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SYMMETRIES OF FLAT RANK TWO DISTRIBUTIONS AND SUB-RIEMANNIAN STRUCTURES

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ABSTRACT. Flat sub-Riemannian structures are local approximations — nilpotentizations — of sub-Riemannian structures at regular points. Lie algebras of symmetries of flat maximal growth distributions and sub-Riemannian structures of rank two are computed in dimensions 3, 4, and 5.

1. SUB-RIEMANNIAN STRUCTURES

A *sub-Riemannian geometry* is a triple $(M, \Delta, \langle \cdot, \cdot \rangle)$, where M is a smooth manifold, $\Delta \subset TM$ is a smooth distribution on M, $\Delta = \{\Delta_q \subset T_qM \mid q \in M\}$, and $\langle \cdot, \cdot \rangle$ is an inner product in Δ that smoothly depends on a point in M, $\langle \cdot, \cdot \rangle = \{\langle \cdot, \cdot \rangle_q$— an inner product in $\Delta_q \mid q \in M\}$. The pair $(\Delta, \langle \cdot, \cdot \rangle)$ is a *sub-Riemannian structure* on M; if $\dim M = n$ and $\dim \Delta_q = k$, $q \in M$, then we say that $(\Delta, \langle \cdot, \cdot \rangle)$ is a (k, n)-*structure*. The number k is called the *rank* of the distribution Δ or the structure $(\Delta, \langle \cdot, \cdot \rangle)$.

In this work, we are interested in the special class of distributions and sub-Riemannian structures called *flat*. Let G be a connected simply connected nilpotent Lie group. Suppose that its Lie algebra $\mathfrak{g}$ is graded:

$$\mathfrak{g} = \mathfrak{g}^1 \oplus \mathfrak{g}^2 \oplus \cdots \oplus \mathfrak{g}^s,$$
$$[\mathfrak{g}^i, \mathfrak{g}^j] \subset \mathfrak{g}^{i+j}, \quad \mathfrak{g}^r = \{0\} \ \forall r > s,$$

and generated as a Lie algebra by its component of degree 1:

$$\mathrm{Lie}(\mathfrak{g}^1) = \mathfrak{g}.$$

Then

$$\Delta = \mathfrak{g}^1$$

can be considered as a completely nonintegrable (bracket-generating) left-invariant distribution on the Lie group G. We call such a distribution Δ *flat*. Further, if Δ is equipped with a left-invariant inner product $\langle \cdot, \cdot \rangle$ obtained from an inner product in $\mathfrak{g}^1$, then $(\Delta, \langle \cdot, \cdot \rangle)$ is called a *flat* sub-Riemannian structure on G. Flat sub-Riemannian structures arise as local approximations — nilpotentizations — of arbitrary sub-Riemannian structures at regular points (see [2], [3] for details).

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For a distribution $\Delta \subset TM$ its *Lie flag* is defined as follows:

$$\Delta \subset \Delta^2 = \Delta + [\Delta, \Delta] \subset \Delta^3 = \Delta^2 + [\Delta, \Delta^2] \subset \cdots \subset TM$$

(here Δ denotes also the $C^\infty(M)$-module of vector fields on M tangent to the distribution Δ). Then the *growth vector* of Δ at a point $q \in M$ is the vector

$$(n_1, n_2, n_3, \dots), \qquad n_i = \dim \Delta^i(q).$$

For a flat distribution $\Delta \subset TG$ we may restrict ourselves to left-invariant vector fields on the Lie group G:

$$\Delta^i(q) = (\mathfrak{g}^1 \oplus \cdots \oplus \mathfrak{g}^i)(q),$$

and the growth vector is constant and takes the form

$$(n_1, n_2, n_3, \dots), \qquad n_i = \sum_{j=1}^{i} \dim \mathfrak{g}^j.$$

Two flat distributions (sub-Riemannian structures) are called *isomorphic* if there exists an isomorphism of Lie algebras that maps the first distribution (respectively, sub-Riemannian structure) onto the second one, in other words, if they are isomorphic as left-invariant objects on G.

The aim of this work is to study symmetries of flat distributions and sub-Riemannian structures in dimensions $(2, n)$, $n = 3, 4, 5$, for maximal growth vectors. More precisely, we consider the following cases:

Dimension	Growth vector
$(2, 3)$	$(2, 3)$
$(2, 4)$	$(2, 3, 4)$
$(2, 5)$	$(2, 3, 5)$

Of course, the maximum growth condition is essential only for dimension (2, 5) since for dimensions (2, 3) and (2, 4) the growth vectors are uniquely determined.

We are interested in the maximum growth case since it is generic: a generic distribution has the maximum possible growth at a generic point. The case (2, 3, 5) is important for applications: sub-Riemannian structures with the growth vector (2, 3, 5) appear in the following systems:

1) a pair of bodies rolling one on another without slipping or twisting [1], [10]; in particular, the sphere rolling on a plane, the plate-ball problem [8];

2) a car with 2 off-hooked trailers [9], [12].

In conclusion, we notice that the results of this work on symmetries of flat distributions are not new. In particular, symmetries of the flat (2, 3, 5) distribution were known to E. Cartan [6]. However, this result does not seem to be presented in the modern terminology elsewhere.

The results on symmetries of flat sub-Riemannian structures are new.

2. Symmetries

In this work, the term "smooth" means C^∞. Given a smooth vector field $X \in \mathrm{Vec}(M)$, we denote by e^{tX} its flow, by $(e^{tX})_*$ the differential (push-forward) action of the flow on vector fields, and by $(e^{tX})^*$ the pull-back action of the flow on forms.

A vector field $X \in \mathrm{Vec}(M)$ is called an (infinitesimal) *symmetry*:
1) of a distribution Δ on M if its flow preserves Δ:

$$(e^{tX})_*\Delta = \Delta, \qquad t \in \mathbb{R};$$

2) of a sub-Riemannian structure $(\Delta, \langle\cdot,\cdot\rangle)$ on M if its flow preserves both Δ and $\langle\cdot,\cdot\rangle$:

$$(e^{tX})_*\Delta = \Delta, \quad (e^{tX})^*\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle, \qquad t \in \mathbb{R}.$$

The Lie algebras of symmetries of a distribution Δ, or a sub-Riemannian structure $(\Delta, \langle\cdot,\cdot\rangle)$, will be denoted by $\mathrm{Sym}(\Delta)$, respectively $\mathrm{Sym}(\Delta, \langle\cdot,\cdot\rangle)$.

Any left-invariant object on a Lie group G (e.g., a vector field, a distribution, or a sub-Riemannian structure) is, by definition, preserved by left translations on G. On the other hand, the flow of a right-invariant vector field X on G acts as a left translation:

$$e^{tX}(g) = e^{tX}(\mathrm{Id})g, \qquad g \in G, \quad t \in \mathbb{R},$$

where Id is the identity element of G. So any right-invariant vector field is an infinitesimal symmetry of any left-invariant object. In particular, for any left-invariant distribution Δ and left-invariant sub-Riemannian structure $(\Delta, \langle\cdot,\cdot\rangle)$ we have

$$\mathfrak{g}_r \subset \mathrm{Sym}(\Delta, \langle\cdot,\cdot\rangle) \subset \mathrm{Sym}(\Delta). \tag{2.1}$$

Here $\mathfrak{g}_r$ is the Lie algebra of right-invariant vector fields on G, which is isomorphic to the Lie algebra $\mathfrak{g}$ of G.

Symmetries of distributions and sub-Riemannian structures can be computed via the following statement.

Proposition 1. *Let $X \in \mathrm{Vec}(M)$.*

(1) $X \in \mathrm{Sym}(\Delta)$ *iff* $\mathrm{ad}\, X(\Delta) \subset \Delta$*, or, equivalently,*

$$\mathrm{ad}\, X \in \mathrm{gl}(\Delta). \tag{2.2}$$

(2) $X \in \mathrm{Sym}(\Delta, \langle\cdot,\cdot\rangle)$ *iff*

$$\mathrm{ad}\, X \in \mathrm{so}(\Delta, \langle\cdot,\cdot\rangle). \tag{2.3}$$

Any right-invariant vector field X on a Lie group G commutes with any left-invariant vector field; thus,

$$\mathrm{ad}\, X|_\Delta = 0$$

for a left-invariant distribution Δ on G. This gives another proof of inclusion (2.1).

Remark 1. Inclusion (2.2) means that for any vector field $\xi \in \mathrm{Vec}(M)$ tangent to the distribution Δ, the Lie bracket $[X, \xi]$ is tangent to Δ as well:

$$\xi \in \Delta \quad \Rightarrow \quad [X, \xi] \in \Delta.$$

That is, there is defined a linear mapping

$$\mathrm{ad}\, X \,:\, \Delta \to \Delta, \qquad \mathrm{ad}\, X \,:\, \xi \mapsto [X, \xi]. \tag{2.4}$$

In terms of a local basis, condition (2.2) reads as follows. For any point $q_0 \in M$ and any local basis $\xi_1, \dots, \xi_k$ of the distribution Δ in a neighborhood $q_0 \in O \subset M$:

$$\Delta_q = \mathrm{span}(\xi_1(q), \dots, \xi_k(q)), \qquad q \in O,$$

there exist smooth functions $c_{ij} = c_{ij}(q)$ defined on O such that

$$[X, \xi_i](q) = \sum_{j=1}^{k} c_{ji}(q)\xi_j(q), \qquad q \in O, \quad i = 1, \dots, k. \tag{2.5}$$

Similarly, inclusion (2.3) means that the linear mapping (2.4) is skew-symmetric with respect to the inner product $\langle \cdot, \cdot \rangle$. In terms of a local basis: for any point $q_0 \in M$ and any local orthonormal basis of the sub-Riemannian structure $(\Delta, \langle \cdot, \cdot \rangle)$ in a neighborhood O of q_0,

$$\Delta_q = \operatorname{span}(\xi_1(q), \dots, \xi_k(q)),$$
$$\langle \xi_i(q), \xi_j(q) \rangle = \delta_{ij}, \qquad q \in O, \quad i,\ j = 1, \dots, k,$$

equality (2.5) is satisfied for some smooth functions $c_{ij} = c_{ij}(q)$ defined on O such that the matrix $C = C(q) = (c_{ij})_{i,j=1}^{k}$ is skew-symmetric:

$$C^* = -C, \qquad q \in O.$$

The preceding equality is equivalent to the following one:

$$\langle [X, \xi_i], \xi_j \rangle + \langle \xi_i, [X, \xi_j] \rangle = 0, \qquad i,\ j = 1, \dots, k.$$

Now we prove Proposition 1.

Proof. Statement (1) is well known; see, e.g., Theorem 3.1 [4]. We prove statement (2).

Let $\xi_1, \dots, \xi_k$ and $\eta_1, \dots, \eta_k$ be local orthonormal bases of the sub-Riemannian structure $(\Delta, \langle \cdot, \cdot \rangle)$ near points $q \in M$ and $q_t = e^{tX}(q)$ respectively. Fix any pair of indices $i, j \in \{1, \dots, k\}$ and define a smooth function depending on a parameter t:

$$\varphi_t = \left\langle (e^{tX})_* \xi_i, (e^{tX})_* \xi_j \right\rangle.$$

Necessity. Let $X \in \operatorname{Sym}(\Delta, \langle \cdot, \cdot \rangle)$. Then

$$\varphi_t \equiv \varphi_0 = \delta_{ij};$$

thus,

$$0 = \left. \frac{d}{dt} \right|_{t=0} \varphi_t = \langle -\operatorname{ad} X(\xi_i), \xi_j \rangle + \langle \xi_i, -\operatorname{ad} X(\xi_j) \rangle.$$

The equalities

$$\langle \operatorname{ad} X(\xi_i), \xi_j \rangle + \langle \xi_i, \operatorname{ad} X(\xi_j) \rangle = 0, \qquad i,\ j = 1, \dots, k,$$

mean that the operator $\operatorname{ad} X : \Delta \to \Delta$ is skew-symmetric with respect to the inner product $\langle \cdot, \cdot \rangle$.

Sufficiency. Let $\operatorname{ad} X \in \operatorname{so}(\Delta, \langle \cdot, \cdot \rangle)$. In particular, inclusion (2.2) is satisfied. By item (1) of this proposition, the flow e^{tX} preserves the distribution Δ, i.e.,

$$(e^{tX})_* \xi_i = \sum_{m=1}^{k} a_{mi} \eta_m, \qquad i = 1, \dots, k,$$

for some smooth functions a_{mi} defined in a neighborhood of the point q_t. Then

$$\varphi_t = \left\langle \sum_{m=1}^{k} a_{mi} \eta_m, \sum_{l=1}^{k} a_{lj} \eta_l \right\rangle = \sum_{m,l=1}^{k} a_{mi} a_{lj} \underbrace{\langle \eta_m, \eta_l \rangle}_{=\delta_{ml}} = \sum_{l=1}^{k} a_{li} a_{lj}. \tag{2.6}$$

On the other hand,

$$
\begin{aligned}
\frac{d\varphi_t}{dt} &= \langle -\operatorname{ad} X \circ (e^{tX})_*\xi_i, (e^{tX})_*\xi_j\rangle + \langle (e^{tX})_*\xi_i, -\operatorname{ad} X \circ (e^{tX})_*\xi_j\rangle \\
&= -\left\langle \operatorname{ad} X \left(\sum_{m=1}^{k} a_{mi}\eta_m \right), \sum_{l=1}^{k} a_{lj}\eta_l \right\rangle \\
&\qquad - \left\langle \sum_{m=1}^{k} a_{mi}\eta_m, \operatorname{ad} X \left(\sum_{l=1}^{k} a_{lj}\eta_l \right) \right\rangle \\
&= -\sum_{m,l=1}^{k} \Bigg(((Xa_{mi})a_{lj} + a_{mi}(Xa_{lj})) \underbrace{\langle \eta_m, \eta_l\rangle}_{=\delta_{ij}} \\
&\qquad + a_{mi}a_{lj} \underbrace{(\langle (\operatorname{ad} X)\eta_m, \eta_l\rangle + \langle \eta_m, (\operatorname{ad} X)\eta_l\rangle)}_{=0} \Bigg) \\
&= -\sum_{l=1}^{k} (Xa_{li})a_{lj} + a_{li}(Xa_{lj}) \\
&= -X \left(\sum_{l=1}^{k} a_{li}a_{lj} \right).
\end{aligned}
$$

