p = 0 \pmod{2K} \\ p = K \pmod{2K} \end{array} \right. \quad \Leftrightarrow \quad p = Kn, \ n \in \mathbb{Z}.$$

If $\nu \in N_2^-$, then the same result is obtained by the inversion $i : N_2^+ \to N_2^-$. □

Proposition 11.3. *Let $\nu \in N_3$, then*

$$\theta_t = 0 \quad \Leftrightarrow \quad t = 0.$$

Proof. The proof is similar to that of Proposition 11.2 (see details in [34]). □

Proposition 11.4. *Let $\nu \in N_6$. Then*

$$\theta_t = 0 \quad \Leftrightarrow \quad ct = 2\pi n, \quad n \in \mathbb{Z}.$$

Proof. We have $\theta_t = ct$ in the case $\nu \in N_6$. □

11.2. **Roots of the equation $P = 0$ for $\nu \in N_1$.** Using the coordinates

$$\tau = \frac{\sqrt{r}(\varphi_t + \varphi)}{2} = \sqrt{r}\left(\varphi + \frac{t}{2}\right), \quad p = \frac{\sqrt{r}(\varphi_t - \varphi)}{2} = \frac{\sqrt{r}t}{2}, \tag{11.1}$$

and the addition formulas for the Jacobi functions (see [22, 34]), we obtain the following in the case $\nu \in N_1$:

$$\begin{gathered} P_t = \frac{4k \operatorname{sn}\tau \operatorname{dn}\tau f_1(p,k)}{\sqrt{r}\Delta}, \quad \nu \in N_1, \\ f_1(p,k) = \operatorname{sn} p \operatorname{dn} p - (2\operatorname{E}(p) - p)\operatorname{cn} p, \\ \Delta = 1 - k^2 \operatorname{sn}^2 p \operatorname{sn}^2 \tau. \end{gathered} \tag{11.2}$$

In order to describe roots of the equation $f_1(p) = 0$, we need the following statements. We denote by $E(k)$ and $K(k)$ the complete elliptic integrals of the first and second kinds respectively (see [22, 34]).

Proposition 11.5 (see [33, Lemma 2.1])**.** *The equation*

$$2E(k) - K(k) = 0, \quad k \in [0, 1),$$

has a unique root $k_0 \in (0, 1)$. Moreover,

$$k \in [0, k_0) \quad \Rightarrow \quad 2E - K > 0,$$

$$k \in (k_0, 1) \quad \Rightarrow \quad 2E - K < 0.$$

Note that for $k = 1/\sqrt{2}$ we have

$$K = \frac{1}{4\sqrt{\pi}} \left(\Gamma \left(\frac{1}{4} \right) \right)^2, \quad E = \frac{2K^2 + \pi}{4K} \quad \Rightarrow \quad 2E - K = \frac{\pi}{2K} > 0 \quad (11.3)$$

(see [22, p. 89, Chap. 3, Exercise 24]). Thus,

$$\frac{1}{\sqrt{2}} < k_0 < 1. \quad (11.4)$$

Numerical simulations show that $k_0 \approx 0.909$. To the value $k = k_0$, there corresponds the periodic Euler elastica in the form of figure-8 (see Fig. 8).

Proposition 11.6 (see [33, Proposition 2.1])**.** *For any $k \in [0, 1)$, the function*

$$f_1(p, k) = \operatorname{sn} p \operatorname{dn} p - (2\operatorname{E}(p) - p) \operatorname{cn} p$$

has a countable number of roots p_n^1, $n \in \mathbb{Z}$. These roots are odd in n:

$$p_{-n}^1 = -p_n^1, \quad n \in \mathbb{Z};$$

in particular, $p_0^1 = 0$. The roots p_n^1 are localized as follows:

$$p_n^1 \in (-K + 2Kn,\ K + 2Kn), \quad n \in \mathbb{Z}.$$

In particular, the roots p_n^1 are monotone in n:

$$p_n^1 < p_{n+1}^1, \quad n \in \mathbb{Z}.$$

Moreover, for $n \in \mathbb{N}$

$$\begin{aligned} k \in [0, k_0) \quad &\Rightarrow \quad p_n^1 \in (2Kn, K + 2Kn), \\ k = k_0 \quad &\Rightarrow \quad p_n^1 = 2Kn, \\ k \in (k_0, 1) \quad &\Rightarrow \quad p_n^1 \in (-K + 2Kn, 2Kn), \end{aligned}$$

where k_0 is the unique root of the equation $2E(k) - K(k) = 0$ (see Proposition 11.5).

Proposition 11.7 (see [33, Corollary 2.1])**.** *The first positive root $p = p_1^1$ of the equation $f_1(p) = 0$ is localized as follows:*

$$\begin{aligned} k \in [0, k_0) \quad &\Rightarrow \quad p_1^1 \in (2K, 3K), \\ k = k_0 \quad &\Rightarrow \quad p_1^1 = 2K, \\ k \in (k_0, 1) \quad &\Rightarrow \quad p_1^1 \in (K, 2K). \end{aligned}$$

Now we can obtain the following description of roots of the equation $P_t = 0$ for $\nu \in N_1$.

