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COMBINATORICS

Lecture 5: Algebra of combinatorial species

A multitude of algebraic operations can be defined between combinatorial species. In this lecture, we will introduce addition and multiplication that turn the set of (equivalence classes of) species into a commutative semiring with unity.

Definition. Let F and G be two combinatorial species. The *sum of species* is defined as follows. For any finite set U

$$(F+G)[U] = F[U] \cup G[U],$$

where the sets $F[U]$ and $G[U]$ are considered to have no common elements. (If $F[U]$ and $G[U]$ do have common elements, then we should set, for example, $(F+G)[U] = (F[U]\times\{1\} \cup (G[U]\times\{2\})$. For a given bijection, the definition of the map $(F+G)[\sigma]$ is natural.

Theorem. Each of the three generating series associated with the sum of species is equal to the sum of corresponding series:

$$\begin{array}{rcl} (F+G)(x) & = & F(x)+G(x), \\ (\widetilde{F+G})(x) & = & \tilde{F}(x)+\tilde{G}(x), \\ Z_{F+G} & = & Z_F+Z_G. \end{array}$$

Examples.

1. If E_e is the species of even sets and E_o the species of odd sets, then we have

$$E = E_e + E_o.$$

The corresponding relations between the generating functions are:

$$\begin{array}{rcl} e^x & = & \mathrm{ch}(x)+sh(x), \\ \dfrac{1}{1-x} & = & \dfrac{1}{1-x^2}+\dfrac{x}{1-x^2}, \\ \exp\sum\limits_k \dfrac{x_k}{k} & = & \exp\sum\limits_k \dfrac{x_{2k}}{2k}[\mathrm{ch}(x_1+x_3/3+...)+\mathrm{sh}(x_1+x_3/3+...)]. \end{array}$$

2. (The canonical decomposition of a species). For any species F, set

$$F_n[U] = \begin{cases} F[U], & \text{if } |U|=n, \\ \emptyset, & \text{otherwise} \end{cases}.$$

Then

$$F = F_0+F_1+F_2+...$$

Definition. Let F and G be two combinatorial species. The *product of species* is defined as follows. For any finite set U

$$(F \cdot G)[U] = \sum_{V \subset U} F[V] \times G[U \setminus V],$$

sum being taken over all subsets of U, including the empty set and the set U itself. For a given bijection σ, the definition of the map $(F \cdot G)[\sigma]$ is natural.

Theorem. Each of the three generating series associated with the sum of species is equal to the product of the two corresponding series:

$$\begin{aligned}(F \cdot G)(x) &= F(x) \cdot G(x),\\ (\widetilde{F \cdot G})(x) &= \tilde{F}(x) \cdot \tilde{G}(x),\\ Z_{F \cdot G} &= Z_F \cdot Z_G.\end{aligned}$$

Examples.

1. $E^2 = E \cdot E$ is the species of subsets.

2. Consider the species of permutations S. The set on which a permutation acts, can be divided into two parts: the subset of fixed points and the subset where the permutation acts without fixed points. In the language of species, this fact can be written as

$$S = E \cdot \mathrm{Der},$$

where Der is the species of *derangements*, i.e. permutations without fixed points.

Problems.

1. Using the above combinatorial equations, find the expressions of the three generating functions for the species E^2 and Der.

2. Let E_+ be the species of non-empty sets (so that $E = 1 + E_+$). Explain the meaning of the species

$$\mathrm{Bal} = \sum_{k=0}^{\infty} (E_+)^k$$

and find its generating series.

3. Prove that the species of linear orders L and the species of singletons X satisfy the equations

$$L = 1 + XL = \sum_{k=0}^{\infty} X^k = \prod_{k=0}^{\infty} (1 + X^{2^k}).$$