Here Xf denotes the Lie derivative (directional derivative) of the function f along the vector field X:

$$Xf = df(X), \qquad f \in C^\infty(M), \quad X \in \operatorname{Vec}(M).$$

In view of equality (2.6), we obtain that the family of functions φ_t is a solution of the ODE

$$\frac{d\varphi_t}{dt} = -X\varphi_t. \tag{2.7}$$

It is easy to see that this ODE has a unique solution: if φ_t satisfies (2.7), then

$$\frac{d}{dt}\varphi_t(e^{tX}q) = -X\varphi_t(e^{tX}q) + X\varphi_t(e^{tX}q) = 0;$$

thus,

$$\varphi_t(e^{tX}q) \equiv \varphi_0(q),$$

i.e.,

$$\varphi_t(q) = \varphi_0(e^{-tX}q).$$

But $\varphi_0 \equiv \delta_{ij}$; thus,

$$\varphi_t = \langle (e^{tX})_*\xi_i, (e^{tX})_*\xi_j\rangle \equiv \delta_{ij}, \qquad i,\ j = 1, \dots, k,$$

i.e., the field X is an infinitesimal symmetry of the sub-Riemannian structure $(\Delta, \langle\cdot,\cdot\rangle)$. □

3. The Heisenberg case

3.1. **The flat distribution and sub-Riemannian structure.** Let $\mathfrak{g}$ be the three-dimensional Heisenberg algebra, i.e., the (unique) three-dimensional two-step nilpotent Lie algebra:

$$\dim \mathfrak{g} = 3, \quad \dim[\mathfrak{g}, \mathfrak{g}] = 1, \quad \dim[\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]] = 0, \tag{3.1}$$

and let G be the three-dimensional Heisenberg group, i.e., the corresponding connected simply connected Lie group. A flat rank two distribution Δ on G is just any rank two nonintegrable left-invariant distribution on G:

$$\Delta \subset \mathfrak{g}, \quad \dim \Delta = 2, \quad \mathrm{Lie}(\Delta) = \mathfrak{g}.$$

To obtain a flat sub-Riemannian structure on G one has to add any left-invariant inner product $\langle \cdot, \cdot \rangle$ in Δ.

As was indicated in [13], up to isomorphism there exists exactly one flat distribution on the Heisenberg group, and the same is true for flat sub-Riemannian structures. To show this, choose an orthonormal frame:

$$\Delta = \mathrm{span}(\xi_1, \xi_2), \tag{3.2}$$

$$\langle \xi_i, \xi_j \rangle = \delta_{ij}, \ \ i, j = 1, 2. \tag{3.3}$$

Since Δ is nonintegrable,

$$\xi_3 := [\xi_1, \xi_2] \notin \Delta, \tag{3.4}$$

and $\mathfrak{g} = \mathrm{span}(\xi_1, \xi_2, \xi_3)$. Then $[\mathfrak{g}, \mathfrak{g}] = \mathbb{R}\xi_3$, and by virtue of (3.1), $\mathbb{R}\xi_3$ is the center of $\mathfrak{g}$:

$$[\xi_3, \xi_1] = 0, \quad [\xi_3, \xi_2] = 0. \tag{3.5}$$

Consequently, for any flat sub-Riemannian structure on G one can choose a basis ξ_1, ξ_2, ξ_3 in $\mathfrak{g}$ with the multiplication rules (3.4), (3.5). Thus any flat sub-Riemannian structures on the Heisenberg group are isomorphic one to another; the more so this is true for flat distributions.

Any basis ξ_1, ξ_2, ξ_3 in the Heisenberg algebra satisfying conditions (3.4) and (3.5) defines a graduation:

$$\mathfrak{g} = \mathfrak{g}^1 \oplus \mathfrak{g}^2, \qquad \mathfrak{g}^1 = \mathrm{span}(\xi_1, \xi_2), \quad \mathfrak{g}^2 = \mathrm{span}(\xi_3).$$

The multiplication rules (3.4), (3.5) in the Heisenberg algebra $\mathfrak{g}$ are schematically shown in Figure 1.

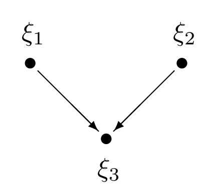


Figure 1. The Heisenberg algebra

The Heisenberg group can be represented by 3×3 upper diagonal matrices:

$$G \cong \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; x, y, z \in \mathbb{R} \right\}$$

with the usual matrix multiplication. This linear representation gives rise to another model of the Heisenberg group:

$$G \cong \mathbb{R}^3_{x,y,z}$$

via the mapping

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3_{x,y,z}.$$

Multiplication in the Lie group $\mathbb{R}^3_{x,y,z}$ is then given by

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \cdot \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 + x_1 y_2 \end{pmatrix},$$

and the vector fields

$$\begin{aligned} \xi_1 &= \frac{\partial}{\partial x}, \\ \xi_2 &= \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}, \\ \xi_3 &= \frac{\partial}{\partial z} \end{aligned} \tag{3.6}$$

form a basis of the Lie algebra of left-invariant vector fields on $\mathbb{R}^3_{x,y,z}$.

Thus we have a model of the Heisenberg group as $\mathbb{R}^3_{x,y,z}$, and the vector fields ξ_1, ξ_2 in (3.6) give a representation of the flat distribution and the flat sub-Riemannian structure in this model since equalities (3.4) and (3.5) are verified.

Now we compute the Lie algebras of symmetries $\mathrm{Sym}(\Delta)$ and $\mathrm{Sym}(\Delta, \langle \cdot, \cdot \rangle)$ with the help of this model.

3.2. **Symmetries of the distribution.**

Theorem 1. *The Lie algebra of symmetries of the flat distribution Δ on the Heisenberg group is parametrized by arbitrary smooth functions of three variables.*

For the model in $\mathbb{R}^3_{x,y,z}$ given by (3.2), (3.6),

$$\mathrm{Sym}(\Delta) = \left\{ X = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y} + R\frac{\partial}{\partial z} \right\}$$

with

$$\begin{aligned} P &= -f_y - x f_z, \\ Q &= f_x, \\ R &= x f_x - f, \end{aligned}$$

where $f = f(x, y, z)$ is an arbitrary smooth function.

The function f is called the generating function of the symmetry X.

Remark 2. It is well known that locally all contact structures in $\mathbb{R}^3$ (i.e., nonintegrable rank two distributions in $\mathbb{R}^3$) are isomorphic. Thus Theorem 1 describes symmetries of a germ of a contact structure in $\mathbb{R}^3$.

Proof. Take an arbitrary vector field in $\mathbb{R}^3_{x,y,z}$: $X = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y} + R\frac{\partial}{\partial z}$, where P, Q, and R are functions of x, y, z. We have

$$[\xi_1, X] = P_x\frac{\partial}{\partial x} + Q_x\frac{\partial}{\partial y} + R_x\frac{\partial}{\partial z}, \tag{3.7}$$

$$[\xi_2, X] = E_P\frac{\partial}{\partial x} + E_Q\frac{\partial}{\partial y} + (E_R - P)\frac{\partial}{\partial z}, \tag{3.8}$$

where

$$E_P = P_y + xP_z, \quad E_Q = Q_y + xQ_z, \quad E_R = R_y + xR_z.$$

We denote by P_x the partial derivative $\frac{\partial P}{\partial x}$, etc. Thus statement (1) of Proposition 1 reads as follows:

$$\begin{pmatrix} P_x \\ Q_x \\ R_x \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \\ x \end{pmatrix}, \qquad \begin{pmatrix} E_P \\ E_Q \\ E_R - P \end{pmatrix} = \gamma \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \delta \begin{pmatrix} 0 \\ 1 \\ x \end{pmatrix}$$

for some real-valued functions $\alpha, \beta, \gamma, \delta$. This system of two vector equations is compatible iff the following system of scalar equations holds:

$$R_x = xQ_x, \tag{3.9}$$

$$P = E_R - xE_Q. \tag{3.10}$$

We integrate the first equation by parts:

$$R = \int xQ_x\,dx = xQ - \int Q\,dx$$

and denote $f = \int Q\,dx$. Then

$$Q = f_x, \tag{3.11}$$

$$R = xQ - f = xf_x - f, \tag{3.12}$$

$$P = R_y + xR_z - x(Q_y + xQ_z) = -f_y - xf_z. \tag{3.13}$$

Thus system (3.9), (3.10) implies system (3.11)–(3.13) for some function $f = f(x, y, z)$. Consequently, if $X \in \mathrm{Sym}(\Delta)$, then system (3.11)–(3.13) holds for some f. Conversely, it is easy to verify that for an arbitrary function f the vector field $X = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y} + R\frac{\partial}{\partial z}$ determined by system (3.11)–(3.13) is a symmetry of the distribution Δ.

The correspondence between symmetries X and their generating functions f is one-to-one since $f = xQ - R$. □

3.3. **Symmetries of the sub-Riemannian structure.**

Theorem 2. *Symmetries of the flat sub-Riemannian structure* $(\Delta, \langle\cdot,\cdot\rangle)$ *on the Heisenberg group form the four-dimensional Diamond Lie algebra*

$$\mathrm{Sym}(\Delta, \langle\cdot,\cdot\rangle) = \mathrm{span}(X_0, X_1, X_2, X_3)$$

with the multiplication rules

$$[X_0, X_1] = -X_2, \quad [X_0, X_2] = X_1, \quad [X_1, X_2] = X_3. \tag{3.14}$$

For the model in $\mathbb{R}^3_{x,y,z}$ *given by (3.2), (3.3), and (3.6), we have*

$$X_0 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + \frac{1}{2}(x^2 - y^2)\frac{\partial}{\partial z}, \tag{3.15}$$

$$X_1 = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}, \tag{3.16}$$

$$X_2 = \frac{\partial}{\partial y}, \tag{3.17}$$

$$X_3 = -\frac{\partial}{\partial z}. \tag{3.18}$$

Remarks. (1) Multiplication rules (3.14) in the Lie algebra $\mathrm{Sym}(\Delta, \langle\cdot,\cdot\rangle)$ are schematically represented in Figure 2, which explains the title Diamond for this Lie algebra.

(2) In terms of Theorem 1, the symmetries $X_0, \ldots, X_3$ have the following generating functions respectively:

$$f_0 = \frac{x^2 + y^2}{2}, \quad f_1 = -y, \quad f_2 = x, \quad f_3 = 1.$$

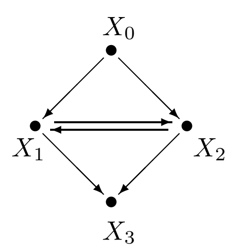


FIGURE 2. $\mathrm{Sym}(\Delta, \langle\cdot,\cdot\rangle)$, the Heisenberg case

The symmetries X_1, X_2, X_3 are just left translations on the Heisenberg group G (compare with Fig. 1), while X_0 is a rotation on G, i.e., a symmetry leaving the identity of G fixed.

Proof. By virtue of the commutation relations (3.7), (3.8), statement (2) of Proposition 1 takes the form

$$\begin{pmatrix} P_x \\ Q_x \\ R_x \end{pmatrix} = \beta \begin{pmatrix} 0 \\ 1 \\ x \end{pmatrix}, \qquad \begin{pmatrix} E_P \\ E_Q \\ E_R - P \end{pmatrix} = -\beta \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

for some real-valued function β. This vector system implies the following one:

$$R_x = xQ_x, \tag{3.19}$$

$$P = E_R, \tag{3.20}$$

$$P_x = 0, \tag{3.21}$$

$$E_Q = 0, \tag{3.22}$$

$$E_P = -Q_x. \tag{3.23}$$

As in Theorem 1, we take $f = f(x,y,z) = \int Q\,dx$. This gives, together with equations (3.19) and (3.20) that

$$Q = f_x, \tag{3.24}$$

$$R = xQ - f = xf_x - f, \tag{3.25}$$

$$P = R_y + xR_z = xf_{xy} - f_y + x^2 f_{xz} - xf_z. \tag{3.26}$$

Equations (3.26), (3.21), (3.24), and (3.22) imply that

$$\begin{aligned} P_x &= xf_{xxy} + xf_{xz} + x^2 f_{xxz} - f_z = 0, \\ E_Q &= f_{xy} + xf_{xz} = 0. \end{aligned} \tag{3.27}$$

That is why

$$P_x - x(E_Q)_x = -f_z = 0;$$

thus,

$$f = f(x,y).$$

Then equation (3.27) gives $f_{xy} = 0$, which means that

$$f = a(x) + b(y).$$

We obtain from (3.26) that $P = -b_y$; hence $E_P = P_y + xP_z = -b_{yy}$. On the other hand, equation (3.24) gives $Q = a_x$. Now (3.23) takes the form $-b_{yy} = -a_{xx}$. But the right-hand side of this equality depends on y, whereas the left-hand one depends on x, which means that they both are constant. We denote this constant by $-c$ and obtain

$$\begin{aligned} b &= \frac{c}{2}y^2 + dy + e, \\ a &= \frac{c}{2}x^2 + gx \end{aligned}$$

for some $c, d, e, g \in \mathbb{R}$. Hence

$$f = \frac{c}{2}(x^2 + y^2) + dy + gx + e$$

and

$$X = (-cy - d)\frac{\partial}{\partial x} + (cx + g)\frac{\partial}{\partial y} + \left(\frac{c}{2}(x^2 - y^2) - dy - e\right)\frac{\partial}{\partial z}. \tag{3.28}$$

To summarize the above computations, if a vector field X is a symmetry of our sub-Riemannian structure, then it has the form (3.28). The converse statement is verified immediately. That is why

$$\operatorname{Sym}(\Delta, \langle\cdot,\cdot\rangle) = \left\{(-cy - d)\frac{\partial}{\partial x} + (cx + g)\frac{\partial}{\partial y} + \left(\frac{c}{2}(x^2 - y^2) - dy - e\right)\frac{\partial}{\partial z} \mid c, d, e, g \in \mathbb{R}\right\}.$$

Now we compute a basis of the 4-dimensional Lie algebra $\mathrm{Sym}(\Delta, \langle\cdot,\cdot\rangle)$.

$$
\begin{aligned}
c = 1,\ d = e = g = 0 \quad &\Rightarrow \quad X_0 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + \frac{1}{2}(x^2 - y^2)\frac{\partial}{\partial z},\\
c = 0,\ d = -1,\ e = g = 0 \quad &\Rightarrow \quad X_1 = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z},\\
c = d = e = 0,\ g = 1 \quad &\Rightarrow \quad X_2 = \frac{\partial}{\partial y},\\
c = d = 0,\ e = 1,\ g = 0 \quad &\Rightarrow \quad X_3 = -\frac{\partial}{\partial z}.
\end{aligned}
$$

Nonzero brackets between the basis vectors are exactly the commutation relations of the Diamond Lie algebra, see (3.14). □

4. The Engel case

4.1. **The Engel algebra and Engel group.** Let $\mathfrak{g}$ be the Engel algebra, i.e., the (unique) four-dimensional 3-step nilpotent Lie algebra, and let G be the Engel group, i.e., the corresponding connected simply connected Lie group. There is a basis ξ_1, ξ_2, ξ_3, ξ_4 in $\mathfrak{g}$ with the only nonzero brackets

$$[\xi_1, \xi_2] = \xi_3, \qquad [\xi_2, \xi_3] = \xi_4.$$

In this basis, multiplication in the Engel algebra is schematically represented by the diagram in Figure 3.