Proposition 11.8. *Let $\nu \in N_1$. Then*

$$P_t = 0 \quad \Leftrightarrow \quad \left[\begin{array}{l} f_1(p) = 0, \\ \operatorname{sn} \tau = 0 \end{array} \right. \quad \Leftrightarrow \quad \left[\begin{array}{l} p = p_n^1, \quad n \in \mathbb{Z}, \\ \operatorname{sn} \tau = 0. \end{array} \right.$$

Proof. Apply Eq. (11.2) and Proposition 11.6. □

11.3. **Roots of the equation $P = 0$ for $\nu \in N_2$.** In order to find the expression for P_t in the case $\nu \in N_2^+$, we apply the transformation of the Jacobi functions $k \mapsto 1/k$ (see Sec. 7.2.2 and (7.14) and (7.15)) to Eq. (11.2):

$$\begin{gathered} P_t = \frac{4k_1 \operatorname{sn}(\tau_1, k_1) \operatorname{dn}(\tau_1, k_1) f_1(p_1, k_1)}{\sqrt{r}(1 - k_1^2 \operatorname{sn}^2(p_1, k_1) \operatorname{sn}^2(\tau_1, k_1))}, \quad \nu \in N_1, \quad k_1 \in (0,1), \\ \tau_1 = \frac{\sqrt{r}(\varphi_t + \varphi)}{2}, \quad p_1 = \frac{\sqrt{r}(\varphi_t - \varphi)}{2}. \end{gathered} \tag{11.5}$$

The both sides of Eq. (11.5) are analytic single-valued functions of the elliptic coordinates (k_1, φ, r) and, therefore, this equality is preserved after an analytic continuation to the domain $k_1 \in (1, +\infty)$, i.e., $\nu \in N_2^+$.

Denote $k_2 = 1/k_1$, then the formulas for the transformation $k \mapsto 1/k$ of the Jacobi functions (7.14) and (7.15) yield the following:

$$\begin{aligned} P_t &= \frac{4\frac{1}{k_2} \operatorname{sn}(\tau_1, \frac{1}{k_2}) \operatorname{cn}(\tau_1, \frac{1}{k_2}) f_1(p_1, \frac{1}{k_2})}{\sqrt{r}(1 - \frac{1}{k_2^2} \operatorname{sn}^2(p_1, \frac{1}{k_2}) \operatorname{sn}^2(\tau_1, \frac{1}{k_2}))} \\ &= \frac{4 \operatorname{sn}(\tau_2, k_2) \operatorname{cn}(\tau_2, k_2) f_2(p_2, k_2)}{\sqrt{r}(1 - k_2^2 \operatorname{sn}^2(p_2, k_2) \operatorname{sn}^2(\tau_2, k_2))}, \end{aligned}$$

where

$$\tau_2 = \frac{\tau_1}{k_2} = \frac{\sqrt{r}(\varphi_t + \varphi)}{2k_2} = \frac{\sqrt{r}(\psi_t + \psi)}{2}, \tag{11.6}$$

$$p_2 = \frac{p_1}{k_2} = \frac{\sqrt{r}(\varphi_t - \varphi)}{2k_2} = \frac{\sqrt{r}(\psi_t - \psi)}{2}, \tag{11.7}$$

and

$$\begin{aligned} f_2(p_2, k_2) = \frac{1}{k_2}[&k_2^2 \operatorname{sn}(p_2, k_2) \operatorname{cn}(p_2, k_2) \\ &+ \operatorname{dn}(p_2, k_2)((2 - k_2^2)p_2 - 2\operatorname{E}(p_2, k_2))]. \end{aligned}$$

Summing up, we have

$$\begin{gathered} P_t = \frac{4 \operatorname{sn}\tau \operatorname{cn}\tau f_2(p, k)}{\sqrt{r}\Delta}, \quad \nu \in N_2^+, \\ f_2(p, k) = \frac{1}{k}[k^2 \operatorname{sn} p \operatorname{cn} p + \operatorname{dn} p((2 - k^2)p - 2\operatorname{E}(p))], \\ \tau = \frac{\sqrt{r}(\psi_t + \psi)}{2}, \quad p = \frac{\sqrt{r}(\psi_t - \psi)}{2}, \quad \Delta = 1 - k^2 \operatorname{sn}^2 p \operatorname{sn}^2 \tau. \end{gathered}$$

We will need the following statement.

Proposition 11.9 (see [33, Proposition 3.1])**.** *The function $f_2(p)$ has no roots $p \neq 0$.*

Proposition 11.10. *Let $\nu \in N_2$. Then*

$$P_t = 0 \quad \Leftrightarrow \quad \left[\begin{array}{l} p = 0 \\ \operatorname{sn}\tau \operatorname{cn}\tau = 0. \end{array} \right.$$

Proof. In the case $\nu \in N_2^+$, we obtain from (11.8) and Proposition 11.9:

$$P_t = 0 \quad \Leftrightarrow \quad \left[\begin{array}{l} f_2(p) = 0 \\ \operatorname{sn}\tau \operatorname{cn}\tau = 0 \end{array} \right. \quad \Leftrightarrow \quad \left[\begin{array}{l} p = 0 \\ \operatorname{sn}\tau \operatorname{cn}\tau = 0. \end{array} \right.$$

The case $\nu \in N_2^-$ is obtained by the inversion $i : N_2^+ \to N_2^-$:

$$P_t = -\frac{4 \operatorname{sn}\tau \operatorname{cn}\tau f_2(p,k)}{\sqrt{r}\Delta}, \quad \nu \in N_2^- \tag{11.8}$$

(see details in [34]), and the statement for the case $\nu \in N_2^-$ is proved. □

11.4. **Roots of the equation $P = 0$ for $\nu \in N_3$.** Passing to the limit $k \to 1 - 0$ in Eqs. (11.8) and (11.8), we obtain the following:

$$P_t = \pm \frac{4 \tanh\tau f_2(p,1)}{\sqrt{r}\cosh\tau(1 - \tanh^2 p \tanh^2 \tau)}, \quad \nu \in N_3^{\pm}, \tag{11.9}$$

$$p = \frac{\sqrt{r}(\varphi_t - \varphi)}{2}, \quad \tau = \frac{\sqrt{r}(\varphi_t + \varphi)}{2}, \tag{11.10}$$

$$f_2(p,1) = \lim_{k \to 1-0} f_2(p,k) = \frac{2p - \tanh p}{\cosh p}. \tag{11.11}$$

Proposition 11.11. *Let $\nu \in N_3$. Then*

$$P_t = 0 \quad \Leftrightarrow \quad \left[\begin{array}{l} p = 0 \\ \tau = 0. \end{array} \right.$$

