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COMBINATORICS

Lecture 6: Composition of combinatorial species

Definition. Let F and G be two combinatorial species such that $G[\emptyset] = \emptyset$ (there are no G-structures on the empty set). The *composition of species* $F \circ G$, also denoted by $F(G)$, is defined as follows. For any finite set U

$$(F \circ G)[U] = \bigcup_{\pi \,\mathrm{Part}(U)} F[\pi] \times \prod_{p \in \pi} G[p],$$

where $\mathrm{Part}(U)$ is the set of all partitions of U, i.e representations of U in the form of a disjoint union of non-empty subsets $U = U_1 \cup ... \cup U_n$. For $\pi = \{U_1, ..., U_n\}$, $F[\pi]$ is the set of all F-structures on the partition (=on the set of subsets $U_1, ..., U_n$). Because of the role of partitions in the definition, this construction is also called *partitional composition.*

For a given bijection σ, the definition of the map $(F \circ G)[\sigma]$ is natural.

Unlike the two previously introduced operations on species, partitional composition is literally compatible with only one generating series $F(x)$. The following theorem holds.

Theorem. The generating series of the partitional composition are computed as follows:

$$\begin{aligned}
(F \circ G)(x) &= F(G(x)),\\
(\widetilde{F \circ G})(x) &= Z_F(\tilde{G}(x), \tilde{G}(x^2), \tilde{G}(x^3), ...\\
Z_{F \circ G}(x_1, x_2, ...) &= Z_F(Z_G(x_1, x_2, ...), Z_G(x_2, x_4, ...), ...).
\end{aligned}$$

The operation on formal series in an infinite number of variables (numbered by natural numbers) that appears in the formula for $Z_{F \circ G}$, is referred to as the *plethystic substitution* of Z_G into Z_F.

Examples.

1. Consider an endofunction, i.e. a mapping $\phi : U \to U$. The points of U can be divided into two types: recurrent points, such that $\phi^k(x) = x$ for some $k \geq 1$, and non-recurrent points. Recurrent points form several cycles, while non-recurrent ones form several trees flowing into these cycles. Therefore. an endofunction if a permutation of rooted trees, and we can write:

$$\mathrm{End} = S \circ R.$$

2. Every permutation is a set of disjoint cycles, therefore

$$S = E \circ C,$$

where E is the species of sets and C is the species of cyclic permutations.

Problems.

1. Find the relations between the generating series of the species End, S, R and S, E, C that follow from the previous combinatorial identities.

2. Prove the isomorphisms $\mathrm{Bal} = L(E_+)$, $\mathrm{Par} = E(E_+)$ and derive enumerative consequences thereof.