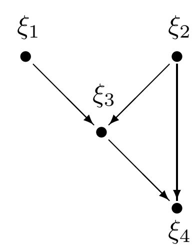


Figure 3. The Engel algebra

The Engel algebra is graded:

$$\mathfrak{g} = \mathfrak{g}^1 \oplus \mathfrak{g}^2 \oplus \mathfrak{g}^3,$$

where

$$\mathfrak{g}^1 = \mathrm{span}(\xi_1, \xi_2), \quad \mathfrak{g}^2 = \mathrm{span}(\xi_3), \quad \mathfrak{g}^3 = \mathrm{span}(\xi_4).$$

4.2. **The flat distribution and sub-Riemannian structure.** As was shown in [7], all flat distributions on the Engel group are isomorphic. Now we prove that, moreover, all flat sub-Riemannian structures on the Engel group are also isomorphic.

Let $(\Delta, \langle\cdot,\cdot\rangle)$ be a flat sub-Riemannian structure on the Engel group G corresponding to a graduation of $\mathfrak{g}$:

$$\begin{aligned}
&\mathfrak{g} = \mathfrak{g}^1 \oplus \mathfrak{g}^2 \oplus \mathfrak{g}^3,\\
&\Delta = \mathfrak{g}^1, \quad \dim \mathfrak{g}^1 = 2,\\
&\mathfrak{g}^2 = [\mathfrak{g}^1, \mathfrak{g}^1], \quad \dim \mathfrak{g}^2 = 1,\\
&\mathfrak{g}^3 = [\mathfrak{g}^1, \mathfrak{g}^2], \quad \dim \mathfrak{g}^3 = 1
\end{aligned}$$

(certainly, these homogeneous components $\mathfrak{g}^i$ should not be the same as in the previous subsection, but their number and dimensions are obviously the same).

Choose any nonzero vector

$$\xi_3 \in \mathfrak{g}^2.$$

The operator

$$\operatorname{ad} \xi_3 \ : \ \mathfrak{g}^1 \to \mathfrak{g}^3$$

has one-dimensional image and thus one-dimensional kernel. We can choose an orthonormal frame in $\mathfrak{g}^1$ so that

$$\begin{aligned}
&\mathfrak{g}^1 = \operatorname{span}(\xi_1, \xi_2),\\
&\langle \xi_i, \xi_j \rangle = \delta_{ij}, \ i, j = 1, 2,\\
&\ker(\operatorname{ad} \xi_3)|_{\mathfrak{g}^1} = \operatorname{span}(\xi_1).
\end{aligned} \tag{4.1}$$

Moreover,

$$[\xi_1, \xi_2] = k\xi_3, \quad k \in \mathbb{R} \setminus \{0\}.$$

Now we denote $k\xi_3$ as ξ_3 and obtain

$$[\xi_1, \xi_2] = \xi_3. \tag{4.2}$$

Finally, the vector

$$\xi_4 = [\xi_2, \xi_3] \tag{4.3}$$

spans the homogeneous component $\mathfrak{g}^3$. Equality (4.1) and the inclusion $\xi_4 \in \mathfrak{g}^3$ mean that all Lie brackets between the vector fields ξ_1, ξ_2, ξ_3, and ξ_4 are zero except those given by (4.2) and (4.3). Consequently, any flat sub-Riemannian structure $(\Delta, \langle\cdot,\cdot\rangle)$ on the Engel group possesses an orthonormal frame with the multiplication rules (4.2) and (4.3). (In the sequel we call such a frame a *standard* left-invariant frame on the Engel group.) This proves the uniqueness of flat sub-Riemannian structures up to an isomorphism. The uniqueness of flat distributions obviously follows.

That is why any particular flat distribution and sub-Riemannian structure can be used to compute the Lie algebras of symmetries $\operatorname{Sym}(\Delta)$ and $\operatorname{Sym}(\Delta, \langle\cdot,\cdot\rangle)$.

4.3. **The model in $\mathbb{R}^4$.** The four-dimensional space $\mathbb{R}^4_{x,y,z,u}$ is the Engel group with the multiplication rule

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ u_1 \end{pmatrix} \cdot \begin{pmatrix} x_2 \\ y_2 \\ z_2 \\ u_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 + x_1 y_2 \\ u_1 + u_2 + y_1 z_2 + x_1 y_1 y_2 + x_1 y_2^2/2 \end{pmatrix}.$$

Then the vector fields

$$\xi_1 = \frac{\partial}{\partial x}, \tag{4.4}$$

$$\xi_2 = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z} + xy\frac{\partial}{\partial u}, \tag{4.5}$$

$$\xi_3 = [\xi_1, \xi_2] = \frac{\partial}{\partial z} + y\frac{\partial}{\partial u},$$

$$\xi_4 = [\xi_2, \xi_3] = \frac{\partial}{\partial u}$$

form a standard left-invariant frame on $\mathbb{R}^4_{x,y,z,u}$.

Thus we have the following model of the flat distribution Δ and sub-Riemannian structure $(\Delta, \langle\cdot,\cdot\rangle)$ on the Engel group in $\mathbb{R}^4_{x,y,z,u}$:

$$\Delta = \operatorname{span}(\xi_1, \xi_2), \tag{4.6}$$

$$\langle \xi_i, \xi_j \rangle = \delta_{ij}, \ \ i, j = 1, 2. \tag{4.7}$$

Now we compute the symmetries $\operatorname{Sym}(\Delta)$ and $\operatorname{Sym}(\Delta, \langle\cdot,\cdot\rangle)$ in this model.

4.3.1. *Symmetries of the distribution.*

Theorem 3. *The Lie algebra of symmetries of the flat distribution Δ on the Engel group is parametrized by functions of 4 variables constant along the canonical vector field.*

For the model in $\mathbb{R}^4_{x,y,z,u}$ given by (4.4)–(4.6), we have

$$\operatorname{Sym}(\Delta) = \left\{ X = S\frac{\partial}{\partial x} + P\frac{\partial}{\partial y} + Q\frac{\partial}{\partial z} + R\frac{\partial}{\partial u} \right\},$$

where

$$S = f_{yy} + 2xf_{yz} + 2xyf_{yu} + xf_u + x^2 f_{zz} + 2x^2 y f_{zu} + x^2 y^2 f_{uu}, \tag{4.8}$$

$$P = -f_z - yf_u, \tag{4.9}$$

$$Q = f_y, \tag{4.10}$$

$$R = yf_y - f, \tag{4.11}$$

and

$$f = f(y, z, u)$$

is an arbitrary smooth function of the variables y, z, u.

Remark 3. It is known that locally all Engel structures in $\mathbb{R}^4$ (i.e., maximal growth rank two distributions in $\mathbb{R}^4$) are isomorphic. Thus Theorem 3 describes symmetries of a germ of an Engel structure in $\mathbb{R}^4$. This question will be continued in Subsection 4.4.

Proof. Take an arbitrary vector field $X = S\frac{\partial}{\partial x} + P\frac{\partial}{\partial y} + Q\frac{\partial}{\partial z} + R\frac{\partial}{\partial u} \in \operatorname{Vec}(\mathbb{R}^4)$. In view of the equalities

$$[\xi_1, X] = S_x\frac{\partial}{\partial x} + P_x\frac{\partial}{\partial y} + Q_x\frac{\partial}{\partial z} + R_x\frac{\partial}{\partial u},$$

$$[\xi_2, X] = E_S\frac{\partial}{\partial x} + E_P\frac{\partial}{\partial y} + (E_Q - S)\frac{\partial}{\partial z} + (E_R - yS - xP)\frac{\partial}{\partial u},$$

where

$$E_S = \xi_2 S = S_y + xS_z + xyS_u, \quad E_P = \xi_2 P = P_y + xP_z + xyP_u,$$
$$E_Q = \xi_2 Q = Q_y + xQ_z + xyQ_u, \quad E_R = \xi_2 R = R_y + xR_z + xyR_u,$$

and by virtue of Proposition 1, a vector field X is a symmetry of the distribution Δ iff

$$\begin{pmatrix} S_x \\ P_x \\ Q_x \\ R_x \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \\ x \\ xy \end{pmatrix}, \tag{4.12}$$

$$\begin{pmatrix} E_S \\ E_P \\ E_Q - S \\ E_R - yS - xP \end{pmatrix} = \gamma \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \delta \begin{pmatrix} 0 \\ 1 \\ x \\ xy \end{pmatrix} \tag{4.13}$$

for some smooth real-valued functions $\alpha, \beta, \gamma, \delta$. These equations for $\alpha, \beta, \gamma, \delta$ are solvable iff the following equalities hold:

$$\begin{aligned} &Q_x = xP_x, \\ &R_x = xyP_x, \\ &E_Q - S = E_P x, \\ &E_R - yS - xP = E_P xy, \end{aligned}$$

which are equivalent to

$$Q_x = xP_x, \tag{4.14}$$
$$R_x = yQ_x, \tag{4.15}$$
$$S = E_Q - xE_P, \tag{4.16}$$
$$E_Q y = E_R - xP. \tag{4.17}$$

Equality (4.15) gives

$$yQ_x = R_x \ \Leftrightarrow \int yQ_x \, dx = \int R_x \, dx \Leftrightarrow yQ = R + f,$$

where

$$f = f(y, z, u)$$

is some smooth function of the variables y, z, u. Thus

$$R = yQ - f. \tag{4.18}$$

Then equality (4.17) gives

$$P = \frac{1}{x}(E_R - yE_Q) = \frac{1}{x}(Q - (f_y + xf_z + xyf_u)). \tag{4.19}$$

Finally, equality (4.14) takes the form

$$Q_x = \frac{1}{x}(Q_x x - Q + f_y),$$

i.e.,

$$Q = f_y. \tag{4.20}$$

This gives, in view of (4.18),

$$R = yf_y - f. \tag{4.21}$$

Now we obtain from (4.19)

$$P = -f_z - y f_u, \tag{4.22}$$

and from (4.16)

$$S = f_{yy} + 2x f_{yz} + 2xy f_{yu} + x f_u + x^2 f_{zz} + 2x^2 y f_{zu} + x^2 y^2 f_{uu}. \tag{4.23}$$

The above computations show that if a vector field $X = S\frac{\partial}{\partial x} + P\frac{\partial}{\partial y} + Q\frac{\partial}{\partial z} + R\frac{\partial}{\partial u}$ is a symmetry of the distribution Δ, then its components S, P, Q, R satisfy equalities (4.23), (4.22), (4.20), (4.21) for some function $f = f(y, z, u)$.

Direct computation shows that given an arbitrary smooth function $f = f(y, z, u)$, any vector field $X = S\frac{\partial}{\partial x} + P\frac{\partial}{\partial y} + Q\frac{\partial}{\partial z} + R\frac{\partial}{\partial u}$ with the components S, P, Q, R determined from equalities (4.23), (4.22), (4.20), (4.21) belongs to $\mathrm{Sym}(\Delta)$.

The correspondence between symmetries X and generating functions f is one-to-one since $f = yQ - R$. $\square$

4.3.2. *Symmetries of the sub-Riemannian structure.*

Theorem 4. *The Lie algebra of symmetries of the flat sub-Riemannian structure $(\Delta, \langle \cdot, \cdot \rangle)$ on the Engel group is the Engel algebra.*

For the model in $\mathbb{R}^4_{x,y,z,u}$ defined by (4.4)–(4.7) we have

$$\mathrm{Sym}(\Delta, \langle \cdot, \cdot \rangle) = \mathrm{span}(X_1, X_2, X_3, X_4),$$

where

$$\begin{aligned}
X_1 &= \frac{\partial}{\partial x} + y\frac{\partial}{\partial z} + \frac{1}{2}y^2\frac{\partial}{\partial u}, \\
X_2 &= \frac{\partial}{\partial y} + z\frac{\partial}{\partial u}, \\
X_3 &= -\frac{\partial}{\partial z}, \\
X_4 &= \frac{\partial}{\partial u}.
\end{aligned}$$

Remark 4. Nonzero Lie brackets in the Lie algebra $\mathrm{Sym}(\Delta, \langle \cdot, \cdot \rangle)$ are described by the scheme in Figure 4; compare with the scheme for the Engel algebra in Figure 3.

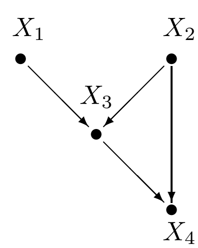


FIGURE 4. $\mathrm{Sym}(\Delta, \langle \cdot, \cdot \rangle)$, the Engel case

Proof. If a vector field $X = S\frac{\partial}{\partial x} + P\frac{\partial}{\partial y} + Q\frac{\partial}{\partial z} + R\frac{\partial}{\partial u} \in \mathrm{Vec}(\mathbb{R}^4)$ is a symmetry of the sub-Riemannian structure $(\Delta, \langle\cdot,\cdot\rangle)$, then equalities (4.12) and (4.13) should hold with

$$\alpha = \delta = 0, \quad \beta = -\gamma.$$

That is why S, P, Q, R satisfy both the old equations (4.14)–(4.17), which mean that X is a symmetry of the distribution Δ, and the following additional equations, which mean that X preserves the inner product $\langle\cdot,\cdot\rangle$ as well:

$$S_x = 0, \tag{4.24}$$

$$E_S = -P_x, \tag{4.25}$$

$$E_P = 0. \tag{4.26}$$

That is why, by Theorem 3, for the components S, P, Q, R, equations (4.8)–(4.11) hold.