Proof. We have

$$(2p - \tanh p)' = 2 - \frac{1}{\cosh^2 p} > 1$$

and, therefore,

$$f_2(p,1) = 0 \quad \Leftrightarrow \quad 2p - \tanh p = 0 \quad \Leftrightarrow \quad p = 0,$$

and the statement follows from (11.9). □

11.5. **Roots of the equation $P = 0$ for $\nu \in N_6$.**

Proposition 11.12. *If $\nu \in N_6$, then $P_t \equiv 0$.*

Proof. We have

$$P_t = x_t \sin\frac{\theta_t}{2} - y_t \cos\frac{\theta_t}{2} = \frac{1}{c}\sin ct \sin\frac{ct}{2} - \frac{1}{c}(1 - \cos ct)\cos\frac{ct}{2} \equiv 0.$$

The proposition is proved. □

The visual meaning of this proposition is simple: an arc of a circle has the same angles with its chord at the initial and terminal points.

11.6. **Roots of the system $y = 0$, $\theta = 0$.** Note that

$$\begin{cases} \theta_t = 0, \\ y_t = 0 \end{cases} \quad \Leftrightarrow \quad \begin{cases} \theta_t = 0, \\ P_t = x_t \sin \dfrac{\theta_t}{2} - y_t \cos \dfrac{\theta_t}{2} = 0, \end{cases}$$

therefore, we can replace the first system by the second one and use our previous results for equations $\theta_t = 0$ and $P_t = 0$.

Proposition 11.13. *Let $\nu \in N_1$. Then*

$$\begin{cases} \theta_t = 0, \\ P_t = 0 \end{cases} \quad \Leftrightarrow \quad \left[\begin{array}{ll} k = k_0, & p = 2Kn, \\ p = p_n^1, & \operatorname{cn} \tau = 0, \\ p = 2Kn, & \operatorname{sn} \tau = 0, \end{array} \right. \qquad n \in \mathbb{Z}.$$

Proof. By virtue of Propositions 11.1 and 11.8, we have

$$\begin{aligned} \begin{cases} \theta_t = 0, \\ P_t = 0 \end{cases} \quad &\Leftrightarrow \quad \begin{cases} p = 2Km \quad \text{or} \quad \operatorname{cn} \tau = 0, \\ p = p_n^1 \quad \text{or} \quad \operatorname{sn} \tau = 0 \end{cases} \\ &\Leftrightarrow \quad \begin{cases} p = 2Km, \\ p = p_n^1, \end{cases} \quad \text{or} \quad \begin{cases} \operatorname{cn} \tau = 0, \\ p = p_n^1, \end{cases} \\ &\quad \text{or} \quad \begin{cases} p = 2Km, \\ \operatorname{sn} \tau = 0, \end{cases} \quad \text{or} \quad \begin{cases} \operatorname{cn} \tau = 0, \\ \operatorname{sn} \tau = 0. \end{cases} \end{aligned}$$

By Proposition 11.6,

$$\begin{cases} p = 2Km, \\ p = p_n^1 \end{cases} \quad \Leftrightarrow \quad p = p_n^1 = 2Kn \quad \Leftrightarrow \quad \begin{cases} k = k_0, \\ p = 2Kn. \end{cases}$$

Now it remains to note that the system $\operatorname{cn} \tau = 0$, $\operatorname{sn} \tau = 0$ is incompatible, and the proof is complete. □

Proposition 11.14. *Let $\nu \in N_2$. Then*

$$\begin{cases} \theta_t = 0, \\ P_t = 0 \end{cases} \quad \Leftrightarrow \quad \left[\begin{array}{ll} p = Kn, & \tau = Km, \\ p = 0, & \end{array} \right. \qquad n, m \in \mathbb{Z}.$$

Proof. Taking into account Propositions 11.2 and 11.10, we obtain

$$\begin{cases} \theta_t = 0, \\ P_t = 0 \end{cases} \quad \Leftrightarrow \quad \begin{cases} p = Kn, \\ p = 0 \quad \text{or} \quad \tau = Km \end{cases} \quad \Leftrightarrow \quad \left[\begin{array}{ll} p = Kn, & \tau = Km, \\ p = 0. & \end{array} \right.$$

The proposition is proved. □

Proposition 11.15. *Let $\nu \in N_3$. Then*

$$\begin{cases} \theta_t = 0, \\ P_t = 0 \end{cases} \quad \Leftrightarrow \quad t = 0.$$

Proof. The proof immediately follows from Propositions 11.3 and 11.11. □

Proposition 11.16. *Let $\nu \in N_6$. Then*

$$\begin{cases} \theta_t = 0, \\ P_t = 0 \end{cases} \quad \Leftrightarrow \quad ct = 2\pi n, \quad n \in \mathbb{Z}.$$

Proof. The proof immediately follows from Propositions 11.4 and 11.12. □

11.7. **Roots of the system $y = 0$, $\theta = \pi$ for $\nu \in N_1$.** The structure of solutions of the system $y_t = 0$, $\theta_t = \pi$ is much more complicated than that of the system $y_t = 0$, $\theta_t = 0$ studied above.

First, for any normal extremal

$$\begin{cases} \theta_t = 0, \\ y_t = \pi \end{cases} \quad \Leftrightarrow \quad \begin{cases} \cos \dfrac{\theta_t}{2} = 0, \\ Q_t = x_t \cos \dfrac{\theta_t}{2} + y_t \sin \dfrac{\theta_t}{2} = 0. \end{cases} \tag{11.12}$$

From now on, we suppose in this subsection that $\nu \in N_1$.

