

Cut locus in sub-Riemannian problem on the group of motions of a plane

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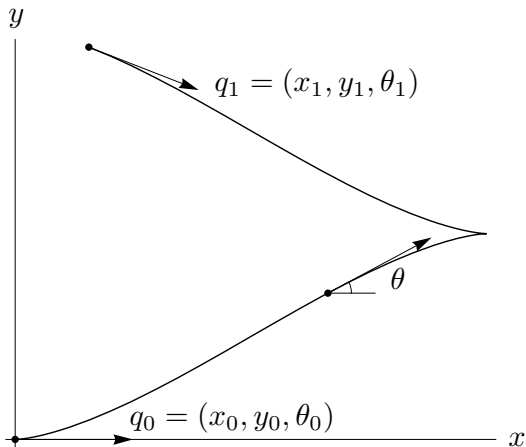
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Problem statement:
Optimal motion of a mobile robot in the plane



$$q(0) = q_0, \quad q(t_1) = q_1, \quad l = \int_0^{t_1} \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{\theta}^2} dt \rightarrow \min$$

Optimal control problem

$$\dot{x} = u \cos \theta, \quad \dot{y} = u \sin \theta, \quad \dot{\theta} = v,$$

$$(x, y) \in \mathbb{R}^2, \quad \theta \in S^1 = \mathbb{R}/(2\pi \mathbb{Z}),$$

$$q = (x, y, \theta) \in M = \mathbb{R}^2 \times S^1,$$

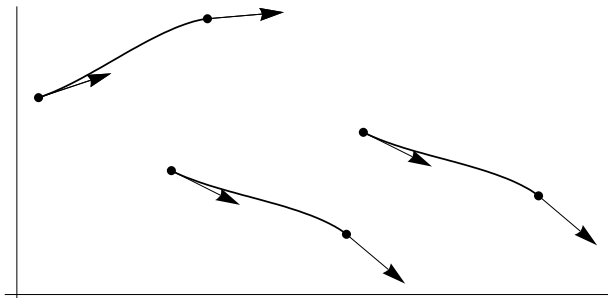
$$(u, v) \in \mathbb{R}^2,$$

$$q(0) = q_0, \quad q(t_1) = q_1,$$

$$I = \int_0^{t_1} \sqrt{u^2 + v^2} dt \rightarrow \min.$$

Continuous symmetries of the problem

- rotations
- translations



Group of motions (rototranslations) of a plane

$$\mathrm{SE}(2) = \mathbb{R}^2 \ltimes \mathrm{SO}(2) = \left\{ \begin{pmatrix} \cos \theta & -\sin \theta & x \\ \sin \theta & \cos \theta & y \\ 0 & 0 & 1 \end{pmatrix} \mid (x, y) \in \mathbb{R}^2, \theta \in S^1 \right\}$$

Left-invariant frame on $\mathrm{SE}(2)$:

$$X_1(q) = qE_{13}, \quad X_2(q) = q(E_{21} - E_{12}), \quad X_3(q) = [X_1, X_2](q) = -qE_{23}.$$

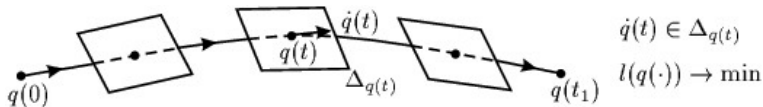
Left-invariant sub-Riemannian problem on SE(2)

$$\dot{q} = uX_1(q) + vX_2(q), \quad q \in \text{SE}(2), \quad (u, v) \in \mathbb{R}^2,$$

$$q(0) = q_0, \quad q(t_1) = q_1,$$

$$l = \int_0^{t_1} \langle \dot{q}, \dot{q} \rangle^{1/2} dt \rightarrow \min,$$

$$\langle X_i, X_j \rangle = \delta_{ij}, \quad i, j = 1, 2.$$



Known results on 3-dimensional sub-Riemannian problems

- Left-invariant problem on the Heisenberg group: global solution (A.Vershik, V.Gershkovich, 1987),
- Contact problems in \mathbb{R}^3 : local study (A.Agrachev, 1996; J.-P.Gauthier, 1996),
- Martinet case: global solution (A.Agrachev, B.Bonnard, M.Chyba, I.Kupka, 1997),
- Left-invariant problems on $SO(3)$, $SU(2)$, $SL(2)$: global solution (U.Boscain, F.Rossi, 2008).

Existence of solutions

- $\dot{q} = uX_1(q) + vX_2(q),$
 $\text{span}(X_1(q), X_2(q), [X_1, X_2](q)) = T_qM \quad \forall q \in M$
 \Rightarrow complete controllability (Rashevskii-Chow theorem)
- Filippov's theorem
 \Rightarrow existence of optimal trajectories $q(t)$.