Equations (4.8) and (4.24) give

$$S_x = 2f_{yz} + 2yf_{yu} + f_u + 2xf_{zz} + 4xyf_{zu} + 2xy^2 f_{uu} = 0.$$

But f does not depend on x; thus we decompose the previous equality in powers of x:

$$2f_{yz} + f_u + 2yf_{yu} = 0, \tag{4.27}$$

$$2f_{zz} + 4yf_{zu} + 2y^2 f_{uu} = 0. \tag{4.28}$$

Analogously, equalities (4.10) and (4.26) lead to

$$f_{zy} + f_u + yf_{yu} + xf_{zz} + 2xyf_{zu} + xy^2 f_{uu} = 0,$$

which decomposes into powers of x:

$$f_{zy} + f_u + yf_{yu} = 0, \tag{4.29}$$

$$f_{zz} + 2yf_{zu} + y^2 f_{uu} = 0. \tag{4.30}$$

We subtract equation (4.27) from the doubled equation (4.29) and obtain

$$f_u = 0,$$

which means

$$f = f(y, z).$$

Now equations (4.29) and (4.30) read

$$f_{yz} = 0, \tag{4.31}$$

$$f_{zz} = 0. \tag{4.32}$$

Then condition (4.31) is equivalent to

$$f = a(y) + b(z),$$

and equality (4.32) gives

$$b(z) = bz + c, \quad b, c \in \mathbb{R}.$$

Consequently,

$$f = a(y) + bz + c. \tag{4.33}$$

Finally, we obtain from (4.8) and (4.33) that

$$S = a''(y),$$

and from (4.9) that

$$P = -b.$$

Then the last yet unused additional equation (4.25) yields

$$a'''(y) = 0;$$

thus,

$$a(y) = ay^2 + dy + \text{const}, \quad a, d \in \mathbb{R}.$$

That is why

$$f = ay^2 + dy + bz + c, \quad a, b, c, d \in \mathbb{R}.$$

Now we recover the components S, P, Q, R from (4.8)–(4.11) and obtain that any vector field $X \in \operatorname{Sym}(\Delta, \langle\cdot,\cdot\rangle)$ must have the form

$$X = 2a\frac{\partial}{\partial x} - b\frac{\partial}{\partial y} + (2ay + d)\frac{\partial}{\partial z} + (ay^2 - bz - c)\frac{\partial}{\partial u}, \quad a, b, c, d \in \mathbb{R}. \tag{4.34}$$

Direct computation verifies that for any $a, b, c, d \in \mathbb{R}$ the vector field X given by (4.34) is a symmetry of the sub-Riemannian structure $(\Delta, \langle\cdot,\cdot\rangle)$. So the Lie algebra $\operatorname{Sym}(\Delta, \langle\cdot,\cdot\rangle)$ is 4-dimensional. We compute its basis:

$$\begin{aligned}
a = \frac{1}{2},\ b = c = d = 0 \quad &\Rightarrow \quad X_1 = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z} + \frac{1}{2}y^2\frac{\partial}{\partial u},\\
a = 0,\ b = -1,\ c = d = 0 \quad &\Rightarrow \quad X_2 = \frac{\partial}{\partial y} + z\frac{\partial}{\partial u},\\
a = b = c = 0,\ d = -1 \quad &\Rightarrow \quad X_3 = -\frac{\partial}{\partial z},\\
a = b = 0,\ c = -1,\ d = 0 \quad &\Rightarrow \quad X_4 = \frac{\partial}{\partial u}.
\end{aligned}$$

Nonzero brackets between the basis vectors are:

$$[X_1, X_2] = X_3, \quad [X_2, X_3] = X_4.$$

Consequently, $\operatorname{Sym}(\Delta, \langle\cdot,\cdot\rangle) = \operatorname{span}(X_1, X_2, X_3, X_4)$ is the Engel algebra. □

4.4. **Engel structure and a transverse contact structure.** An Engel structure on a four-dimensional manifold M_4 (see e.g. [7]) is a rank two maximal growth distribution Δ on M_4, i.e., a rank two distribution Δ with the growth vector $(2, 3, 4)$.

One of the vector fields admissible for an Engel distribution Δ, namely ξ_1, satisfies the property

$$(\operatorname{ad}\xi_1)\Delta^2 \subset \Delta^2.$$

This property determines the vector field ξ_1 uniquely up to a nonvanishing factor. Such a vector field is called a *canonical vector field* of the distribution Δ.

Given an Engel distribution Δ on a four-dimensional manifold M_4, its canonical vector field ξ_1, and a three-dimensional submanifold $N_3 \subset M_4$ transversal to ξ_1, the distribution

$$D = \Delta^2 \cap TN_3$$

defines a contact structure on N_3 (see [7]) called a *transverse contact structure*.

Locally, all Engel structures are isomorphic (see [5], [7]); in particular, a germ of any Engel structure can be represented by the model

$$\Delta = \operatorname{span}(\xi_1, \xi_2), \qquad \xi_1, \xi_2 \in \operatorname{Vec}(\mathbb{R}^4_{x,y,z,u})$$

considered in Subsection 4.3. This implies that the Lie algebras of symmetries of the flat Engel distribution computed in Theorem 3 are Lie algebras of symmetries of a germ of an arbitrary Engel distribution.

On the other hand, all contact structures are also locally isomorphic (the Darboux theorem). In particular, any contact structure on a three-dimensional manifold is locally isomorphic to the flat rank two distribution on the Heisenberg group (see Subsection 3.1), and the Lie algebra of symmetries computed in Theorem 1 is, in fact, the Lie algebra of symmetries of a germ of a contact structure on a three-dimensional manifold.

That is why, comparing Theorems 1 and 3, we arrive at the following proposition.

Theorem 5. *Let Δ be a germ of an Engel distribution on a four-dimensional manifold M_4 with a canonical vector field ξ_1, and let D be a germ of a transverse contact structure on a three-dimensional submanifold $N_3 \subset M_4$ transversal to ξ_1. Then there is a one-to-one correspondence between:*

1) *symmetries of Δ;*
2) *symmetries of D;*
3) *functions $f : M_4 \to \mathbb{R}$ constant along the canonical vector field ξ_1.*

5. The Cartan case

Rank two distributions in the five-dimensional space were studied by E. Cartan [6], and this gave the title of the case we consider in this section.

5.1. **The Lie algebra and Lie group.** Let $\mathfrak{g}$ be the five-dimensional nilpotent three-step Lie algebra with multiplication rules in some basis

$$\mathfrak{g} = \operatorname{span}(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5)$$

as follows:

$$[\xi_1, \xi_2] = \xi_3, \quad [\xi_1, \xi_3] = \xi_4, \quad [\xi_2, \xi_3] = \xi_5$$

(all the remaining brackets are equal to zero). We call such a frame $\xi_1, \dots, \xi_5$ a *standard* left-invariant frame on $\mathfrak{g}$. Multiplication rules in a standard frame of $\mathfrak{g}$ are represented by the scheme in Figure 5.

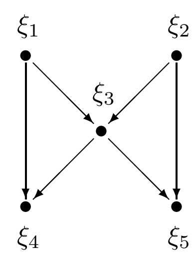


Figure 5. Lie algebra $\mathfrak{g}$, the Cartan case

The Lie algebra $\mathfrak{g}$ is graded:

$$\mathfrak{g} = \mathfrak{g}^1 \oplus \mathfrak{g}^2 \oplus \mathfrak{g}^3,$$

where

$$\mathfrak{g}^1 = \operatorname{span}(\xi_1, \xi_2), \quad \mathfrak{g}^2 = \operatorname{span}(\xi_3), \quad \mathfrak{g}^3 = \operatorname{span}(\xi_4, \xi_5).$$

Denote by G the simply connected Lie group corresponding to $\mathfrak{g}$.

5.2. **The flat distribution and sub-Riemannian structure.** We assert that any flat distribution or sub-Riemannian structure on the Lie group G is isomorphic to the following one defined via a standard frame in $\mathfrak{g}$:

$$\Delta = \operatorname{span}(\xi_1, \xi_2), \qquad \langle \xi_i, \xi_j \rangle = \delta_{ij}, \; i, j = 1, 2. \tag{5.1}$$

As before, we prove the isomorphism for sub-Riemannian structures, and the isomorphism for distributions will follow. Take an arbitrary flat sub-Riemannian structure $(\Delta, \langle \cdot, \cdot \rangle)$ on the Lie group G corresponding to a graduation of $\mathfrak{g}$:

$$\begin{aligned} &\mathfrak{g} = \mathfrak{g}^1 \oplus \mathfrak{g}^2 \oplus \mathfrak{g}^3, \\ &\Delta = \mathfrak{g}^1, \quad \dim \mathfrak{g}^1 = 2, \\ &\mathfrak{g}^2 = [\mathfrak{g}^1, \mathfrak{g}^1], \quad \dim \mathfrak{g}^2 = 1, \\ &\mathfrak{g}^3 = [\mathfrak{g}^1, \mathfrak{g}^2], \quad \dim \mathfrak{g}^3 = 2 \end{aligned}$$

(as in the Engel case, these homogeneous components $\mathfrak{g}^i$ should not be the same as in the previous subsection, but their number and dimensions are obviously the same).

Choose any orthonormal basis as in (5.1). Then the vector

$$\xi_3 = [\xi_1, \xi_2]$$

spans $\mathfrak{g}^2$, and the vectors

$$\xi_4 = [\xi_1, \xi_3], \qquad \xi_5 = [\xi_2, \xi_3]$$

span the homogeneous component $\mathfrak{g}^3$. Thus the orthonormal frame ξ_1, ξ_2 of the sub-Riemannian structure $(\Delta, \langle \cdot, \cdot \rangle)$ generates a standard frame in $\mathfrak{g}$. This proves the uniqueness of flat sub-Riemannian structures on the Lie group G up to an isomorphism; the flat distributions are isomorphic so much the more.

5.3. **The Cartan model.** In this subsection we describe the local model of the flat distribution on the Lie group G due to E. Cartan (this construction was kindly communicated to us by A. A. Agrachev).

Let $\mathfrak{g} = \mathfrak{g}_2$ be the (unique) noncompact real form of the complex simple 14-dimensional Lie algebra $\mathfrak{g}_2^{\mathbb{C}}$, and let G be the corresponding connected simply connected Lie group. We consider $\mathfrak{g}$ as the Lie algebra of left-invariant vector fields on G and choose a basis

$$\mathfrak{g} = \operatorname{span}(Z_1, \dots, Z_{14})$$

so that Z_{13} and Z_{14} span a (two-dimensional) Cartan subalgebra of $\mathfrak{g}$ and $Z_1, \dots, Z_{12}$ correspond to root vectors; see Figure 6. (Compare with the description of the linear representation of $\mathfrak{g}_2$ in the Appendix and Figure 8.)

The vector fields Z_1, Z_2 generate the left-invariant 2-distribution

$$D = \operatorname{span}(Z_1, Z_2)$$

on G and the 5-dimensional nilpotent Lie algebra

$$\mathfrak{n} = \operatorname{Lie}(Z_1, Z_2) = \operatorname{span}(Z_1, \dots, Z_5),$$

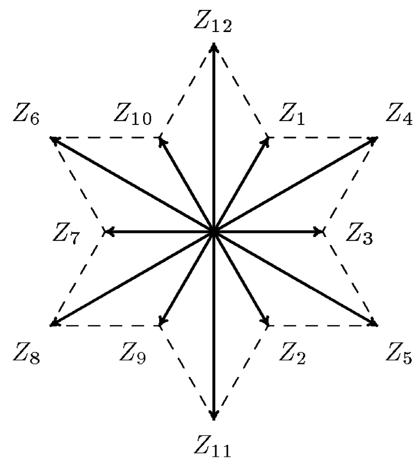


FIGURE 6. Root vectors of $\mathfrak{g}$, the Cartan case

which is obvious from the scheme of root vectors in Figure 6. The same figure shows that the vector subspace of $\mathfrak{g}$ transversal to $\mathfrak{n}$, namely

$$\mathfrak{h} = \operatorname{span}(Z_6, \dots, Z_{14}),$$

is in fact a subalgebra having the property

$$\operatorname{ad} \mathfrak{h}(D) \subset D + \mathfrak{h}. \tag{5.2}$$

That is, the subalgebra $\mathfrak{h}$ preserves the distribution D modulo $\mathfrak{h}$ itself; consequently, we can factorize by $\mathfrak{h}$.

Let H be the local connected subgroup of G corresponding to the Lie subalgebra $\mathfrak{h}$, and

$$M = G/H = \{ xH \mid x \in G \}$$

the left coset space, a smooth 5-dimensional manifold. Take the corresponding projection and its differential

$$\pi \; : \; G \to G/H = M, \quad \pi \; : \; x \mapsto xH, \qquad \pi_* \; : \; \mathfrak{h} \to \mathfrak{g}/\mathfrak{h}$$

and define

$$\Delta = \pi_*(D).$$

By virtue of (5.2), Δ is a correctly defined 2-distribution on M.