In the same way as at the beginning of Sec. 11.2, in the coordinates τ and p given by (11.1) we obtain

$$\begin{aligned} \cos \frac{\theta_t}{2} &= (\operatorname{dn}^2 p - k^2 \operatorname{sn}^2 p \operatorname{cn}^2 \tau)(\operatorname{dn}^2 \tau + k^2 \operatorname{cn}^2 p \operatorname{sn}^2 \tau)/\Delta^2 \\ &= (1 - 2k^2 \operatorname{sn}^2 p + k^2 \operatorname{sn}^2 p \operatorname{sn}^2 \tau)(\operatorname{dn}^2 \tau + k^2 \operatorname{cn}^2 p \operatorname{sn}^2 \tau)/\Delta^2, \\ Q_t &= 2\operatorname{E}(p) - p + k^2 \operatorname{sn}^2 \tau (2 \operatorname{cn} p \operatorname{sn} p \operatorname{dn} p - (2\operatorname{E}(p) - p)(2 - \operatorname{sn}^2 p)). \end{aligned}$$

Thus,

$$\cos \frac{\theta_t}{2} = 0 \quad \Leftrightarrow \quad \operatorname{sn}^2 \tau = (2k^2 \operatorname{sn}^2 p - 1)/(k^2 \operatorname{sn}^2 \tau).$$

Substituting this value for $\operatorname{sn}^2 \tau$ into Q_t, we get rid of the variable τ in the second equation in (11.12):

$$\begin{aligned} Q_t|_{\operatorname{sn}^2 \tau = (2k^2 \operatorname{sn}^2 p - 1)/(k^2 \operatorname{sn}^2 \tau)} &= \frac{2}{\operatorname{sn}^2 p} g_1(p,k), \\ g_1(p,k) &= (1 - k^2 + k^2 \operatorname{cn}^4 p)(2\operatorname{E}(p) - p) \\ &\quad + \operatorname{cn} p \operatorname{sn} p \operatorname{dn} p (2k^2 \operatorname{sn}^2 p - 1). \end{aligned} \tag{11.13}$$

As a consequence, we obtain the following statement (see details in [34]).

Proposition 11.17. *Let $\nu \in N_1$. Then*

$$\begin{cases} \theta_t = \pi, \\ y_t = 0 \end{cases} \quad \Leftrightarrow \quad \begin{cases} \operatorname{sn}^2 \tau = (2k^2 \operatorname{sn}^2 p - 1)/(k^2 \operatorname{sn}^2 \tau), \\ g_1(p,k) = 0. \end{cases} \tag{11.14}$$

Now we study the solvability of the second system in (11.14) and describe its solutions in the domain $\{p \in (0, 2K)\}$. For the study of the global optimality of normal extremal trajectories, it is essential to know the first Maxwell point. By Proposition 11.1, the first Maxwell point corresponding to ε^1 occurs at $p = 2K$ and, therefore, for the study of the global optimal control problem we can restrict ourselves by the domain $\{p \in (0, 2K)\}$. As concerns the local problem, in the forthcoming paper [35] we show that only the Maxwell strata MAX_t^1, MAX_t^2, but not MAX_t^3 are important for the local optimality. But for the global problem, the stratum MAX_t^3 is very important: in fact, extremal trajectories lose global optimality on this stratum, i.e., MAX_t^3 provides a part of the cut locus [35, 36].

The second system in (11.14) is compatible iff the equation $g_1(p,k) = 0$ has solutions (p,k) such that

$$0 \le \frac{2k^2 \operatorname{sn}^2 p - 1}{k^2 \operatorname{sn}^2 p} \le 1,$$

or, which is equivalent,

$$2k^2 \operatorname{sn}^2 p - 1 \ge 0. \tag{11.15}$$

After the change of variable

$$p = F(u,k) = \int_0^u \frac{dt}{\sqrt{1 - k^2 \sin^2 t}} \quad \Leftrightarrow \quad u = \operatorname{am}(p,k), \tag{11.16}$$

where $\operatorname{am}(p,k)$ is the Jacobi amplitude (see [22, 34]), we obtain

$$\begin{aligned} g_1(p,k) &= h_1(u,k), \\ h_1(u,k) &= (1 - k^2 + k^2 \cos^4 u)(2E(u,k) - F(u,k)) \\ &\quad + \cos u \sin u \sqrt{1 - k^2 \sin^2 u}(2k^2 \sin^2 u - 1). \end{aligned} \tag{11.17}$$

Denote

$$\begin{aligned} h_2(u,k) = \frac{h_1(u,k)}{1 - k^2 + k^2 \cos^4 u} &= 2E(u,k) - F(u,k) \\ &\quad + \frac{\cos u \sin u \sqrt{1 - k^2 \sin^2 u}(2k^2 \sin^2 u - 1)}{1 - k^2 + k^2 \cos^4 u}; \end{aligned} \tag{11.18}$$

a direct computation yields

$$\frac{\partial h_2}{\partial u} = \frac{\sin^2 u \sqrt{2 - k^2 + k^2 \cos 2u}}{4\sqrt{2}(1 - k^2 + k^2 \cos^4 u)^2} \, a_1(u,k), \tag{11.19}$$

$$a_1(u,k) = c_0 + c_1 \cos 2u + c_2 \cos^2 2u, \tag{11.20}$$

$$c_0 = 8 - 10k^2 + 4k^4, \quad c_1 = 4k^2(3 - 2k^2), \quad c_2 = 2k^2(2k^2 - 1).$$

One can prove the following statement.

Proposition 11.18. (1) *The set*

$$\left\{(u,k) \in \mathbb{R} \times \left[\frac{1}{\sqrt{2}}, 1\right] \mid a_1(u,k) = 0\right\}$$

is a smooth curve.

