$\{c_{ij}^k\}$ кососимм. отн. i,j,k

$\mu \in \mathfrak{g}^* \otimes \mathfrak{g}^* \otimes \mathfrak{g}$ Пусть $e_1^*, \dots, e_n^*$ – базис $\mathfrak{g}^*$, двойств. к e_i.

$$\mu = \sum c_{ij}^k \, e_i^* \otimes e_j^* \otimes e_k$$

$\mathfrak{g}^* \xleftrightarrow{\sim} \mathfrak{g}$, $x \in \mathfrak{g} \mapsto \Phi(x, \cdot)$

$e_i \mapsto e_i^*$ $J \in \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$ соотв. μ $\sum c_{ijk} e_i \otimes e_j \otimes e_k$

$J \in \Lambda^3 \mathfrak{g}$ $\overline{e_i}$ – базис $\mathfrak{g}$ двойств. к $\{e_i\}$ в смыс Φ

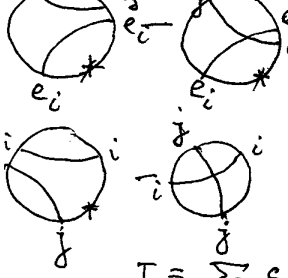
$$\Phi(e_i, \overline{e_j}) = \delta_{ij}$$

2. $f_1, \dots, f_n$ – другой базис $f_i = \sum a_{ij} e_j$

$$\overline{f_i} = \sum \overline{a}_{ij} \overline{e_j}, \quad \Phi(f_i, \overline{f_j}) = \Phi\Big(\sum_k a_{ik} e_k, \sum_\ell \overline{a}_{j\ell} \overline{e_\ell}\Big)$$

$$= \sum_k a_{ik} \overline{a}_{jk} = \delta_{ij}$$

② $e_i \oplus e_j \to e_i$ $\sum e_i e_j \overline{e_i}\,\overline{e_j}$, $\sum f_i f_j \overline{f_i}\,\overline{f_j} = \sum_{k,\ell,s,t} a_{ik} e_k \cdot a_{j\ell} e_\ell \cdot \overline{a}_{is} \overline{e_s} \cdot \overline{a}_{jt} \overline{e_t} = \sum_{k,\ell} e_k e_\ell \overline{e_k}\,\overline{e_\ell}$

③  ортоном. для Φ

$$\sum_{i,j,\dots} \dots [e_i, e_j] \dots e_j \dots e_i \dots \quad (\mathrm{I})$$

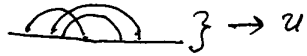
$$\sum_{i,j} \dots e_i \dots [e_i, e_j] \dots e_j \quad (\mathrm{II})$$

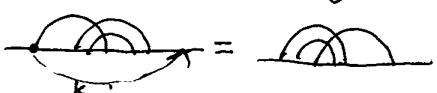
$$I = \sum_{i,j,k} c_{ijk} (\dots e_k \dots e_j \dots e_i \dots)$$

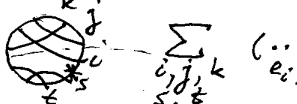
$$II = \sum_{i,j,k} c_{ijk} (\dots e_i \dots e_k \dots e_j \dots) =$$

$$= \sum e_{kij} \dots e_k \dots e_j \dots e_i \dots$$

Упр. 1 Вывести т. Якоби из сущ. ad-инв. метрики.

① $A^\ell \{$  $\} \to U(\mathfrak{g})$, $A \simeq A^\ell$



④ 

$$\sum_{i,j,k} (\dots e_i, e_j, e_k) (\dots e_s, e_k)$$

Упр. 2 $\mathrm{Im}\varphi \subset Z(U(\mathfrak{g}))$

Примеры вычисл.