We choose a vector field basis in Δ as follows. Let N be the local subgroup of G corresponding to the nilpotent subalgebra $\mathfrak{n}$. Since

$$\mathfrak{n} \cap \mathfrak{h} = \{0\},$$

then the restriction

$$\pi|_N \; : \; N \to M$$

is a diffeomorphism. Denote the inverse diffeomorphism by

$$\tau \; : \; M \to N, \qquad \tau = (\pi|_N)^{-1}$$

and define the vector fields on M by

$$\xi_i(q) = \pi_* Z_i(x), \qquad x = \tau(q), \quad i = 1, 2.$$

First,

$$\Delta_q = \mathrm{span}(\xi_1(q), \xi_2(q)), \quad q \in M.$$

Second, the vector fields ξ_i, $i = 1, 2$, are π-related to the vector fields Z_i, $i = 1, 2$, respectively. Consequently,

$$\mathrm{Lie}(\xi_1, \xi_2) \cong \mathrm{Lie}(Z_1, Z_2) = \mathfrak{n}.$$

To summarize, $\Delta \subset TM$ is a 2-distribution on the 5-dimensional manifold M, and admissible vector fields of D form the 5-dimensional nilpotent Lie algebra $\mathfrak{n}$. That is why the distribution Δ is a (local) model for a flat (2,5)-distribution. We call it the *Cartan model*.

Now we compute some symmetries of the distribution Δ with the help of the Cartan model.

The left-invariant distribution D is preserved by all left translations on the Lie group G. The flow of a *right*-invariant vector field on G is realized by *left* translations on G; that is why all right-invariant vector fields on G are infinitesimal symmetries of D:

$$\mathfrak{g}_r \subset \mathrm{Sym}(D)$$

(we denote by $\mathfrak{g}_r$ the Lie algebra of right-invariant vector fields on G).

Now we project these symmetries to M. The action of the group G by left translations is naturally projected from G to its homogeneous space $M = G/H$; hence, right-invariant vector fields on G are correctly projected to the vector fields

$$\pi_*(\mathfrak{g}_r) \subset \mathrm{Vec}(M).$$

The left translations of G on M preserve the distribution Δ; thus,

$$\pi_*(\mathfrak{g}_r) \subset \mathrm{Sym}(\Delta).$$

In order to show that the projection π_* does not send any symmetry from $\mathfrak{g}_r$ to zero, suppose the contrary:

$$\exists\ v \in \mathfrak{g}_r \ :\ v(x) \in \ker \pi_*|_x \quad \forall x \in G.$$

We apply this inclusion at the identity $e \in G$ and see that

$$v \in \mathfrak{h}_r$$

(we denote by $\mathfrak{h}_r$ the Lie algebra of all right-invariant vector fields on G that are tangent at the identity e to the subgroup H). On the other hand,

$$\ker \pi_*|_x = \mathfrak{h}(x)$$

(in the right-hand side stands the vector space obtained by the values of the left-invariant vector fields from $\mathfrak{h}$ at the point $x \in G$). Consequently,

$$v \in \mathfrak{h}_r \cap \mathfrak{h}.$$

But

$$\mathfrak{h}_r \cap \mathfrak{h} = \{0\}$$

since the Lie algebra $\mathfrak{g}$ is simple. This means that $v = 0$. Thus,

$$\dim \pi_*(\mathfrak{g}_r) = 14$$

and

$$\pi_*(\mathfrak{g}_r) \cong \mathfrak{g}_r \cong \mathfrak{g} \subset \mathrm{Sym}(\Delta). \tag{5.3}$$

That is, the Cartan model yields that locally a flat (2,5)-distribution has a 14-dimensional algebra of symmetries isomorphic to the Lie algebra $\mathfrak{g}_2$. In Subsubsection 5.4.1 we show that inclusion (5.3) is in fact equality.

5.4. **The model in** $\mathbb{R}^5$**.** The five-dimensional space $\mathbb{R}^5_{x,y,z,u,v}$ endowed with the multiplication rule

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ u_1 \\ v_1 \end{pmatrix} \cdot \begin{pmatrix} x_2 \\ y_2 \\ z_2 \\ u_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 + x_1 y_2 \\ u_1 + u_2 + z_1 y_2 + x_1 y_2^2/2 \\ v_1 + v_2 + 2x_1 z_2 + x_1^2 y_2 \end{pmatrix}$$

becomes the five-dimensional nilpotent Lie group G described in Subsection 5.1 with the standard left-invariant frame

$$\xi_1 = \frac{\partial}{\partial x}, \tag{5.4}$$

$$\xi_2 = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z} + z\frac{\partial}{\partial u} + x^2\frac{\partial}{\partial v}, \tag{5.5}$$

$$\xi_3 = [\xi_1, \xi_2] = \frac{\partial}{\partial z} + 2x\frac{\partial}{\partial v},$$

$$\xi_4 = [\xi_1, \xi_3] = 2\frac{\partial}{\partial v},$$

$$\xi_5 = [\xi_2, \xi_3] = -\frac{\partial}{\partial u}.$$

Thus we can use the corresponding model of the flat distribution and sub-Riemannian structure on the group G:

$$\Delta = \operatorname{span}(\xi_1, \xi_2), \tag{5.6}$$

$$\langle \xi_i, \xi_j \rangle = \delta_{ij}, \; i, j = 1, 2. \tag{5.7}$$

Now we compute the symmetries $\operatorname{Sym}(\Delta)$ and $\operatorname{Sym}(\Delta, \langle \cdot, \cdot \rangle)$ with the help of this model.

5.4.1. *Symmetries of the distribution.*

Theorem 6. *The Lie algebra of symmetries of the flat distribution Δ on the five-dimensional nilpotent Lie group G described in Subsection 5.1 is the* 14*-dimensional Lie algebra $\mathfrak{g}_2$, that is, the* (*unique*) *noncompact real form of the complex exceptional simple Lie algebra $\mathfrak{g}_2^{\mathbb{C}}$.*

For the model of Δ in $\mathbb{R}^5$ defined by (5.4)–(5.6), we have

$$\operatorname{Sym}(\Delta) = \operatorname{span}(Y_1, \dots, Y_{14}),$$

where

$$
\begin{aligned}
Y_1 &= \frac{1}{36}\left(2\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial z} + y^2\frac{\partial}{\partial u} + 4z\frac{\partial}{\partial v}\right),\\
Y_2 &= -3\frac{\partial}{\partial y},\\
Y_3 &= \frac{1}{12}\left(\frac{\partial}{\partial z} + y\frac{\partial}{\partial u}\right),\\
Y_4 &= -\frac{1}{324}\frac{\partial}{\partial v},\\
Y_5 &= \frac{1}{12}\frac{\partial}{\partial u},\\
Y_6 &= \frac{1}{3}\Bigg((6x^2y^2 - 12xyz + 24z^2 - 18yv)\frac{\partial}{\partial x} + (6xy^3 - 24zy^2 + 36yu)\frac{\partial}{\partial y}\\
&\qquad +(3x^2y^3 - 12yz^2 - 9y^2v + 36zu)\frac{\partial}{\partial z}\\
&\qquad +(6xy^3z - 12y^2z^2 - 3y^3v + 36u^2)\frac{\partial}{\partial u}\\
&\qquad + (2x^3y^3 - 36yzv + 16z^3 + 36uv)\frac{\partial}{\partial v}\Bigg),\\
Y_7 &= 3\Bigg((-4x^2y + 4xz + 6v)\frac{\partial}{\partial x} + (-6xy^2 + 16yz - 12u)\frac{\partial}{\partial y}\\
&\qquad +(-3x^2y^2 + 4z^2 + 6vy)\frac{\partial}{\partial z} + (-6xy^2z + 8yz^2 + 3y^2v)\frac{\partial}{\partial u}\\
&\qquad + (-2x^3y^2 + 12zv)\frac{\partial}{\partial v}\Bigg),\\
Y_8 &= 36\Bigg((6x^2z - 9xv)\frac{\partial}{\partial x} + (18xu - 12z^2)\frac{\partial}{\partial y} + (9x^2u - 9zv)\frac{\partial}{\partial z}\\
&\qquad +(18xzu - 8z^3 - 9uv)\frac{\partial}{\partial u} + (6x^3u - 9v^2)\frac{\partial}{\partial v}\Bigg),\\
Y_9 &= 9\Bigg(2x^2\frac{\partial}{\partial x} + (6xy - 8z)\frac{\partial}{\partial y} + (3x^2y - 3v)\frac{\partial}{\partial z}\\
&\qquad +(6xyz - 4z^2 - 3yv)\frac{\partial}{\partial u} + 2x^3y\frac{\partial}{\partial v}\Bigg),\\
Y_{10} &= \frac{1}{3}\left((xy - 4z)\frac{\partial}{\partial x} - y^2\frac{\partial}{\partial y} - (zy + 3u)\frac{\partial}{\partial z} - 3yu\frac{\partial}{\partial u} - 4z^2\frac{\partial}{\partial v}\right),\\
Y_{11} &= -9\left(6x\frac{\partial}{\partial y} + 3x^2\frac{\partial}{\partial z} + (6xz - 3v)\frac{\partial}{\partial u} + 2x^3\frac{\partial}{\partial v}\right),\\
Y_{12} &= \frac{1}{324}\left(6y\frac{\partial}{\partial x} + 3y^2\frac{\partial}{\partial z} + y^3\frac{\partial}{\partial u} + (12yz - 12u)\frac{\partial}{\partial v}\right),\\
Y_{13} &= -x\frac{\partial}{\partial x} - z\frac{\partial}{\partial z} - u\frac{\partial}{\partial u} - 2v\frac{\partial}{\partial v},\\
Y_{14} &= -x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}.
\end{aligned}
$$

Corollary 1. *The vector fields $Y_1, \dots, Y_{14} \in \operatorname{Vec}(\mathbb{R}^5)$ given in Theorem 6 provide a faithful representation of the Lie algebra $\mathfrak{g}_2$.*

The proof of this theorem reduces to the following two independent lemmas.

Lemma 5.1. $\operatorname{Sym}(\Delta) = \operatorname{span}(Y_1, \dots, Y_{14})$.

Lemma 5.2. $\operatorname{span}(Y_1, \dots, Y_{14}) \cong \mathfrak{g}_2$.

Proof of Lemma 5.1. We take an arbitrary smooth vector field

$$Y = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y} + R\frac{\partial}{\partial z} + S\frac{\partial}{\partial u} + T\frac{\partial}{\partial v} \in \operatorname{Vec}(\mathbb{R}^5),$$

and compute the brackets

$$[\xi_1, Y] = P_x\frac{\partial}{\partial x} + Q_x\frac{\partial}{\partial y} + R_x\frac{\partial}{\partial z} + S_x\frac{\partial}{\partial u} + T_x\frac{\partial}{\partial v},$$
$$[\xi_2, Y] = E_P\frac{\partial}{\partial x} + E_Q\frac{\partial}{\partial y} + (E_R - P)\frac{\partial}{\partial z} + (E_S - R)\frac{\partial}{\partial u} + (E_T - 2xP)\frac{\partial}{\partial v},$$

where

$$E_P = \xi_2 P = P_y + xP_z + zP_u + x^2 P_v, \tag{5.8}$$
$$E_Q = \xi_2 Q = Q_y + xQ_z + zQ_u + x^2 Q_v, \tag{5.9}$$
$$E_R = \xi_2 R = R_y + xR_z + zR_u + x^2 R_v,$$
$$E_S = \xi_2 S = S_y + xS_z + zS_u + x^2 S_v,$$
$$E_T = \xi_2 T = T_y + xT_z + zT_u + x^2 T_v.$$

By Proposition 1, a vector field Y is a symmetry of the distribution Δ iff

$$\begin{pmatrix} P_x \\ Q_x \\ R_x \\ S_x \\ T_x \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \\ x \\ z \\ x^2 \end{pmatrix}, \quad \begin{pmatrix} E_P \\ E_Q \\ E_R - P \\ E_S - R \\ E_T - 2xP \end{pmatrix} = \gamma \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \delta \begin{pmatrix} 0 \\ 1 \\ x \\ z \\ x^2 \end{pmatrix} \tag{5.10}$$

for some smooth real-valued functions $\alpha, \beta, \gamma, \delta$.

These vector equations are solvable in $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ iff the following equalities hold:

$$R_x = xQ_x, \tag{5.11}$$
$$S_x = zQ_x, \tag{5.12}$$
$$T_x = x^2 Q_x, \tag{5.13}$$
$$P = E_R - xE_Q, \tag{5.14}$$
$$R = E_S - zE_Q, \tag{5.15}$$
$$x^2 E_Q - 2xE_R + E_T = 0. \tag{5.16}$$

Equality (5.12) is obviously equivalent to

$$S = zQ + \varphi \tag{5.17}$$

for some smooth function

$$\varphi = \varphi(y, z, u, v).$$

Now we can compute R in terms of Q and φ:

$$\begin{aligned} E_S &= \xi_2 S = \xi_2(zQ+\varphi) = z\xi_2 Q + xQ + \xi_2\varphi \\ &= zE_Q + xQ + \varphi_y + x\varphi_z + z\varphi_u + x^2\varphi_v; \end{aligned}$$

hence by (5.15),

$$R = E_S - zE_Q = xQ + \varphi_y + x\varphi_z + z\varphi_u + x^2\varphi_v. \tag{5.18}$$

We differentiate the previous equality with respect to x:

$$R_x = xQ_x + Q + \varphi_z + 2x\varphi_v,$$

and in view of (5.11) obtain

$$Q = -\varphi_z - 2x\varphi_v. \tag{5.19}$$

Then (5.17) is rewritten in the form

$$S = zQ + \varphi = \varphi - z\varphi_z - 2xz\varphi_v \tag{5.20}$$

and (5.13) as

$$T_x = x^2 Q_x = -2x^2\varphi_v.$$

We integrate the previous equation with respect to x:

$$T = -\frac{2}{3}x^3\varphi_v + \psi, \quad \psi = \psi(y,z,u,v). \tag{5.21}$$

Taking into account equalities (5.19), (5.18), (5.20), and (5.21), we see that conditions (5.11)–(5.16) are equivalent to the following ones:

$$P = E_R - xE_Q, \tag{5.22}$$

$$Q = -\varphi_z - 2x\varphi_v, \tag{5.23}$$

$$R = \varphi_y + z\varphi_u - x^2\varphi_v, \tag{5.24}$$

$$S = \varphi - z\varphi_z - 2xz\varphi_v, \tag{5.25}$$

$$T = -\frac{2}{3}x^3\varphi_v + \psi, \tag{5.26}$$

$$x^2E_Q - 2xE_R + E_T = 0. \tag{5.27}$$

Thus all components of our vector field P, Q, R, S, T are uniquely determined by two functions $\varphi = \varphi(y,z,u,v)$ and $\psi = \psi(y,z,u,v)$ that satisfy equation (5.27). Now we substitute the expressions of P, Q, R, S, T in terms of φ, ψ into this equation and find independent parameters that determine φ, ψ.