(2) *There exists a function*

$$u_{a_1} : \left[\frac{1}{\sqrt{2}}, 1\right] \to \left(\frac{\pi}{4}, \frac{\pi}{2}\right], \quad u = u_{a_1}(k),$$

such that

$$k = \frac{1}{\sqrt{2}},\ 1 \qquad \Rightarrow \qquad u_{a_1}(k) = \frac{\pi}{2},$$
$$k \in \left(\frac{1}{\sqrt{2}}, 1\right) \qquad \Rightarrow \qquad u_{a_1}(k) \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right),$$

and for $k = \dfrac{1}{\sqrt{2}}, 1,$

$$a_1(u,k) = 0 \quad \Leftrightarrow \quad u = u_{a_1}(k) + \pi n = \frac{\pi}{2} + \pi n, \tag{11.21}$$

while for $k \in \left(\dfrac{1}{\sqrt{2}}, 1\right)$,

$$a_1(u,k) = 0 \quad \Leftrightarrow \quad \left[\begin{array}{l} u = u_{a_1}(k) + 2\pi n \\ u = \pi - u_{a_1}(k) + 2\pi n. \end{array}\right. \tag{11.22}$$

Moreover,

$$u_{a_1} \in C\left[\frac{1}{\sqrt{2}}, 1\right] \cap C^\infty\left(\frac{1}{\sqrt{2}}, 1\right). \tag{11.23}$$

Proof. The proof follows from the implicit function theorem (see details in [34]). □

Similarly to Proposition 11.18, one can prove the following statement.

Lemma 11.1. *There exist numbers* $k_* \in \left(\dfrac{1}{\sqrt{2}}, k_0\right)$, $u_* \in \left(\dfrac{\pi}{2}, \dfrac{3\pi}{4}\right)$ *and a function*

$$u_{h_1} : [k_*, 1] \to \left(\frac{\pi}{2}, \frac{3\pi}{4}\right)$$

such that for $k \in [k_*, 1]$, $u \in (0, \pi - u_{a_1}(k))$,

$$h_1(u,k) = 0 \quad \Leftrightarrow \quad u = u_{h_1}(k).$$

Moreover, $u_{h_1} \in C[k_*, 1] \cap C^\infty(k_*, 1)$ *and* $u_{h_1}(k_*) = u_*$ *and* $u_{h_1}(k_0) = u(1) = \pi/2$.

Numerical simulations give $k_* \approx 0.841$ and $u_* \approx 1.954$ (see a plot of the elastica corresponding to $k = k_*$ in [34]).

Lemma 11.1 describes the first solutions of the equation $h_1(u,k) = 0$ in u for $k \in [k_*, 1]$.

Proposition 11.19. *Let the function* $g_1(p,k)$ *be given by* (11.13).

(1) *The set*

$$\gamma_{g_1} = \{(p,k) \mid k \in (0,1),\ p \in (0, 2K(k)),\ g_1(p,k) = 0\}$$

is a smooth connected curve.

(2) *The curve* γ_{g_1} *does not intersect the domain*

$$\{(p,k) \mid k \in (0,k_*),\ p \in (0, 2K(k))\}.$$

(3) *The function*

$$p = p_{g_1}(k) = F(u_1(k), k), \quad p_{g_1} \in C^\infty(k_*, 1),$$

satisfies the condition

$$\min\{p > 0 \mid g_1(p,k) = 0\} = p_{g_1}(k), \quad k \in [k_*, 1).$$

The function $p = p_{g_1}(k)$ *satisfies the bounds*:

$$\begin{aligned} k \in [k_*, k_0) \quad &\Rightarrow \quad p_{g_1}(k) \in \left(K, \frac{3}{2}K\right), \\ k = k_0 \quad &\Rightarrow \quad p_{g_1}(k) = K, \\ k \in (k_0, 1) \quad &\Rightarrow \quad p_{g_1}(k) \in \left(\frac{1}{2}K, K\right). \end{aligned}$$

(4) *For any* $k \in [k_*, 1)$,

$$p = p_{g_1}(k) \quad \Rightarrow \quad 2k^2 \operatorname{sn}^2(p,k) - 1 \in (0, 1].$$

(5) *If* $k \in (0, k_*)$, *then the system of equations* (11.14) *has no solutions* (p, τ) *such that* $p \in (0, 2K(k))$. *If* $k \in [k_*, 1)$, *then the minimum* $p \in (0, 2K(k))$ *such that system* (11.14) *has a solution* (p, τ) *is* $p = p_{g_1}(k)$.

Proof. The proof follows from the implicit function theorem (see details in [34]). □

Thus, we have described the first solution of system (11.14) obtained in Proposition 11.17.

11.8. **Roots of the system $y = 0$, $\theta = \pi$ for $\nu \in N_2$.** Similarly to Proposition 11.17, we have the following statement.

Proposition 11.20.

$$\begin{cases} \theta_t = \pi, \\ y_t = 0 \end{cases} \quad \Leftrightarrow \quad \begin{cases} \operatorname{sn}^2 \tau = \dfrac{2 \operatorname{sn}^2 p - 1}{k^2 \operatorname{sn}^2 p}, \\ g_1(p,k) = 0, \end{cases} \tag{11.24}$$

where

$$\begin{aligned} g_1(p,k) = \frac{1}{k}[&k^2 \operatorname{cn} p \operatorname{sn} p \operatorname{dn} p (2 \operatorname{sn}^2 p - 1) \\ &+ (1 - 2\operatorname{sn}^2 p + k^2 \operatorname{sn}^4 p)(2\operatorname{E}(p) - (2 - k^2)p)]. \end{aligned} \tag{11.25}$$

Proof. Let $\nu \in N_2^+$. We apply the equivalence relation (11.12). Further, in order to obtain expressions for $\cos\dfrac{\theta_t}{2}$ and Q_t through the variables τ_2, p_2 given by (11.6), (11.7), we apply the transformation of the Jacobi functions $k \mapsto 1/k$ in the same way as we did in Sec. 11.3 and obtain

$$\cos\frac{\theta_t}{2} = \frac{(1 - 2\operatorname{sn}^2 p + k^2 \operatorname{sn}^2 p \operatorname{sn}^2 \tau)(\operatorname{cn}^2 \tau + \operatorname{dn}^2 p \operatorname{sn}^2 \tau)}{\Delta^2}, \tag{11.26}$$

$$\begin{aligned} Q_t &= \frac{1}{k}[(2\operatorname{E}(p) - (2-k^2)p) \\ &\quad + \operatorname{sn}^2 \tau(2k^2 \operatorname{cn} p \operatorname{sn} p \operatorname{dn} p - (2\operatorname{E}(p) - (2-k^2)p)(2 - k^2 \operatorname{sn}^2 p)]. \end{aligned}$$