Pontryagin maximum principle

- Abnormal extremal trajectories constant.
- Normal extremals:

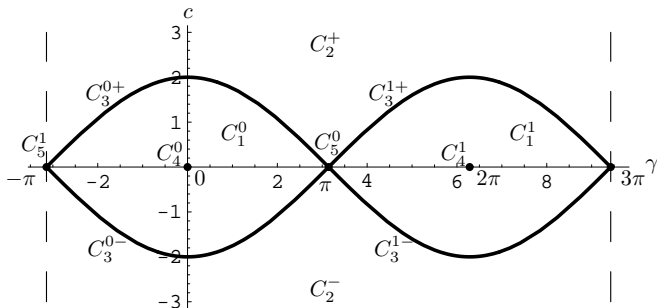
$$\begin{aligned}\dot{\gamma} &= c, & \dot{c} &= -\sin \gamma, & (\gamma, c) &\in C \cong (2S^1_\gamma) \times \mathbb{R}_c, \\ \dot{x} &= \sin \frac{\gamma}{2} \cos \theta, & \dot{y} &= \sin \frac{\gamma}{2} \sin \theta, & \dot{\theta} &= -\cos \frac{\gamma}{2}.\end{aligned}$$

- Arc length parametrization:

$$\dot{x}^2 + \dot{y}^2 + \dot{\theta}^2 \equiv 1 \quad \Rightarrow \quad l = t_1 \rightarrow \min$$

Decomposition of phase cylinder of pendulum $C = \cup_{i=1}^5 C_i$

- Energy integral $E = c^2/2 - \cos \gamma \in [-1, +\infty)$
- $C_1 = \{\lambda \in C \mid E \in (-1, 1)\}$,
- $C_2 = \{\lambda \in C \mid E \in (1, +\infty)\}$,
- $C_3 = \{\lambda \in C \mid E = 1, c \neq 0\}$,
- $C_4 = \{\lambda \in C \mid E = -1\}$,
- $C_5 = \{\lambda \in C \mid E = 1, c = 0\}$.



Parametrisation of extremal trajectories

- $\lambda = (\gamma, c) \in C_1 \Rightarrow$

$$\theta_t = s_1(\operatorname{am} \varphi - \operatorname{am} \varphi_t) \pmod{2\pi},$$

$$x_t = (s_1/k)[\operatorname{cn} \varphi(\operatorname{dn} \varphi - \operatorname{dn} \varphi_t) + \operatorname{sn} \varphi(t + E(\varphi) - E(\varphi_t))],$$

$$y_t = (1/k)[\operatorname{sn} \varphi(\operatorname{dn} \varphi - \operatorname{dn} \varphi_t) - \operatorname{cn} \varphi(t + E(\varphi) - E(\varphi_t))].$$

- $\lambda = (\gamma, c) \in C_2 \Rightarrow$

$$\cos \theta_t = k^2 \operatorname{sn} \psi \operatorname{sn} \psi_t + \operatorname{dn} \psi \operatorname{dn} \psi_t,$$

$$\sin \theta_t = k(\operatorname{sn} \psi \operatorname{dn} \psi_t - \operatorname{dn} \psi \operatorname{sn} \psi_t),$$

$$x_t = s_2 k[\operatorname{dn} \psi(\operatorname{cn} \psi - \operatorname{cn} \psi_t) + \operatorname{sn} \psi(t/k + E(\psi) - E(\psi_t))],$$

$$y_t = s_2[k^2 \operatorname{sn} \psi(\operatorname{cn} \psi - \operatorname{cn} \psi_t) - \operatorname{dn} \psi(t/k + E(\psi) - E(\psi_t))].$$

- $\lambda = (\gamma, c) \in C_3 \cup C_4 \cup C_5 \Rightarrow$ hyperbolic and linear functions.

Extremal trajectories: generic cases

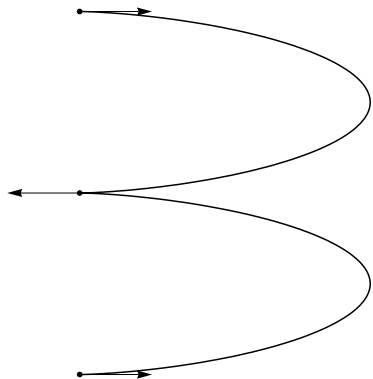


Figure: $\lambda \in C_1$

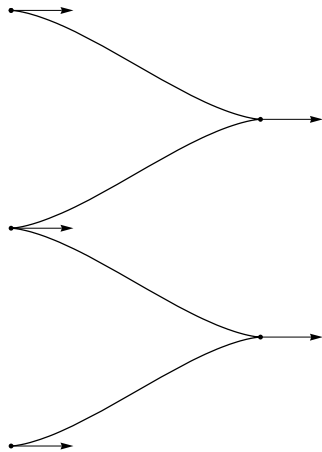


Figure: $\lambda \in C_2$

Extremal trajectories: special cases

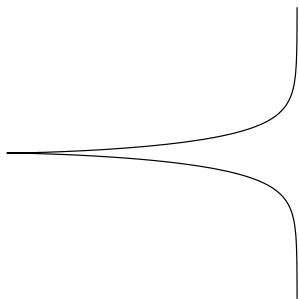


Figure: $\lambda \in C_3$

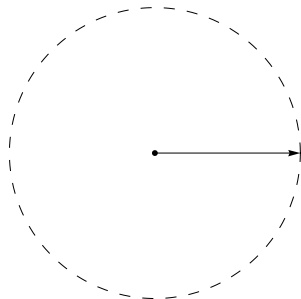


Figure: $\lambda \in C_4$



Figure: $\lambda \in C_5$

Optimality of extremal trajectories

$q(t)$ is **locally** optimal:

$$\begin{aligned} \exists \varepsilon > 0 \quad \forall \text{ trajectory } \tilde{q} : \quad & \|\tilde{q} - q\|_C < \varepsilon, \\ & q(0) = \tilde{q}(0), \quad q(t_1) = \tilde{q}(\tilde{t}_1) \quad \Rightarrow \quad t_1 \leq \tilde{t}_1 \end{aligned}$$