In terms of φ, ψ, equality (5.27) takes the form

$$\begin{aligned} x^5\left(-\frac{2}{3}\varphi_{vv}\right) + x^4\left(-\frac{5}{3}\varphi_{zv}\right) + x^3\left(-\frac{8}{3}\varphi_{yv} - \varphi_{zz} - \frac{8}{3}z\varphi_{uv}\right) \\ + x^2(-3\varphi_{yz} - 3z\varphi_{zu} - 2\varphi_u + \psi_v) \\ + x(-2\varphi_{yy} - 4z\varphi_{yu} - 2z^2\varphi_{uu} + \psi_z) + (\psi_y + z\psi_u) = 0. \end{aligned}$$

Recall that φ does not depend on x; that is why

$$\varphi_{vv} = 0, \tag{5.28}$$
$$\varphi_{zv} = 0, \tag{5.29}$$
$$\varphi_{yv} + \frac{3}{8}\varphi_{zz} + z\varphi_{uv} = 0, \tag{5.30}$$
$$3\varphi_{yz} + 3z\varphi_{zu} + 2\varphi_u - \psi_v = 0, \tag{5.31}$$
$$2\varphi_{yy} + 4z\varphi_{yu} + 2z^2\varphi_{uu} - \psi_z = 0, \tag{5.32}$$
$$\psi_y + z\psi_u = 0. \tag{5.33}$$

Equations (5.28), (5.29) mean that φ_v does not depend on v, z:

$$\varphi_v = \alpha(y, u);$$

thus,

$$\varphi = v\alpha(y, u) + \beta(y, z, u) \tag{5.34}$$

for some functions $\alpha(y, u)$ and $\beta(y, z, u)$. In view of the previous equality, conditions (5.30)–(5.33) take the form

$$\alpha_y + \frac{3}{8}\beta_{zz} + z\alpha_u = 0, \tag{5.35}$$
$$3\beta_{yz} + 3z\beta_{zu} + 2v\alpha_u + 2\beta_u - \psi_v = 0, \tag{5.36}$$
$$2v\alpha_{yy} + 2\beta_{yy} + 4zv\alpha_{yu} + 4z\beta_{yu} + 2z^2v\alpha_{uu} + 2z^2\beta_{uu} - \psi_z = 0, \tag{5.37}$$
$$\psi_y + z\psi_u = 0. \tag{5.38}$$

Differentiation of equation (5.35) with respect to z gives

$$\frac{3}{8}\beta_{zzz} = -\alpha_u.$$

Then we integrate the previous equality three times with respect to z and obtain

$$\beta = -\frac{4}{9}z^3\alpha_u + \frac{1}{2}z^2\gamma + z\delta + \sigma \tag{5.39}$$

for some functions

$$\gamma = \gamma(y, u), \quad \delta = \delta(y, u), \quad \sigma = \sigma(y, u).$$

We substitute expression (5.39) into (5.35) and obtain

$$\gamma = -\frac{8}{3}\alpha_y;$$

thus,

$$\beta = -\frac{4}{9}z^3\alpha_u - \frac{4}{3}z^2\alpha_y + z\delta + \sigma, \tag{5.40}$$

and this equality is equivalent to (5.35).

Substitution of the expression for β from (5.40) to equalities (5.36) and (5.37) leads, after some transformations, to

$$\begin{aligned}\psi_v &= z^3\left(-\frac{44}{9}\alpha_{uu}\right)+z^2\left(-\frac{44}{3}\alpha_{yu}\right)+z(-8\alpha_{yy}+5\delta_u)\\ &\quad+(3\delta_y+2v\alpha_u+2\sigma_u), \qquad (5.41)\\ \psi_z &= z^5\left(-\frac{8}{9}\alpha_{uuu}\right)+z^4\left(-\frac{40}{9}\alpha_{yuu}\right)+z^3\left(-\frac{56}{9}\alpha_{yyu}+2\delta_{uu}\right)\\ &\quad+z^2\left(-\frac{8}{3}\alpha_{yyy}+4\delta_{yu}+2v\alpha_{uu}+2\sigma_{uu}\right)\\ &\quad+z(2\delta_{yy}+4v\alpha_{yu}+4\sigma_{yu})+(2v\alpha_{yy}+2\sigma_{yy}). \qquad (5.42)\end{aligned}$$

We differentiate:

$$\begin{aligned}\psi_{vz} &= -\frac{44}{3}z^2\alpha_{uu}+2z\left(-\frac{44}{3}\alpha_{yu}\right)+(-8\alpha_{yy}+5\delta_u),\\ \psi_{zv} &= 2z^2\alpha_{uu}+4z\alpha_{yu}+2\alpha_{yy},\end{aligned}$$

equate these mixed derivatives and the terms near equal powers of z in them, and obtain

$$\alpha_{uu}=0, \qquad (5.43)$$

$$\alpha_{yu}=0, \qquad (5.44)$$

$$\alpha_{yy}=\frac{1}{2}\delta_u. \qquad (5.45)$$

Equations (5.43) and (5.44) mean that

$$\alpha_u=c, \quad c\in\mathbb{R};$$

thus,

$$\alpha=cu+\pi(y). \qquad (5.46)$$

Condition (5.45) then reads

$$\pi_{yy}=\frac{1}{2}\delta_u;$$

that is why

$$\delta=2u\pi_{yy}+\lambda(y). \qquad (5.47)$$

In view of (5.46) and (5.47), equalities (5.41) and (5.42) are rewritten as

$$\begin{aligned}\psi_v &= z(2\pi_{yy})+6u\pi_{yyy}+3\lambda_y+2cv+2\sigma_u, \qquad (5.48)\\ \psi_z &= z^2\left(\frac{16}{3}\pi_y^{(3)}+2\sigma_{uu}\right)+z(4u\pi_y^{(4)}+2\lambda_{yy}+4\sigma_{yu})\\ &\quad+(2v\pi_{yy}+2\sigma_{yy}), \qquad (5.49)\end{aligned}$$

the first of which after integration with respect to v gives

$$\psi=cv^2+v(2z\pi_{yy}+6u\pi_{yyy}+3\lambda_y+2\sigma_u)+\tau(y,z,u). \qquad (5.50)$$

We differentiate the previous equality with respect to z and obtain

$$\psi_z=2v\pi_{yy}+\tau_z,$$

then compare with (5.49) and get

$$\tau_z = z^2\left(\frac{16}{3}\pi_y^{(3)} + 2\sigma_{uu}\right) + z(4u\pi_y^{(4)} + 2\lambda_{yy} + 4\sigma_{yu}) + 2\sigma_{yy}.$$

Integrating with respect to z leads to

$$\tau = z^3\left(\frac{16}{9}\pi_y^{(3)} + \frac{2}{3}\sigma_{uu}\right) + z^2(2u\pi_y^{(4)} + \lambda_{yy} + 2\sigma_{yu}) + 2\sigma_{yy}z + \varepsilon(y,u).$$

We substitute this into (5.50) and obtain

$$\begin{aligned} \psi &= cv^2 + v(2z\pi_y^{(2)} + 6u\pi_y^{(3)} + 3\lambda_y + 2\sigma_u) + z^3\left(\frac{16}{9}\pi_y^{(3)} + \frac{2}{3}\sigma_{uu}\right) \\ &\quad + z^2(2u\pi_y^{(4)} + \lambda_{yy} + 2\sigma_{yu}) + 2\sigma_{yy}z + \varepsilon. \end{aligned} \tag{5.51}$$

We substitute this expression for ψ into (5.38) and get

$$\begin{aligned} &z^4\left(\frac{2}{3}\sigma_{uuu}\right) + z^3\left(\frac{34}{9}\pi_y^{(4)} + \frac{8}{3}\sigma_{yuu}\right) + z^2(2u\pi_y^{(5)} + \lambda_y^{(3)} + 4\sigma_{yyu}) \\ &\quad + zv(8\pi_y^{(3)} + 2\sigma_{uu}) + z(2\sigma_{yyy} + \varepsilon_u) + v(6u\pi_y^{(4)} + 3\lambda_y^{(2)} + 2\sigma_{yu}) + \varepsilon_y = 0. \end{aligned}$$

We equate terms near powers of z and v to zero and obtain

$$\sigma_{uuu} = 0, \tag{5.52}$$

$$\frac{34}{9}\pi_y^{(4)} + \frac{8}{3}\sigma_{yuu} = 0, \tag{5.53}$$

$$2u\pi_y^{(5)} + \lambda_y^{(3)} + 4\sigma_{yyu} = 0, \tag{5.54}$$

$$8\pi_y^{(3)} + 2\sigma_{uu} = 0, \tag{5.55}$$

$$2\sigma_{yyy} + \varepsilon_u = 0, \tag{5.56}$$

$$6u\pi_y^{(4)} + 3\lambda_y^{(2)} + 2\sigma_{yu} = 0, \tag{5.57}$$

$$\varepsilon_y = 0. \tag{5.58}$$

Equation (5.58) means that

$$\varepsilon = \varepsilon(u),$$

and equation (5.53) yields

$$\sigma = \frac{1}{2}u^2\theta(y) + u\rho(y) + \gamma(y). \tag{5.59}$$

Then equalities (5.53)–(5.58) are rewritten as

$$\frac{34}{9}\pi_y^{(4)} + \frac{8}{3}\theta_y = 0, \tag{5.60}$$

$$2u\pi_y^{(5)} + \lambda_y^{(3)} + 4u\theta_y^{(2)} + 4\rho_y^{(2)} = 0, \tag{5.61}$$

$$8\pi_y^{(3)} + 2\theta = 0, \tag{5.62}$$

$$u^2\theta_y^{(3)} + 2u\rho_y^{(3)} + 2\gamma_y^{(3)} + \varepsilon_u = 0, \tag{5.63}$$

$$6u\pi_y^{(4)} + 3\lambda_y^{(2)} + 2u\theta_y + 2\rho_y = 0. \tag{5.64}$$

We differentiate (5.62):

$$8\pi_y^{(4)} + 2\theta_y = 0,$$

which gives in combination with (5.60):

$$\pi_y^{(4)} = \theta_y = 0.$$

Thus,

$$\theta = a, \qquad a \in \mathbb{R}, \tag{5.65}$$

and equations (5.60)–(5.64) take the form

$$\lambda_y^{(3)} + 4\rho_y^{(2)} = 0, \tag{5.66}$$

$$8\pi_y^{(3)} + 2a = 0, \tag{5.67}$$

$$2u\rho_y^{(3)} + 2\gamma_y^{(3)} + \varepsilon_u = 0, \tag{5.68}$$

$$3\lambda_y^{(2)} + 2\rho_y = 0. \tag{5.69}$$

Then (5.67) implies

$$\pi = -\frac{1}{24}ay^3 + by^2 + dy + f, \quad b, d, f \in \mathbb{R}, \tag{5.70}$$

and equations (5.66)–(5.69) are equivalent to

$$\begin{aligned} &\lambda_y^{(3)} = 0, \\ &\rho_y^{(2)} = 0, \\ &2\gamma_y^{(3)} + \varepsilon_y = 0, \\ &3\lambda_y^{(2)} + 2\rho_y = 0. \end{aligned}$$

That is why

$$\begin{aligned} &\lambda = -\frac{1}{3}ny^2 + ly + m, \\ &\rho = ny + p, \\ &\gamma = \frac{1}{12}ky^3 + qy^2 + ry + s, \\ &\varepsilon = -ku + t \end{aligned}$$

for some

$$n, l, m, p, k, q, r, s, t \in \mathbb{R}.$$

Now we recover the functions φ and ψ via (5.34), (5.40), (5.46), (5.47), (5.51), (5.59), (5.65), and (5.70):

$$\begin{aligned} \varphi \;=\;& \frac{1}{6}ay^2z^2 - \frac{1}{24}ay^3v + by^2v - \frac{4}{9}cz^3 - \frac{8}{3}byz^2 - \frac{1}{2}ayzu \\ & -\frac{1}{3}ny^2z + \frac{1}{12}ky^3 + cuv + dyv - \frac{4}{3}dz^2 + 4bzu + lyz + \frac{1}{2}au^2 \\ & +nyu + qy^2 + fv + mz + pu + ry + s, \\ \psi \;=\;& -\frac{1}{2}ayzv + \frac{2}{9}az^3 + cv^2 + 4bzv + \frac{1}{2}auv + \frac{4}{3}nz^2 + kyz + 3lv \\ & +2pv + 4qz - ku + t, \\ & \qquad\qquad a, b, c, d, f, n, l, m, p, k, q, r, s, t \in \mathbb{R}. \end{aligned}$$

To summarize, we proved that

$$\mathrm{Sym}(\Delta) \subset \left\{ Y = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y} + R\frac{\partial}{\partial z} + S\frac{\partial}{\partial u} + T\frac{\partial}{\partial v} \right\},$$

where

$$P = \frac{1}{12}ax^2y^2 - \frac{4}{3}bx^2y - \frac{2}{3}cx^2z - \frac{1}{6}axyz - \frac{2}{3}dx^2 - \frac{1}{3}nxy + \frac{4}{3}bxz$$

$$+cxv + \frac{1}{3}az^2 - \frac{1}{4}ayv + (p+2l)x + \frac{1}{2}ky + \frac{4}{3}nz + 2bv + 2q, \tag{5.71}$$

$$Q = \frac{1}{12}axy^3 - 2bxy^2 - \frac{1}{3}azy^2 - 2dxy - 2cxu + \frac{4}{3}cz^2 + \frac{16}{3}byz + \frac{1}{3}ny^2$$

$$+\frac{1}{2}ayu - 2fx - ly + \frac{8}{3}dz - 4bu - m, \tag{5.72}$$

$$R = \frac{1}{24}ax^2y^3 - bx^2y^2 - dx^2y - cx^2u - \frac{1}{6}ayz^2 - \frac{1}{8}ay^2v - fx^2 + \frac{1}{3}nyz$$

$$+2byz + \frac{4}{3}bz^2 + \frac{1}{2}azu + \frac{1}{4}ky^2 + czv + 2qy + lz + nu$$

$$+dv + pz + r, \tag{5.73}$$

$$S = \frac{1}{12}axy^3z - 2bxy^2z - \frac{1}{6}ay^2z^2 - \frac{1}{24}avy^3 - 2cxzu - 2dxyz + by^2v$$

$$+\frac{8}{3}byz^2 + \frac{1}{12}ky^3 + \frac{8}{9}cz^3 - 2fxz + qy^2 + nyu + \frac{1}{2}au^2 + cuv + dyv$$

$$+\frac{4}{3}dz^2 + ry + pu + fv + s, \tag{5.74}$$

$$T = \frac{1}{36}ax^3y^3 - \frac{2}{3}bx^3y^2 - \frac{2}{3}dx^3y - \frac{2}{3}cx^3u - \frac{2}{3}fx^3 - \frac{1}{2}ayzv + \frac{2}{9}az^3$$

$$+\frac{4}{3}nz^2 + kyz + 4bzv + \frac{1}{2}auv + cv^2 + 4qz - ku + (3l+2p)v + t. \tag{5.75}$$