Consequently, $\cos\dfrac{\theta_t}{2} = 0 \quad \Leftrightarrow \quad \operatorname{sn}^2 \tau = \dfrac{2\operatorname{sn}^2 p - 1}{k^2 \operatorname{sn}^2 p}$. Then a direct computation yields

$$Q_t|_{\operatorname{sn}^2 \tau = (2\operatorname{sn}^2 p - 1)/(k^2 \operatorname{sn}^2 p)} = \frac{2}{k^2 \operatorname{sn}^2 p} g_1(p,k),$$

where the function $g_1(p,k)$ is defined in (11.25). The statement of this proposition is proved for $\nu \in N_2^+$, and for $\nu \in N_2^-$ it is obtained via the inversion $i : N_2^+ \to N_2^-$. $\square$

Now we study the solvability of the system of equations (11.24) in the domain $p \in (0, K)$. This bound on p is given by the minimum $p = K$ for points in MAX_t^2 (see Proposition 11.2).

Proposition 11.21. *Let the function $g_1(p,k)$ be given by* (11.25)*. Then for any $k \in (0,1)$, $p \in (0, K(k))$ we have $g_1(p,k) > 0$.*

Proof. The proof follows by the standard monotonicity argument (see details in [34]). $\square$

In fact, numerical simulations show that the equation $g_1(p,k) = 0$ has solutions $p > K$.

11.9. Roots of the system $y = 0$, $\theta = \pi$ for $\nu \in N_3$.

Proposition 11.22. *If $\nu \in N_3$, then the system of equations $y_t = 0$, $\theta_t = \pi$ is incompatible for $t > 0$.*

Proof. Let $\nu \in N_3^+$. We pass to the limit $k \to 1 - 0$ in Propositions 11.20 and 11.21 and obtain that the system of equations $y_t = 0$, $\theta_t = \pi$ has no roots for $p \in (0, K(1-0))$, $p = \sqrt{r}t/2$. But $K(1-0) = \lim\limits_{k \to 1-0} K(k) = +\infty$. Thus, the system in question is incompatible for $t > 0$ and $\nu \in N_3^+$. The same result for $\nu \in N_3^-$ follows via the inversion $i : N_3^+ \to N_3^-$. $\square$

11.10. **Roots of the system $y = 0$, $\theta = \pi$ for $\nu \in N_6$.**

Proposition 11.23. *If $\nu \in N_6$, then the system of equations $y_t = 0$, $\theta_t = \pi$ is incompatible.*

Proof. As always, we can restrict ourselves by the case $\nu \in N_6^+$. Then it is obvious that the system is incompatible:

$$y_t = \frac{1 - \cos ct}{c} = 0, \qquad \theta_t = ct = \pi + 2\pi k.$$

The proposition is proved. $\square$

11.11. **Complete description of Maxwell strata.** Now we can summarize our previous results and obtain the following statement.

Theorem 11.1. (1.1) $N_1 \cap \mathrm{MAX}_t^1 = \{\nu \in N_1 \mid p = 2Kn,\ \operatorname{cn}\tau \neq 0\}$,

(1.2) $N_1 \cap \mathrm{MAX}_t^2 = \{\nu \in N_1 \mid p = p_n^1,\ \operatorname{sn}\tau \neq 0\}$,

(1.3+) $N_1 \cap \mathrm{MAX}_t^{3+} = \{\nu \in N_1 \mid (k,p) = (k_0, 2Kn)$ *or* $(p = p_n^1, \operatorname{cn}\tau = 0)$ *or* $(p = 2Kn, \operatorname{sn}\tau = 0)\}$,

(1.3−) $N_1 \cap \mathrm{MAX}_t^{3-} = \{\nu \in N_1 \mid g_1(p,k) = 0,\ \operatorname{sn}^2\tau = (2k^2 \operatorname{sn}^2 p - 1)/(k^2 \operatorname{sn}^2 p)\}$,[1] $N_1 \cap \mathrm{MAX}_t^{3-} \cap \{p \in (0, 2K)\} = \{k \in [k_*, 1),\ p = p_{g_1}(k),\ \operatorname{sn}^2\tau = (2k^2 \operatorname{sn}^2 p - 1)/(k^2 \operatorname{sn}^2 p)\}$,[2],

(2.1) $N_2 \cap \mathrm{MAX}_t^1 = \{\nu \in N_2 \mid p = Kn,\ \operatorname{cn}\tau \operatorname{sn}\tau \neq 0\}$,

(2.2) $N_2 \cap \mathrm{MAX}_t^2 = \emptyset$,

(2.3+) $N_2 \cap \mathrm{MAX}_t^{3+} = \{\nu \in N_2 \mid p = Kn,\ \operatorname{sn}\tau \operatorname{cn}\tau = 0\}$,

(2.3−) $N_2 \cap \mathrm{MAX}_t^{3-} = \{\nu \in N_2 \mid g_1(p,k) = 0,\ \operatorname{sn}^2\tau = (2\operatorname{sn}^2 p - 1)/(k^2 \operatorname{sn}^2 p)\}$,[3], $N_2 \cap \mathrm{MAX}_t^{3-} \cap \{p \in (0,K)\} = \emptyset$,

(3.1) $N_3 \cap \mathrm{MAX}_t^1 = \emptyset$,

(3.2) $N_3 \cap \mathrm{MAX}_t^2 = \emptyset$,

(3.3+) $N_3 \cap \mathrm{MAX}_t^{3+} = \emptyset$,

(3.3−) $N_3 \cap \mathrm{MAX}_t^{3-} = \emptyset$,

(6.1) $N_6 \cap \mathrm{MAX}_t^1 = \{\nu \in N_6 \mid ct = 2\pi n\}$,

(6.2) $N_6 \cap \mathrm{MAX}_t^2 = \emptyset$,

(6.3+) $N_6 \cap \mathrm{MAX}_t^{3+} = \{\nu \in N_6 \mid ct = 2\pi n\}$,

(6.3−) $N_6 \cap \mathrm{MAX}_t^{3-} = \emptyset$.