$q(t)$ is **globally** optimal:

$$\forall \text{ trajectory } \tilde{q} : \quad q(0) = \tilde{q}(0), \quad q(t_1) = \tilde{q}(\tilde{t}_1) \quad \Rightarrow \quad t_1 \leq \tilde{t}_1$$

Loss of optimality

- Strong Legendre condition:

$$\frac{\partial^2 h_u^{-1}}{\partial u^2} < 0 \quad \Rightarrow \quad \text{short arcs } q(t) \text{ are optimal.}$$

- Cut time:

$$t_{\text{cut}}(q) = \sup\{t > 0 \mid q(s) \text{ is optimal for } s \in [0, t]\}.$$

Reasons for loss of optimality: (1) Maxwell point

Maxwell point q_t :

\exists extremal trajectory $\tilde{q}_s \neq q_s : q_0 = \tilde{q}_0, q_t = \tilde{q}_t$

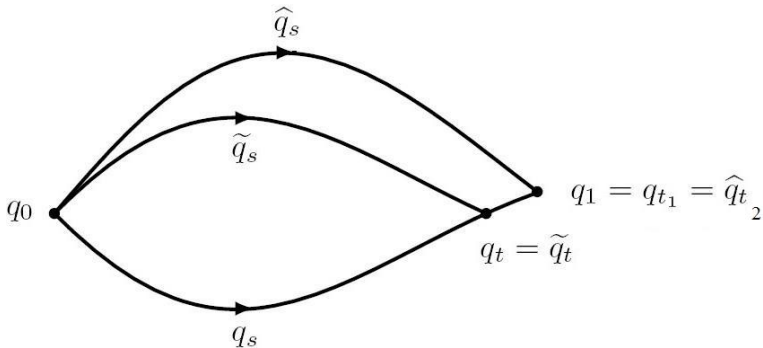
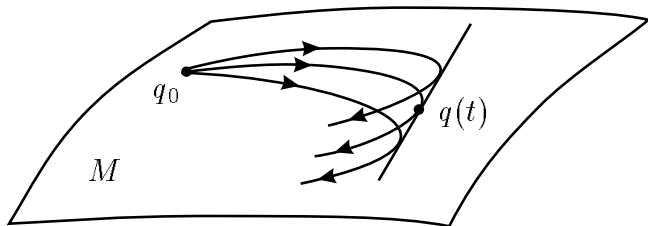


Figure: $t_2 < t_1$

Reasons for loss of optimality: (2) Conjugate point

$q_t \in$ envelope of the family of extremal trajectories



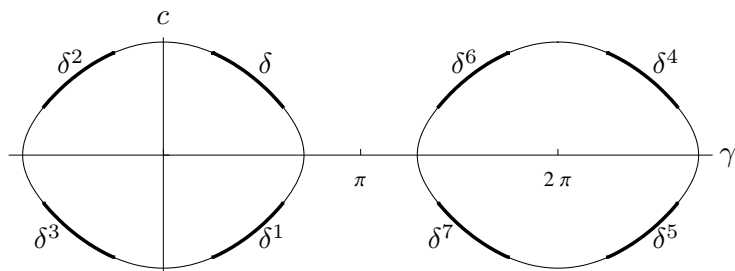
$$t_{\text{cut}} \leq \min(t_{\text{Max}}, t_{\text{conj}})$$

Reflections ε^i in the phase cylinder of pendulum $\ddot{\gamma} = -\sin \gamma$

- Group of symmetries of parallelepiped

$$G = \{\text{Id}, \varepsilon^1, \dots, \varepsilon^7\} = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2.$$

- Action of reflections $\varepsilon^i : \delta \mapsto \delta^i$ on trajectories of pendulum:



Action of reflections ε^i on curves (x_t, y_t) modulo rotations

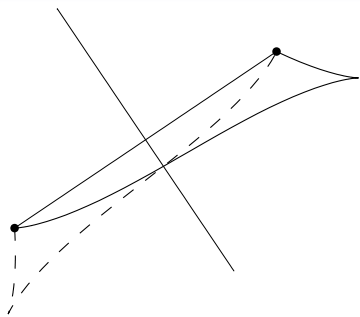


Figure: $\varepsilon^1, \varepsilon^2$

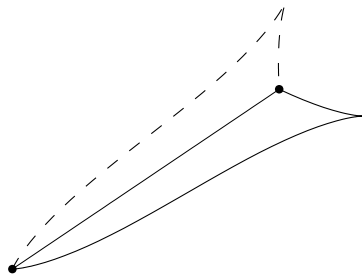


Figure: $\varepsilon^4, \varepsilon^7$

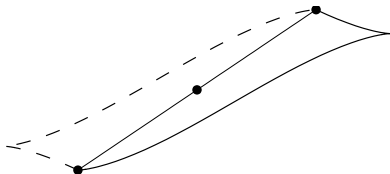


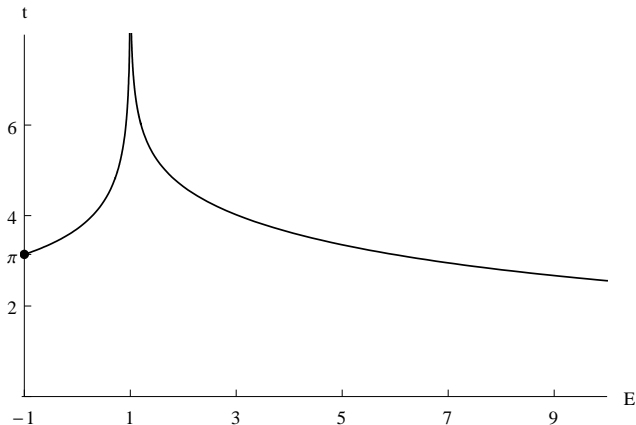
Figure: $\varepsilon^5, \varepsilon^6$

Maxwell points corresponding to reflections

- Fixed points of reflections ε^i :

$$t = t_{\varepsilon^i}^n, \quad i = 1, 2, \dots, 7, \quad n = 1, 2, \dots$$

- Upper bound of cut time: $t_{\text{cut}} \leq \mathbf{t} := \min(t_{\varepsilon^i}^1)$.
- Plot of function $\mathbf{t} = \mathbf{t}(E)$:



Exponential mapping and conjugate points

- Exponential mapping

$$\text{Exp} : (\lambda, t) = (\gamma, c, t) \mapsto q(t),$$

$$\text{Exp} : N = C \times \mathbb{R}_+ \rightarrow M$$

- q — conjugate point $\iff q$ — critical value of Exp

- $\text{Exp}(\gamma, c, t) = (x, y, \theta)$

- $\frac{\partial(x, y, \theta)}{\partial(\gamma, c, t)} = 0$

Bounds of conjugate time

- Trajectories without inflexion points:

$$\lambda \in C_1 \cup C_3 \cup C_4 \cup C_5 \quad \Rightarrow \quad t_{\text{conj}}^1(\lambda) = +\infty.$$

- Trajectories with inflexion points:

$$\lambda \in C_2 \quad \Rightarrow \quad t_{\varepsilon^6}^1(\lambda) \geq t_{\text{conj}}^1(\lambda) \geq t_{\varepsilon^2}^1(\lambda) = \mathbf{t}(\lambda).$$

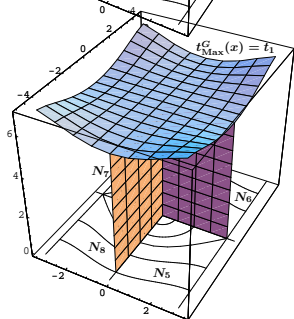
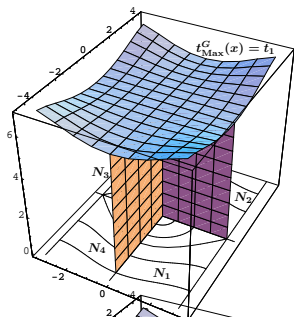
Stratifications in preimage and image of exponential mapping

- $\hat{N} = \{(\lambda, t) \in C \times \mathbb{R}_+ \mid t \leq \mathbf{t}(\lambda)\}, \quad \hat{M} = M \setminus \{q_0\}$
- $\text{Exp} : \hat{N} \rightarrow \hat{M}$ surjective
- $\hat{N} = \cup_{i \in I} N_i, \quad N_i \cap N_j = \emptyset$ for $i \neq j \in I$
- $\hat{M} = \cup_{i \in I} M_i, \quad I = J \cup K, \quad J \cap K = \emptyset$
- $\forall i \neq j \in J \quad M_i \cap M_j = \emptyset$
- $\forall i \in K \exists! j \in K, j \neq i : M_i = M_j$
- N_i, M_i smooth manifolds of $\dim \in \{0, \dots, 3\}$
- $\#I = 66, \quad \#J = 32, \quad \#K = 34.$

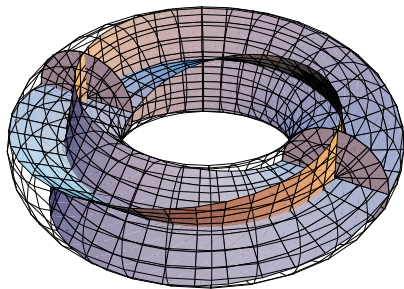
Global structure of exponential mapping

- $\text{Exp} : N_i \rightarrow M_i$ **diffeomorphism** $\forall i \in I$.
- $\widehat{M} = \text{Max} \cup \widetilde{M}$, $\text{Max} = \cup_{i \in K} M_i$, $\widetilde{M} = \cup_{i \in J} M_i$
- $\widehat{N} = N_{\text{Max}} \cup \widetilde{N}$, $N_{\text{Max}} = \cup_{i \in K} N_i$, $\widetilde{N} = \cup_{i \in J} N_i$
- $\text{Exp} : \widetilde{N} \rightarrow \widetilde{M}$ bijection
- $\text{Exp} : N_{\text{Max}} \rightarrow \text{Max}$ double mapping