The basis of the Lie algebra $\mathrm{Sym}(\Delta)$ presented in the formulation of Theorem 6 is obtained for the following values of parameters (we write only nonzero values of the parameters $k, q, m, r, s, t, a, b, c, d, f, n, p, l$):

$$q = \frac{1}{36} \Rightarrow Y_1, \tag{5.76}$$

$$m = 3 \Rightarrow Y_2, \tag{5.77}$$

$$r = \frac{1}{12} \Rightarrow Y_3, \tag{5.78}$$

$$t = -\frac{1}{324} \Rightarrow Y_4, \tag{5.79}$$

$$\begin{aligned} s = \frac{1}{12} &\Rightarrow Y_5, \\ a = 24 &\Rightarrow Y_6, \\ b = 9 &\Rightarrow Y_7, \\ c = -324 &\Rightarrow Y_8, \\ d = -27 &\Rightarrow Y_9, \\ n = -1 &\Rightarrow Y_{10}, \\ f = 27 &\Rightarrow Y_{11}, \\ k = \frac{1}{27} &\Rightarrow Y_{12}, \\ p = -1 &\Rightarrow Y_{13}, \\ p = 1, l = -1 &\Rightarrow Y_{14}. \end{aligned} \tag{5.80}$$

Immediate verification shows that the vector fields $Y_1, \dots, Y_{14}$ are linearly independent.

All vector fields $Y_1, \dots, Y_{14}$ are indeed symmetries of the distribution Δ since the conditions of Proposition 1 hold:

$$
\begin{array}{llll}
[Y_i, \xi_1] & = 0, & [Y_i, \xi_2] & = 0, \qquad i = 1, \dots, 5, \\
[Y_6, \xi_1] & = (-4xy^2 + 4yz)\xi_1 & [Y_6, \xi_2] & = (6x^2y - 12xz + 6v)\xi_1 \\
 & \qquad -2y^3\xi_2, & & \qquad +(2xy^2 + 4yz - 12u)\xi_2, \\
[Y_7, \xi_1] & = (24xy - 12z)\xi_1 & [Y_7, \xi_2] & = -18x^2\xi_1 \\
 & \qquad +18y^2\xi_2, & & \qquad -(12xy + 12z)\xi_2, \\
[Y_8, \xi_1] & = (-432xz + 324v)\xi_1 & [Y_8, \xi_2] & = 108x^3\xi_1 \\
 & \qquad -648u\xi_2, & & \qquad +216xz\xi_2, \\
[Y_9, \xi_1] & = -36x\xi_1 - 54y\xi_2, & [Y_9, \xi_2] & = 18x\xi_2, \\
[Y_{10}, \xi_1] & = -\frac{1}{3}y\xi_1, & [Y_{10}, \xi_2] & = x\xi_1 + \frac{2}{3}y\xi_2, \\
[Y_{11}, \xi_1] & = 54\xi_2, & [Y_{11}, \xi_2] & = 0, \\
[Y_{12}, \xi_1] & = 0, & [Y_{12}, \xi_2] & = -\frac{1}{54}\xi_1, \\
[Y_{13}, \xi_1] & = \xi_1, & [Y_{13}, \xi_2] & = 0, \\
[Y_{14}, \xi_1] & = \xi_1, & [Y_{14}, \xi_2] & = -\xi_2.
\end{array}
$$

Thus,

$$\mathrm{Sym}(\Delta) = \mathrm{span}(Y_1, \dots, Y_{14}),$$

and Lemma 5.1 is completely proved. □

Proof of Lemma 5.2. Nonzero brackets in the Lie algebra $\mathrm{Sym}(\Delta)$ in the basis $Y_1, \dots, Y_{14}$ given in the formulation of Theorem 6 are shown in Table 1.

TABLE 1. Multiplication in $\mathrm{Sym}(\Delta)$, the Cartan case

$$
\begin{array}{lll}
[Y_3, Y_{10}] = -2Y_1, & [Y_3, Y_9] = 2Y_2, & [Y_3, Y_2] = 3Y_5, \\
[Y_3, Y_1] = -3Y_4, & [Y_3, Y_8] = Y_9, & [Y_3, Y_6] = -Y_{10}, \\
[Y_{10}, Y_9] = -2Y_7, & [Y_{10}, Y_7] = -3Y_6 & [Y_{10}, Y_1] = 3Y_{12}, \\
[Y_{10}, Y_5] = Y_3, & [Y_{10}, Y_{11}] = -Y_9, & [Y_9, Y_7] = 3Y_8, \\
[Y_9, Y_2] = -3Y_{11}, & [Y_9, Y_{12}] = Y_{10}, & [Y_9, Y_4] = -Y_3, \\
[Y_7, Y_2] = -2Y_9 & [Y_7, Y_1] = 2Y_{10}, & [Y_7, Y_5] = -Y_2, \\
[Y_7, Y_4] = Y_1, & [Y_2, Y_1] = -2Y_3 & [Y_2, Y_{12}] = -Y_1, \\
[Y_2, Y_6] = Y_7, & [Y_1, Y_8] = -Y_7, & [Y_1, Y_{11}] = Y_2, \\
[Y_{12}, Y_8] = Y_6, & [Y_{12}, Y_5] = -Y_4, & [Y_8, Y_5] = Y_{11}, \\
[Y_{11}, Y_4] = Y_5 & [Y_{11}, Y_6] = -Y_8, & [Y_4, Y_6] = Y_{12}, \\
[Y_{13}, Y_3] = Y_3, & [Y_{13}, Y_9] = -Y_9 & [Y_{13}, Y_7] = -Y_7, \\
[Y_{13}, Y_1] = Y_1, & [Y_{13}, Y_{12}] = Y_{12}, & [Y_{13}, Y_8] = -2Y_8, \\
[Y_{13}, Y_5] = Y_5, & [Y_{13}, Y_{11}] = -Y_{11}, & [Y_{13}, Y_4] = 2Y_4, \\
[Y_{13}, Y_6] = -Y_6, & [Y_{14}, Y_{10}] = Y_{10}, & [Y_{14}, Y_9] = -Y_9, \\
[Y_{14}, Y_2] = -Y_2, & [Y_{14}, Y_1] = Y_1, & [Y_{14}, Y_{12}] = 2Y_{12}, \\
[Y_{14}, Y_8] = -Y_8, & [Y_{14}, Y_5] = -Y_5, & [Y_{14}, Y_{11}] = -2Y_{11}, \\
[Y_{14}, Y_4] = Y_4, & [Y_{14}, Y_6] = Y_6, &
\end{array}
$$

$$
\begin{array}{ll}
[Y_3, Y_7] = -2Y_{13} + Y_{14}, & [Y_{10}, Y_2] = Y_{13} - 2Y_{14}, \\
[Y_9, Y_1] = Y_{13} + Y_{14}, & [Y_{12}, Y_{11}] = -Y_{14}, \\
[Y_8, Y_4] = Y_{13}, & [Y_5, Y_6] = -Y_{13} + Y_{14}.
\end{array}
$$

The required isomorphism

$$F : \mathrm{Sym}(\Delta) \to \mathfrak{g}_2$$

is defined on the bases of these Lie algebras by the following matrix:

$Y \in \mathrm{Sym}(\Delta)$	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6	Y_7
$F(Y) \in \mathfrak{g}_2$	X_{-e_3}	X_{-e_2}	X_{e_1}	X_{-f_2}	X_{f_3}	X_{-f_3}	X_{-e_1}

$Y \in \mathrm{Sym}(\Delta)$	Y_8	Y_9	Y_{10}	Y_{11}	Y_{12}	Y_{13}	Y_{14}
$F(Y) \in \mathfrak{g}_2$	X_{f_2}	X_{e_3}	X_{e_2}	X_{-f_1}	X_{f_1}	H_1	H_2

See the appendix for a description of the Lie algebra $\mathfrak{g}_2$ and its basis H_1, H_2, $X_{\pm e_i}$, $X_{\pm f_i}$, $i = 1, 2, 3$. The map F is indeed an isomorphism of $\mathrm{Sym}(\Delta)$ and $\mathfrak{g}_2$ since the multiplication Tables 1 and 3 (see the appendix) for these Lie algebras are isomorphic. □

5.4.2. *Symmetries of the sub-Riemannian structure.*

Theorem 7. *The Lie algebra of symmetries of the flat sub-Riemannian structure $(\Delta, \langle\cdot,\cdot\rangle)$ on the five-dimensional nilpotent Lie group G described in Subsection 5.1 is the six-dimensional Lie algebra*

$$\mathrm{Sym}(\Delta, \langle\cdot,\cdot\rangle) = \mathrm{span}(X_0, \dots, X_5)$$

with the following multiplication rules for the basis elements:

$$\begin{array}{ll} [X_0, X_1] = -X_2, & [X_0, X_2] = X_1, \\ [X_0, X_4] = -X_5, & [X_0, X_5] = X_4, \\ [X_1, X_2] = X_3, & \\ [X_1, X_3] = X_4, & [X_2, X_3] = X_5. \end{array} \tag{5.81}$$

For the model of $(\Delta, \langle\cdot,\cdot\rangle)$ in $\mathbb{R}^5$ defined by (5.4)–(5.7) we have

$$X_0 = -\frac{1}{54} Y_{11} - 54 Y_{12} \tag{5.82}$$

and

$$X_1 = -3Y_1, \quad X_2 = \frac{1}{18} Y_2, \quad X_3 = -\frac{1}{3} Y_3, \quad X_4 = 3Y_4, \quad X_5 = \frac{1}{18} Y_5, \tag{5.83}$$

where the vector fields $Y_1, \dots, Y_5$ are defined in Theorem 6.

Remark 5. Multiplication rules (5.81) in the Lie algebra $\mathrm{Sym}(\Delta, \langle\cdot,\cdot\rangle)$ are schematically represented in Figure 7 (we draw X_0 twice to obtain a planar graph).

Proof. By the proof of Theorem 6, a vector field

$$X = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y} + R\frac{\partial}{\partial z} + S\frac{\partial}{\partial u} + T\frac{\partial}{\partial v} \in \mathrm{Vec}(\mathbb{R}^5)$$

is a symmetry of the distribution Δ iff the functions P, Q, R, S, T have the form (5.71)–(5.75). Moreover, X also preserves the inner product $\langle\cdot,\cdot\rangle$ iff P, Q, R, S, T satisfy, in addition to (5.71)–(5.75), the following extra equations:

$$P_x = 0, \tag{5.84}$$

$$Q_x = -E_P, \tag{5.85}$$

$$E_Q = 0 \tag{5.86}$$

(the notation E_P, E_Q is introduced in (5.8), (5.9)).

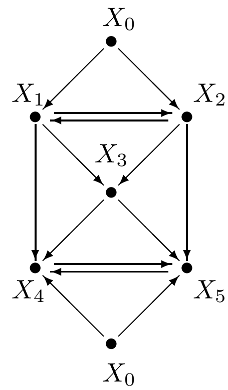


FIGURE 7. $\mathrm{Sym}(\Delta, \langle \cdot, \cdot \rangle)$, the Cartan case

But conditions (5.84)–(5.86) mean that

$$a = b = c = d = n = p = l = 0, \quad f = \frac{1}{4}k,$$

i.e.,

$$\begin{aligned}
P &= \frac{1}{2}ky + 2q, \\
Q &= -\frac{1}{2}kx - m, \\
R &= -\frac{1}{4}kx^2 + \frac{1}{4}ky^2 + 2qy + r, \\
S &= \frac{1}{12}ky^3 - \frac{1}{2}kxz + qy^2 + ry + \frac{1}{4}kv + s, \\
T &= -\frac{1}{6}kx^3 + kyz + 4qz - ku + t.
\end{aligned}$$

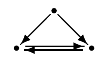
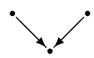
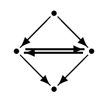
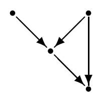
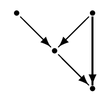
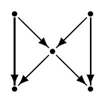
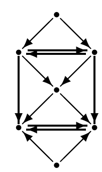
For $k = -2$, $q = m = r = s = t = 0$ we obtain the vector field X_0 defined by (5.82). The remaining basis vector fields X_i, $i = 1, \dots, 5$, are determined by (5.83), (5.76). Then $\mathrm{Sym}(\Delta, \langle \cdot, \cdot \rangle) = \mathrm{span}(X_0, \dots, X_5)$, and the commutation rules (5.81) are directly verified. □

6. GENERAL PICTURE

We summarize the above computations of symmetries of flat rank-two distributions and sub-Riemannian structures in Table 2.