Proof. It remains to compile the corresponding items of Theorem 10.1 with appropriate propositions of this section:

(1.1) $\Rightarrow$ Proposition 11.1;

(1.2) $\Rightarrow$ Proposition 11.8;

(1.3+) $\Rightarrow$ Proposition 11.13;

(1.3−) $\Rightarrow$ Propositions 11.17 and 11.19;

(2.1) $\Rightarrow$ Proposition 11.2;

[1] The function $g_1(p,k)$ is given by (11.13)

[2] k_* and $p_{g_1}(k)$ are described in Proposition 11.19

[3] The function $g_1(p,k)$ is given by (11.25)

(2.2) $\Rightarrow$ Proposition 11.10;
(2.3+) $\Rightarrow$ Proposition 11.14;
(2.3−) $\Rightarrow$ Proposition 11.20 and 11.21;
(3.1) $\Rightarrow$ Proposition 11.3;
(3.2) $\Rightarrow$ Proposition 11.11;
(3.3+) $\Rightarrow$ Proposition 11.15;
(3.3−) $\Rightarrow$ Proposition 11.22;
(6.1) $\Rightarrow$ Proposition 11.4;
(6.2) $\Rightarrow$ item (6.2) of Theorem 10.1;
(6.3+) $\Rightarrow$ Proposition 11.16;
(6.3−) $\Rightarrow$ Proposition 11.23. □

12. Upper bound of the cut time

Let q_s, $s > 0$, be an extremal trajectory of an optimal control problem of the form (5.9)–(5.11). The *cut time* for the trajectory q_s is defined as follows:

$$t_{\text{cut}} = \sup\{t_1 > 0 \mid q_s \text{ is optimal on } [0, t_1]\}.$$

For normal extremal trajectories $q_s = \operatorname{Exp}_s(\lambda)$, the cut time becomes a function of the initial covector λ:

$$t_{\text{cut}} : N = T^*_{q_0}M \to [0, +\infty], \quad t = t_{\text{cut}}(\lambda).$$

Short arcs of regular extremal trajectories are optimal and, therefore, $t_{\text{cut}}(\lambda) > 0$ for any $\lambda \in N$. On the other hand, some extremal trajectories can be optimal on an arbitrarily long segment $[0, t_1]$, $t_1 \in (0, +\infty)$; in this case $t_{\text{cut}} = +\infty$.

Denote the first Maxwell time as follows:

$$t_1^{\text{MAX}}(\lambda) = \inf\{t > 0 \mid \lambda \in \text{MAX}_t\}.$$

By Proposition 10.1, a normal extremal trajectory q_s cannot be optimal after a Maxwell point and, therefore,

$$t_{\text{cut}}(\lambda) \leq t_1^{\text{MAX}}(\lambda). \tag{12.1}$$

Now we return to the Euler elastic problem. For this problem, we can define the first instant in the Maxwell sets MAX^i, $i = 1, 2, 3$:

$$t_1^{\text{MAX}^i}(\lambda) = \inf\{t > 0 \mid \lambda \in \text{MAX}^i_t\}.$$

Since $t_1^{\text{MAX}}(\lambda) \leq t_1^{\text{MAX}^i}(\lambda)$, we obtain from inequality (12.1):

$$t_{\text{cut}}(\lambda) \leq \min(t_1^{\text{MAX}^i}(\lambda)), \quad i = 1, 2, 3.$$

Now we combine this inequality with the results of Sec. 11 and obtain an upper bound of the cut time in the Euler elastic problem. To this end, we define the following function:

$$\mathbf{t} : N \to (0, +\infty], \qquad \lambda \mapsto \mathbf{t}(\lambda),$$

$$
\begin{aligned}
&\lambda \in N_1 \quad \Rightarrow \quad \mathbf{t} = \frac{2}{\sqrt{r}} p_1(k), \\
&p_1(k) = \min(2K(k), p_1^1(k)) = \begin{cases} 2K(k), & k \in (0, k_0], \\ p_1^1(k), & k \in [k_0, 1), \end{cases} \qquad (12.2) \\
&\lambda \in N_2 \quad \Rightarrow \quad \mathbf{t} = \frac{2k}{\sqrt{r}} p_1(k), \quad p_1(k) = K(k), \\
&\lambda \in N_6 \quad \Rightarrow \quad \mathbf{t} = \frac{2\pi}{|c|}, \\
&\lambda \in N_3 \cup N_4 \cup N_5 \cup N_7 \quad \Rightarrow \quad \mathbf{t} = +\infty.
\end{aligned}
$$

Theorem 12.1. *Let $\lambda \in N$. We have*

$$t_{\mathrm{cut}}(\lambda) \le \mathbf{t}(\lambda) \tag{12.3}$$

in the following cases:

(1) $\lambda = (k, p, \tau) \in N_1$, $\operatorname{cn} \tau \operatorname{sn} \tau \neq 0$, *or*
(2) $\lambda \in N \setminus N_1$.