$\mathfrak{g} = \mathfrak{gl}_N$ $\{e_{ij}\}$ i $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ j

$$[e_{ij}, e_{k\ell}] = \delta_{jk} e_{i\ell} - \delta_{i\ell} e_{kj}$$

$\Phi(a,b) = \mathrm{tr}(ab)$, $\Phi(e_{ij}, e_{k\ell}) = \delta_{jk}\delta_{i\ell}$

$\overline{e}_{ij} = e_{ji}$ – закон двойств.

Теор: $Z(U(\mathfrak{gl}_N))$ порождено $z_1, z_2, \dots$

$$z_1 = \sum e_{ii}$$

$$z_2 = \sum_{i,j} e_{ij} e_{ji}$$

$$z_k = \sum_{i_1 \dots i_k} e_{i_1 i_2} e_{i_2 i_3} \dots e_{i_k i_1} \quad (n^k \text{ слаг.})$$

Примеры φ для $\mathfrak{gl}_N$

$e_{ji} \oplus e_{ij}$ $\sum_{ij} e_{ij} e_{ji} = z_2$ $\bigcirc_{k\ell} \mapsto z_2^2$

$ji \oplus ij$ $\mapsto \sum e_{ij} \underline{e_{k\ell} e_{ji}} e_{\ell k} = \sum e_{ij} e_{ji} e_{k\ell} e_{\ell k} +$

$+ \sum e_{ij} [e_{k\ell}, e_{ji}] e_{\ell k} = z_2^2 + \sum e_{ij} (\delta_{\ell j} e_{ki} - \delta_{ik} e_{j\ell}) e_{\ell k}$

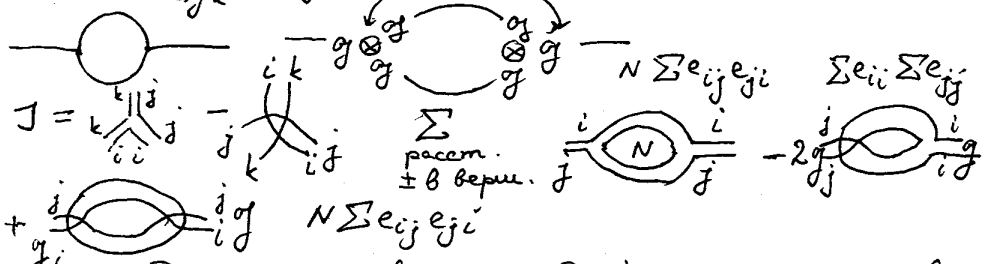
$= z_2^2 + \sum_{ijk} (e_{ij} e_{ki} e_{jk}) - \sum_{ij\ell} e_{ij} e_{j\ell} e_{\ell i} = \sum_{ijk} (e_{ij} e_{ki} e_{jk} - e_{ij} e_{jk} e_{ki})$

$= z_2^2 + \sum e_{ij} [e_{ki}, e_{jk}] = z_2^2 + \sum e_{ij} (\delta_{ij} e_{kk} - e_{ji}) = z_2^2 + z_1^2 - z_0 z_3$

где $z_0 = N$

 $\mapsto z_0 z_2 - z_1^2$ $\bigcirc \mapsto 2 z_0 z_2 - 2 z_1^2$

$$J = \sum_{i,j,k} (e_{ij} \otimes e_{jk} \otimes e_{ki} - e_{ij} \otimes e_{ki} \otimes e_{jk}) \quad \mathfrak{gl}_N$$



$N \sum e_{ij} e_{ji}$ $\sum e_{ii} \sum e_{jj}$

$-2 \sum e_{ij} e_{ji}$

$+ \sum e_{ij} e_{ji}$ $N \sum e_{ij} e_{ji}$

$\mathrm{Im}\varphi \subset$ ad-инв часть $S(\mathfrak{g})$ мн-ен от эл-тов $\mathfrak{g}$ комм. $e_1 \dots e_n$

$\mathrm{ad}_x (f(e_1, \dots, e_n)) = 0 \ \forall x$ $[x, e_i]$ комм.

ad инв. часть $S(\mathfrak{g})$ – кольцо, порожд. прав. Лейбница.

(z_i)