For completeness, we include the two-dimensional (Riemannian) case: any vector field in $G = \mathbb{R}^2$ preserves the rank-two distribution TG, and the flat Riemannian structure is preserved by the Euclidean group of motions of the plane. In the Heisenberg case, $n = 3$, symmetries of flat distributions are parametrized by arbitrary smooth functions of three variables, and the flat sub-Riemannian structure is preserved by the four-dimensional Lie algebra: in addition to three independent left translations on the Heisenberg group, there is one additional rotation in

TABLE 2. Symmetries of flat $(2, n)$-distributions and sub-Riemannian structures

n	$(\Delta, \langle\cdot,\cdot\rangle)$	$\mathrm{Sym}(\Delta)$	dim	$\mathrm{Sym}(\Delta, \langle\cdot,\cdot\rangle)$	dim
2		$\mathrm{Vec}(\mathbb{R}^2)$	∞		3
3		$f(x, y, z)$	∞		4
4		$f(y, z, v)$	∞		4
5		$\mathfrak{g}_2$	14		6

this group. In the Engel case, $n = 4$, the Lie algebra $\mathrm{Sym}(\Delta)$ is parametrized by functions of four variables constant along the canonical vector field of the Engel distribution (which is here taken to be $\xi_1 = \frac{\partial}{\partial x}$ as in the model of Subsection 4.3). As for symmetries of the flat sub-Riemannian structure, there is only the "trivial" four-dimensional group of left translations on the Engel group. In the Cartan case, $n = 5$, there is the 14-dimensional Lie algebra $\mathfrak{g}_2$ of symmetries of the flat distribution; and the flat sub-Riemannian structure is preserved by five "trivial" left translations on the 5-dimensional nilpotent Lie group and one additional rotation on this group.

7. Appendix: Linear representation of $\mathfrak{g}_2$

For completeness of exposition, we describe here a faithful representation of the simple exceptional complex Lie algebra $\mathfrak{g}_2^{\mathbb{C}}$ and its unique noncompact real form by 7×7 complex skew-symmetric matrices (we follow [11], Lecture 14).

We denote by

$$E_{ij}, \quad i, j = 1, \dots, 7,$$

the 7×7 matrix with all zero entries except the only identity entry in the i-th row and j-th column; introduce also the skew-symmetric matrices — the basis elements of the Lie algebra so(7):

$$E_{[i,j]} = \frac{1}{2}(E_{ij} - E_{ji}), \quad i, j = 1, \dots, 7, \; i < j.$$

Then

$$\mathfrak{g}_2^{\mathbb{C}} = \mathrm{span}_{\mathbb{C}}(P_0, \dots, P_6, Q_0, \dots, Q_6),$$

where

$$\begin{aligned} P_0 &= 2(E_{[3,2]} + E_{[6,7]}), & Q_0 &= 2(E_{[4,5]} + E_{[6,7]}),\\ P_1 &= E_{[1,3]} + E_{[5,7]}, & Q_1 &= E_{[6,4]} + E_{[5,7]},\\ P_2 &= E_{[2,1]} + E_{[7,4]}, & Q_2 &= E_{[6,5]} + E_{[7,4]},\\ P_3 &= E_{[1,4]} + E_{[7,2]}, & Q_3 &= E_{[3,6]} + E_{[7,2]},\\ P_4 &= E_{[5,1]} + E_{[3,7]}, & Q_4 &= E_{[2,6]} + E_{[3,7]},\\ P_5 &= E_{[1,7]} + E_{[3,5]}, & Q_5 &= E_{[4,2]} + E_{[3,5]},\\ P_6 &= E_{[6,1]} + E_{[4,3]}, & Q_6 &= E_{[5,2]} + E_{[4,3]}. \end{aligned}$$

Matrices of the form

$$H = aP_0 + bQ_0 = aE_{[3,2]} + bE_{[4,5]} + cE_{[7,6]}, \quad a + b + c = 0, \tag{7.1}$$

form a two-dimensional Abelian subalgebra $\mathfrak{h}$ of $\mathfrak{g}_2^{\mathbb{C}}$ ($\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}_2^{\mathbb{C}}$).

Introduce the elements

$$\begin{aligned} U_{\pm 1} &= (2P_2 - Q_2) \pm i(2P_1 - Q_1), & V_{\pm 1} &= Q_2 \mp iQ_1,\\ U_{\pm 2} &= (2P_4 - Q_4) \pm i(2P_3 - Q_3), & V_{\pm 2} &= Q_4 \pm iQ_3,\\ U_{\pm 3} &= (2P_6 - Q_6) \pm i(2P_5 - Q_5), & V_{\pm 3} &= Q_6 \pm iQ_5, \end{aligned}$$

which span $\mathfrak{g}_2^{\mathbb{C}}$ together with $\mathfrak{h}$.

In the dual space $\mathfrak{h}^*$ we choose the basis e_1, e_2 dual to the basis P_0, Q_0. Then for each element (7.1) of the space $\mathfrak{h}$ we have

$$e_1(H) = a, \quad e_2(H) = b, \quad e_3(H) = c,$$

where

$$e_3 = -(e_1 + e_2).$$

Assume that the dual space $\mathfrak{h}^*$ is a Euclidean space with Cartesian coordinates in which the vectors e_1, e_2 have the coordinates

$$e_1 = \left(\frac{\sqrt{6}}{3}, 0\right), \quad e_2 = \left(-\frac{\sqrt{6}}{6}, \frac{\sqrt{2}}{2}\right).$$

Introduce also the vectors

$$f_1 = e_2 - e_3, \quad f_2 = e_3 - e_3, \quad f_3 = e_1 - e_2.$$

Then we obtain the following 12 vectors in the plane $\mathfrak{h}^*$:

$$\mathbf{G}_2 = \{\pm e_1, \pm e_2, \pm e_3, \pm f_1, \pm f_2, \pm f_3\};$$

see Figure 8.

Now we choose the following elements in $\mathfrak{g}_2^{\mathbb{C}}$:

$$X_\alpha = \begin{cases} U_{\pm k} & \text{if} \quad \alpha = \pm e_k,\\ V_{\pm k} & \text{if} \quad \alpha = \pm f_k, \end{cases}$$

$$H_\alpha = \frac{2}{a^2 + b^2 + c^2}(aE_{[3,2]} + bE_{[4,5]} + cE_{[7,6]}).$$

Then

$$\mathfrak{g}_2^{\mathbb{C}} = \operatorname{span}(\mathfrak{h}; X_\alpha, \alpha \in \mathbf{G}_2)$$

and multiplication rules in the Lie algebra $\mathfrak{g}_2^{\mathbb{C}}$ take the following simple form.

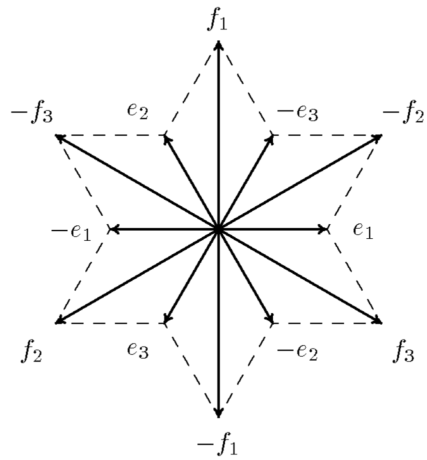


FIGURE 8. Root vectors of $\mathfrak{g}_2^{\mathbb{C}}$

Proposition 2 ([11])**.** *For any vectors $\alpha, \beta \in \mathbf{G}_2$, the following relations hold:*

$$
\begin{aligned}
&[H, X_\alpha] = i\alpha(H)X_\alpha, \quad H \in \mathfrak{h}, \\
&[X_\alpha, X_{-\alpha}] = iH_\alpha, \\
&[X_\alpha, X_\beta] = 0 \quad \text{if } \alpha + \beta \neq 0 \text{ and } \alpha + \beta \notin \mathbf{G}_2, \\
&[X_\alpha, X_\beta] = N_{\alpha,\beta} X_{\alpha+\beta} \text{ if } \alpha + \beta \in \mathbf{G}_2\,.
\end{aligned}
$$

Here $N_{\alpha,\beta}$ are some integers whose absolute values satisfy

$$|N_{\alpha,\beta}| = p + 1,$$

where p is the least integer such that for any $j = 0, 1, \ldots, p$ the vector $\beta - j\alpha$ belongs to $\mathbf{G}_2$.

Remark 6. Direct computations with the matrices X_α yield the following values for the coefficients $N_{\alpha,\beta}$:

$N_{\alpha,\beta}$	e_1	e_2	e_3	$-e_1$	$-e_2$	$-e_3$	f_1	f_2	f_3	$-f_1$	$-f_2$	$-f_3$
e_1	0	−2	2		3	−3	0	1	0	0	0	−1
e_2	2	0	−2	−3		3	0	0	1	−1	0	0
e_3	−2	2	0	3	−3		1	0	0	0	−1	0
$-e_1$		3	−3	0	−2	2	0	0	−1	0	1	0
$-e_2$	−3		3	2	0	−2	−1	0	0	0	0	1
$-e_3$	3	−3		−2	2	0	0	−1	0	1	0	0
f_1	0	0	−1	0	1	0	0	1	−1		0	0
f_2	−1	0	0	0	0	1	−1	0	1	0		0
f_3	0	−1	0	1	0	0	1	−1	0	0	0	
$-f_1$	0	1	0	0	0	−1		0	0	0	1	−1
$-f_2$	0	0	1	−1	0	0	0		0	−1	0	1
$-f_3$	1	0	0	0	−1	0	0	0		1	−1	0

In fact, this table is directly extracted from the commutation relations from Table 3.

TABLE 3. Multiplication in $\mathfrak{g}_2$

$[X_{e_1}, X_{e_2}] = -2X_{-e_3}$,	$[X_{e_1}, X_{e_3}] = 2X_{-e_2}$,	$[X_{e_1}, X_{-e_2}] = 3X_{f_3}$,
$[X_{e_1}, X_{-e_3}] = -3X_{-f_2}$,	$[X_{e_1}, X_{f_2}] = X_{e_3}$,	$[X_{e_1}, X_{-f_3}] = -X_{e_2}$,
$[X_{e_2}, X_{e_3}] = -2X_{-e_1}$,	$[X_{e_2}, X_{-e_1}] = -3X_{-f_3}$	$[X_{e_2}, X_{-e_3}] = 3X_{f_1}$,
$[X_{e_2}, X_{f_3}] = X_{e_1}$,	$[X_{e_2}, X_{-f_1}] = -X_{e_3}$,	$[X_{e_3}, X_{-e_1}] = 3X_{f_2}$,
$[X_{e_3}, X_{-e_2}] = -3X_{-f_1}$,	$[X_{e_3}, X_{f_1}] = X_{e_2}$,	$[X_{e_3}, X_{-f_2}] = -X_{e_1}$,
$[X_{-e_1}, X_{-e_2}] = -2X_{e_3}$,	$[X_{-e_1}, X_{-e_3}] = 2X_{e_2}$,	$[X_{-e_1}, X_{f_3}] = -X_{-e_2}$,
$[X_{-e_1}, X_{-f_2}] = X_{-e_3}$,	$[X_{-e_2}, X_{-e_3}] = -2X_{e_1}$,	$[X_{-e_2}, X_{f_1}] = -X_{-e_3}$,
$[X_{-e_2}, X_{-f_3}] = X_{-e_1}$,	$[X_{-e_3}, X_{f_2}] = -X_{-e_1}$,	$[X_{-e_3}, X_{-f_1}] = X_{-e_2}$,
$[X_{f_1}, X_{f_2}] = X_{-f_3}$,	$[X_{f_1}, X_{f_3}] = -X_{-f_2}$,	$[X_{f_2}, X_{f_3}] = X_{-f_1}$,
$[X_{-f_1}, X_{-f_2}] = X_{f_3}$,	$[X_{-f_1}, X_{-f_3}] = -X_{f_2}$,	$[X_{-f_2}, X_{-f_3}] = X_{f_1}$,
$[H_1, X_{e_1}] = X_{e_1}$,	$[H_1, X_{e_3}] = -X_{e_3}$,	$[H_1, X_{-e_1}] = -X_{-e_1}$,
$[H_1, X_{-e_3}] = X_{-e_3}$,	$[H_1, X_{f_1}] = X_{f_1}$,	$[H_1, X_{f_2}] = -2X_{f_2}$,
$[H_1, X_{f_3}] = X_{f_3}$,	$[H_1, X_{-f_1}] = -X_{-f_1}$,	$[H_1, X_{-f_2}] = 2X_{-f_2}$,
$[H_1, X_{-f_3}] = -X_{-f_3}$,	$[H_2, X_{e_2}] = X_{e_2}$,	$[H_2, X_{e_3}] = -X_{e_3}$,
$[H_2, X_{-e_2}] = -X_{-e_2}$,	$[H_2, X_{-e_3}] = X_{-e_3}$	$[H_2, X_{f_1}] = 2X_{f_1}$,
$[H_2, X_{f_2}] = -X_{f_2}$,	$[H_2, X_{f_3}] = -X_{f_3}$,	$[H_2, X_{-f_1}] = -2X_{-f_1}$
$[H_2, X_{-f_2}] = X_{-f_2}$,	$[H_2, X_{-f_3}] = X_{-f_3}$,	

$[X_{e_1}, X_{-e_1}] = -2H_1 + H_2$,	$[X_{e_2}, X_{-e_2}] = H_1 - 2H_2$,
$[X_{e_3}, X_{-e_3}] = H_1 + H_2$,	$[X_{f_1}, X_{-f_1}] = -H_2$,
$[X_{f_2}, X_{-f_2}] = H_1$,	$[X_{f_3}, X_{-f_3}] = -H_1 + H_2$.

The vectors

$$H_1 = -iP_0, \quad H_2 = -iQ_0$$

form a basis of the Cartan subalgebra $\mathfrak{h}$. Moreover, the elements

$$X_\alpha, \alpha \in \mathbf{G}_2; \quad H_1, H_2$$

make up a Cartan-Weyl basis of $\mathfrak{g}_2^{\mathbb{C}}$ with real structure constants. Then the set of elements of $\mathfrak{g}_2^{\mathbb{C}}$ invariant with respect to the complex conjugation relative to this basis, i.e.,

$$X \mapsto \overline{X},$$

form the (unique) real noncompact form of the complex Lie algebra $\mathfrak{g}_2^{\mathbb{C}}$ (see e.g. [14]):

$$\mathfrak{g}_2 = \{ X \in \mathfrak{g}_2^{\mathbb{C}} \mid \overline{X} = X \}.$$

We have

$$\mathfrak{g}_2 = \operatorname{span}_{\mathbb{R}}(X_\alpha, \alpha \in \mathbf{G}_2; H_1, H_2),$$

and nonzero brackets between these basis vectors are given in Table 3.

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