Global structure of exponential mapping



Exp
→



Cut time and cut points

$$t_{\text{cut}}(\lambda) = \mathbf{t}(\lambda) = \begin{cases} t_{\varepsilon_5}^1 = 2K(k) = T/2, & \lambda \in C_1, \\ t_{\varepsilon_2}^1 = 2kp_1^1(k) \in (T, 2T), & \lambda \in C_2, \\ +\infty, & \lambda \in C_3 \cup C_5, \\ t_{\varepsilon_5}^1 = \pi = T/2, & \lambda \in C_4 \end{cases}$$

$$p = p_1^1(k) : \quad \text{cn}(p, k)(E(p, k) - p) - \text{dn}(p, k) \text{sn}(p, k) = 0$$

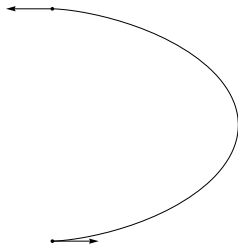


Figure: $\lambda \in C_1$

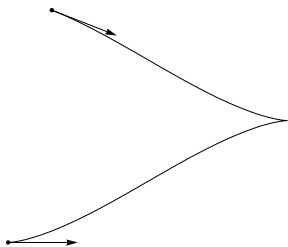


Figure: $\lambda \in C_2$

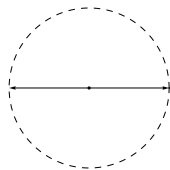


Figure: $\lambda \in C_4$

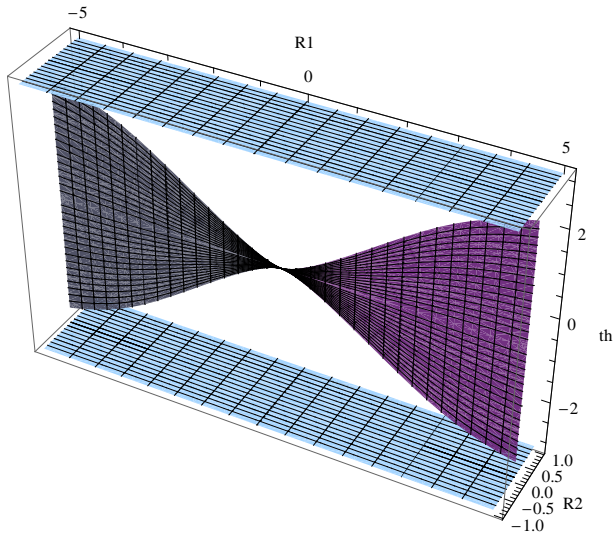
Maxwell strata

- $\text{Max} = \text{Max}_{\text{loc}} \cup \text{Max}_{\text{glob}}$
- $\text{Max}_{\text{glob}} = \{q \in M \mid \theta = \pi\}$
- $\text{Max}_{\text{loc}} = \{q \in M \mid \theta \in (-\pi, \pi), R_2 = 0, |R_1| > R_1^1(|\theta|)\},$
 $R_1 = y \cos \frac{\theta}{2} - x \sin \frac{\theta}{2}, \quad R_2 = x \cos \frac{\theta}{2} + y \sin \frac{\theta}{2},$
 $R_1^1(\theta) = 2(p_1^1(k) - E(p_1^1(k), k)),$
 $k = k_1^1(\theta)$ inverse of $\theta = k \operatorname{sn}(p_1^1(k), k).$
- $q_1 \in \text{Max} \Rightarrow 2$ optimal trajectories,
- $q_1 \in M \setminus \text{Max} \Rightarrow 1$ optimal trajectory.

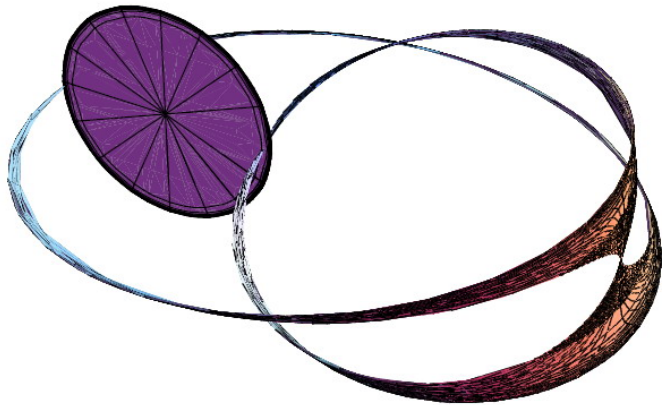
Cut locus

- $\text{Cut} = \{\text{Exp}(\lambda, t) \mid \lambda \in N, t = t_{\text{cut}}(\lambda)\}$
- $\text{Cut} = \text{cl}(\text{Max}) \setminus \{q_0\} = \text{Cut}_{\text{loc}} \cup \text{Cut}_{\text{glob}}$
- $\text{Cut}_{\text{loc}} = \text{cl}(\text{Max}_{\text{loc}}) \setminus \{q_0\}$
- $\text{Cut}_{\text{glob}} = \text{Max}_{\text{glob}}$

Cut locus in rectifying coordinates (R_1, R_2, θ)