Proof. (1) Let $\lambda = (k, p, \tau) \in N_1$, $\operatorname{cn} \tau \operatorname{sn} \tau \neq 0$. Then Theorem 11.1 yields the following:

$$
\begin{aligned}
&k \in (0, k_0] \quad \Rightarrow \quad \mathbf{t}(\lambda) = \frac{2}{\sqrt{r}} 2K = t_1^{\mathrm{MAX}^1}(\lambda), \\
&k \in (k_0, 1) \quad \Rightarrow \quad \mathbf{t}(\lambda) = \frac{2}{\sqrt{r}} p_1^1(k) = t_1^{\mathrm{MAX}^2}(\lambda).
\end{aligned}
$$

(2) Let $\lambda = (k, p, \tau) \in N_2$, then we obtain from Theorem 11.1:

$$
\begin{aligned}
&\operatorname{sn} \tau \operatorname{cn} \tau \neq 0 \quad \Rightarrow \quad \mathbf{t}(\lambda) = \frac{2K(k)k}{\sqrt{r}} = t_1^{\mathrm{MAX}^1}(\lambda), \\
&\operatorname{sn} \tau \operatorname{cn} \tau = 0 \quad \Rightarrow \quad \mathbf{t}(\lambda) = \frac{2K(k)k}{\sqrt{r}} = t_1^{\mathrm{MAX}^{3+}}(\lambda).
\end{aligned}
$$

If $\lambda = (\beta, c, r) \in N_6$, then Theorem 11.1 implies that

$$\mathbf{t}(\lambda) = \frac{2\pi}{|c|} = t_1^{\mathrm{MAX}^1}(\lambda) = t_1^{\mathrm{MAX}^3}(\lambda).$$

If $\lambda \in N_3$, then there is nothing to prove since $\mathbf{t}(\lambda) = +\infty$.

If $\lambda \in N_4 \cup N_5 \cup N_7$, then there is also nothing to prove since, in this case, the extremal trajectory q_s is optimal on the whole ray $s \in [0, +\infty)$, and $t_{\mathrm{cut}}(\lambda) = \mathbf{t}(\lambda) = +\infty$. □

In the proof of Theorem 12.1, we used the explicit description (12.2) of the function $p_1(k) = \min(2K(k), p_1^1(k))$ which directly follows from Proposition 11.5.

Remarks. (1) For critical elasticae ($\lambda \in N_3$), the upper bound (12.3) of the cut time becomes trivial: $t_{\mathrm{cut}}(\lambda) \leq \mathbf{t}(\lambda) = +\infty$. We conjecture that the critical elasticae are optimal forever ($t_{\mathrm{cut}}(\lambda) = +\infty$). For straight lines ($\lambda \in N_4 \cup N_5 \cup N_7$) this is obvious.

(2) In the forthcoming work [35], we prove that if $\lambda = (k, p, \tau) \in N_1$ and $\operatorname{cn}\tau \operatorname{sn}\tau = 0$, then the corresponding point $q_t = \operatorname{Exp}_t(\lambda)$, $t = \mathbf{t}(\lambda)$ is conjugate and, therefore, the trajectory q_s is not optimal for $s > \mathbf{t}(\lambda)$; consequently, $t_{\mathrm{cut}}(\lambda) \leq \mathbf{t}(\lambda)$ (cf. item (1) of Theorem 12.1). Therefore, the bound (12.3) is valid for *all* $\lambda \in N$.

Note the different role of the Maxwell strata MAX^{3+} and MAX^{3-} for the upper bound of the cut time obtained in Theorem 12.1. On the one hand, the stratum MAX^{3+} generically does not give better bound of the cut time than the strata MAX^1, MAX^2 since generically

$$t_1^{\mathrm{MAX}^{3+}} = \min\left(t_1^{\mathrm{MAX}^1}, t_1^{\mathrm{MAX}^2}\right)$$

(see Theorem 11.1). This follows mainly from the fact that the system of equations determining the stratum MAX^{3+} consists of equations determining the strata MAX^1 and MAX^2:

$$\begin{cases} y_t = 0, \\ \theta_t = 0 \end{cases} \quad \Leftrightarrow \quad \begin{cases} P_t = 0, \\ \theta_t = 0 \end{cases}$$

(see Theorem 10.1).

The situation with the stratum MAX^{3-} is drastically different. By item (1.3−) of Theorem 11.1, we have

$$\nu = (k, p, \tau) \in N_1 \cap \mathrm{MAX}^{3-}, \tag{12.4}$$

$$k \in [k_*, 1), \quad p = p_{g_1}(k), \quad \operatorname{sn}^2 \tau = \frac{2k^2 \operatorname{sn}^2 p - 1}{k^2 \operatorname{sn}^2 p} \in [0, 1]. \tag{12.5}$$

Moreover, Proposition 11.19, Theorem 11.1, and Proposition 11.6 imply that

$$\begin{aligned} k \in [k_*, k_0) \quad &\Rightarrow \quad p_{g_1}(k) < \frac{3}{2}K < 2K = p_1(k), \\ k = k_0 \quad &\Rightarrow \quad p_{g_1}(k) = K < 2K = p_1(k), \\ k \in (k_0, 1) \quad &\Rightarrow \quad p_{g_1}(k) < K < p_1^1(k) = p_1(k). \end{aligned}$$

That is, $p_{g_1}(k) < p_1(k)$ for all $k \in [k_*, 1)$ and, consequently,

$$t_1^{\mathrm{MAX}^{3-}}(\lambda) < \mathbf{t}(\lambda) = \min\left(t_1^{\mathrm{MAX}^1}(\lambda), t_1^{\mathrm{MAX}^2}(\lambda)\right)$$

for all $\lambda = \nu$ defined by (12.4), (12.5).

It is natural to conjecture that for such λ, we have

$$t_{\mathrm{cut}}(\lambda) = t_1^{\mathrm{MAX}^{3-}}(\lambda) \tag{12.6}$$

and we will prove this equality in the forthcoming work [35].

However, the covectors $\lambda = \nu$ defined by (12.4) and (12.5) form a codimension-2 subset of N and, therefore, Eq. (12.6) defines the cut time for a codimension-1 subset of extremal trajectories. The question on an exact description of the cut time for arbitrary extremal trajectories is under investigation now.

An essential progress in the description of the cut time was achieved via the study of the global properties of the exponential mapping. Moreover, a precise description of locally optimal extremal trajectories (i.e., stable Euler elasticae) was obtained due to the detailed study of conjugate points. These results will be presented in [35].

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