Cut locus: global view

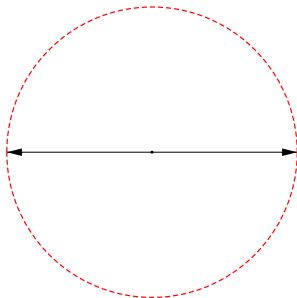


Optimal solutions

$$x_1 \neq 0, \quad y_1 = 0, \quad \theta_1 = 0$$

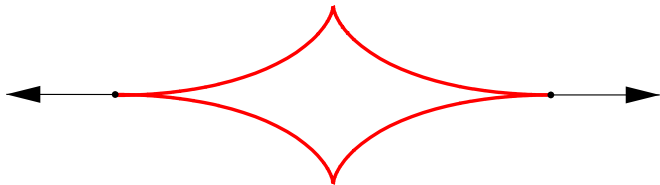


$$x_1 = 0, \quad y_1 = 0, \quad \theta_1 \neq 0$$



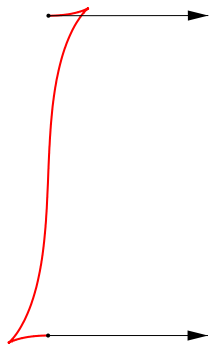
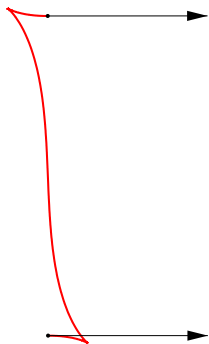
Optimal solutions

$$x_1 \neq 0, \quad y_1 = 0, \quad \theta_1 = \pi$$



Optimal solutions

$$x_1 = 0, \quad y_1 \neq 0, \quad \theta_1 = 0$$



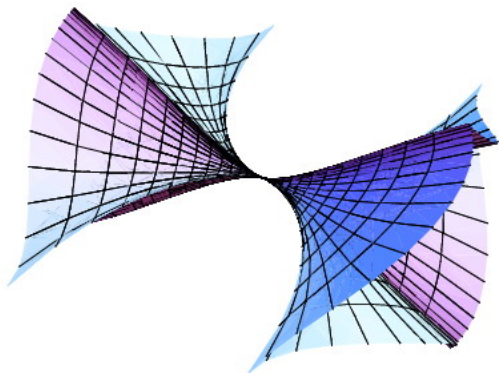
Optimal solutions

Generic boundary conditions:

systems of equations in Jacobi's functions \Rightarrow

\Rightarrow software (MATHEMATICA).

Sub-Riemannian caustic $\{\text{Exp}(\lambda, t) \mid \lambda \in N, t = t_{\text{conj}}^1(\lambda)\}$

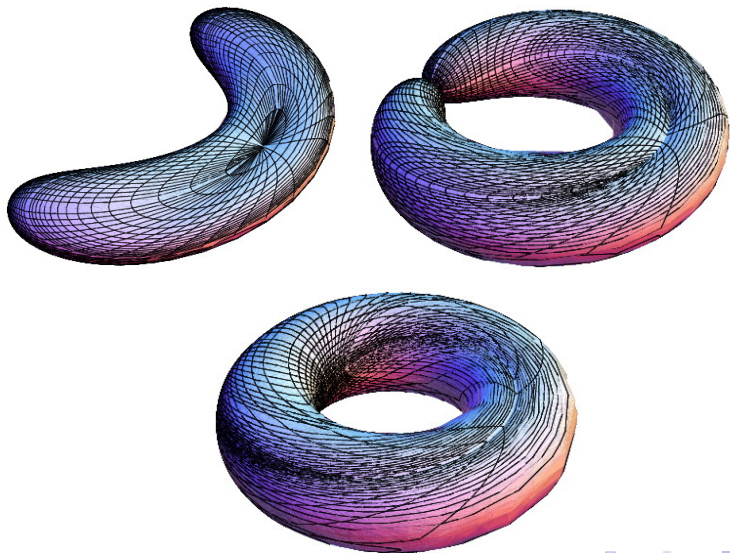


Sub-Riemannian spheres

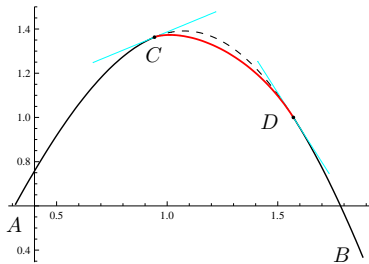
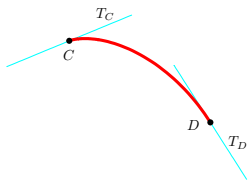
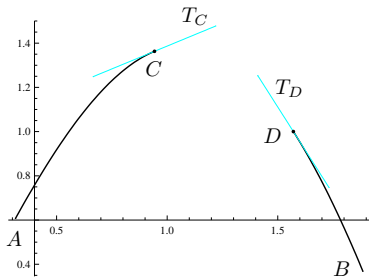
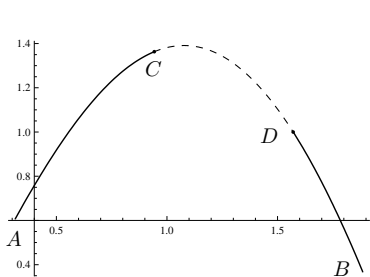
- $d(q_0, q_1) = \inf\{l(q(\cdot)) \mid q(0) = q_0, q(t_1) = q_1\},$
- $S_R = \{q \in M \mid d(q_0, q) = R\},$
- $R = 0 \Rightarrow S_R = \{q_0\},$
- $R \in (0, \pi) \Rightarrow S_R \cong S^2,$
- $R = \pi \Rightarrow S_R \cong S^2/\{N = S\},$
- $R > \pi \Rightarrow S_R \cong T^2.$

Global structure of sub-Riemannian spheres:

$$R < \pi, \quad R = \pi, \quad R > \pi$$



Application: Antropomorphic restoration of curves

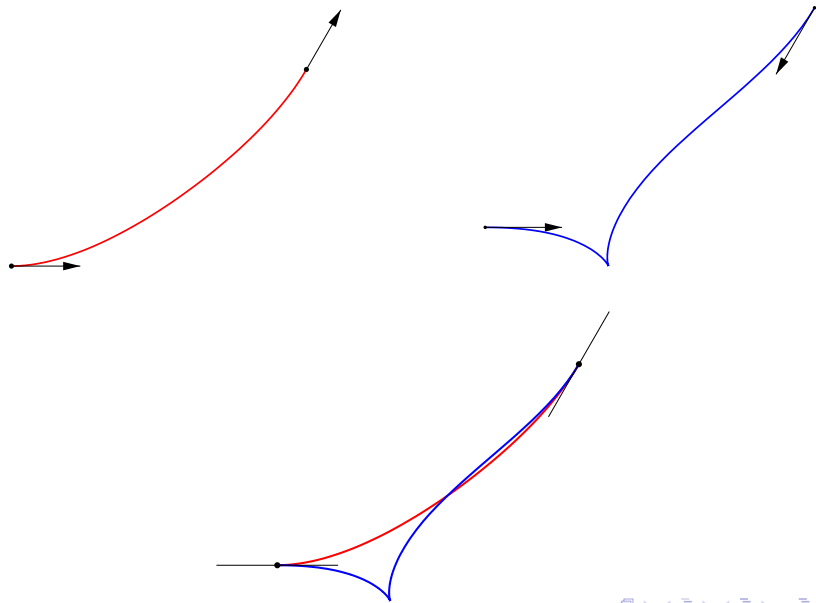


Neurogeometry and sub-Riemannian problem on $\mathbb{R}^2 \times \mathbb{R}P^1$

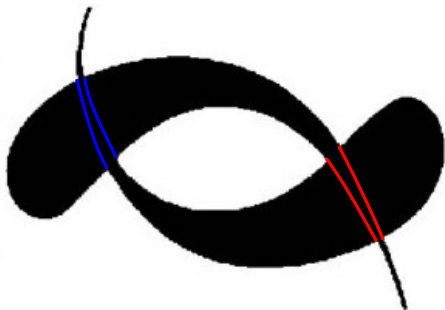
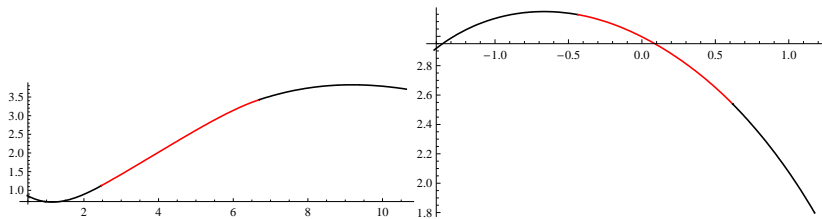
- J.Petitot, The neurogeometry of pinwheels as a sub-Riemannian contact structure, *J. Physiology - Paris* 97 (2003), 265–309.
- J.Petitot, *Neurogeometrie de la vision — Modeles mathematiques et physiques des architectures fonctionnelles*, 2008, Editions de l'Ecole Polytechnique.

$$\begin{aligned} \dot{x} &= u \cos \theta, & \dot{y} &= u \sin \theta, & \dot{\theta} &= v, \\ q &= (x, y, \theta), & (x, y) &\in \mathbb{R}^2, & \theta &\in \mathbb{R}P^1 = \mathbb{R}/(\pi \mathbb{Z}), \\ (u, v) &\in \mathbb{R}^2, \\ q(0) &= q_0, & q(t_1) &= q_1, \\ l &= \int_0^{t_1} \sqrt{u^2 + v^2} dt \rightarrow \min. \end{aligned}$$

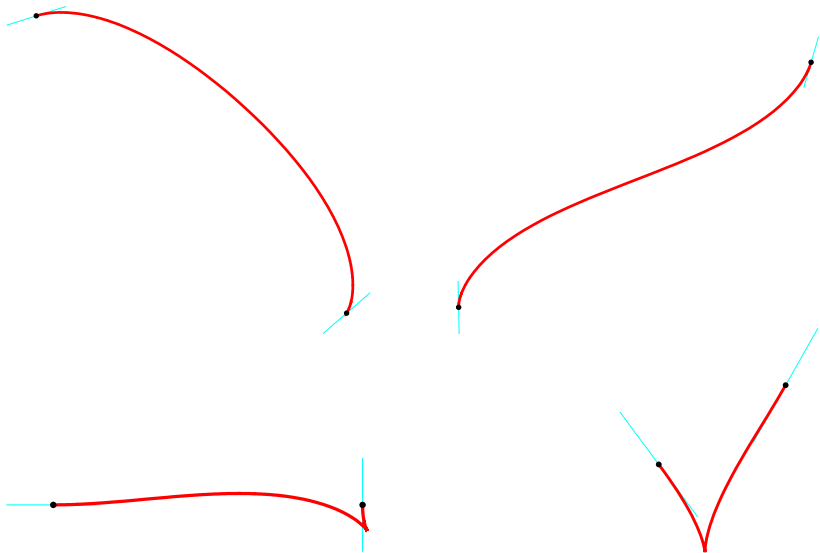
Optimal solution for the problem on $\mathbb{R}^2 \times \mathbb{R}P^1$



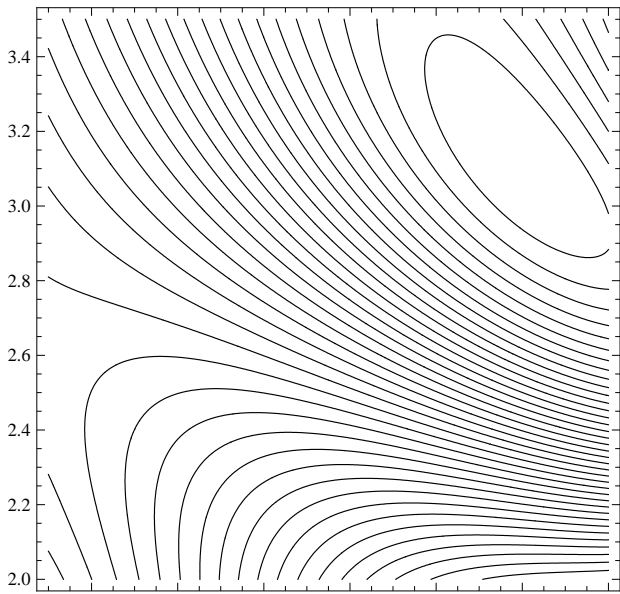
Restored curves



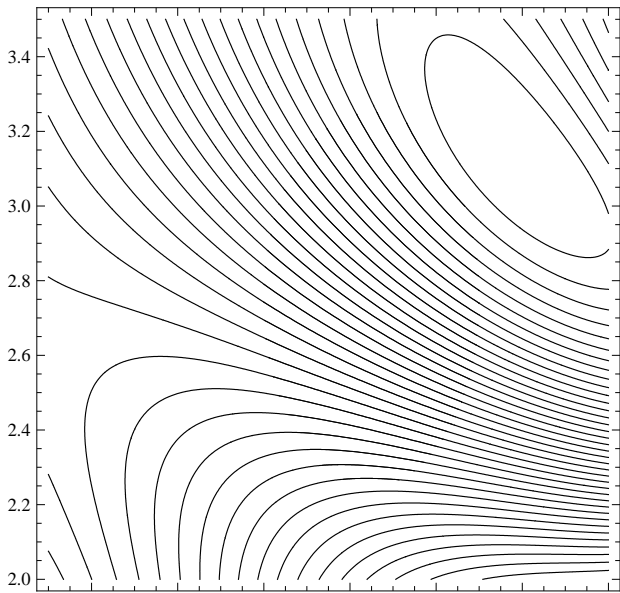
Smooth and non-smooth arcs



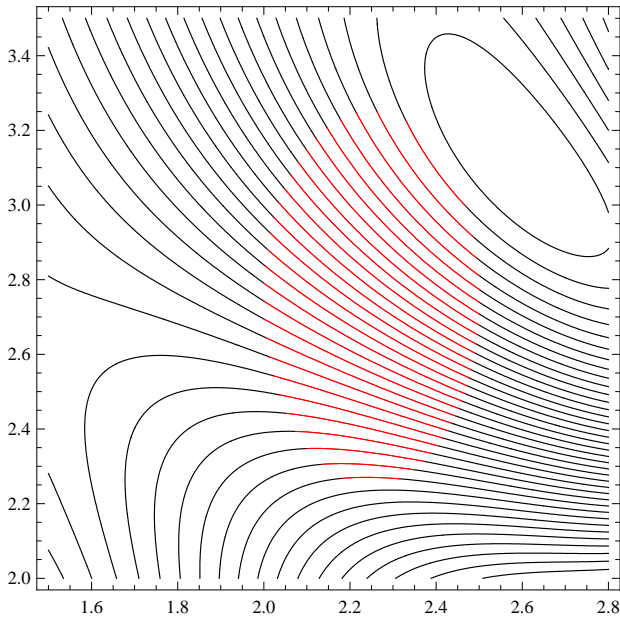
Initial family of curves



Restored family of curves



Restored family of curves



Publications

<http://www.botik.ru/PSI/CPRC/sachkov/>

[1] I. Moiseev, Yu. L. Sachkov, Maxwell strata in sub-Riemannian problem on the group of motions of a plane, *ESAIM: COCV*, accepted, available at arXiv:0807.4731 [math.OC].

[2] Yu. L. Sachkov, Conjugate and cut time in sub-Riemannian problem on the group of motions of a plane, *ESAIM: COCV*, accepted, available at arXiv:0903.0727 [math.OC].

[3] Yu. L. Sachkov, Cut locus and optimal synthesis in the sub-Riemannian problem on the group of motions of a plane, *submitted*, available at arXiv:0903.0727 [math.OC].