S.Duzhin

COMBINATORICS

Lecture 9: Virtual species

The set Spe of (equivalence classes of) all combinatorial species suplied with the operations of addition and multiplication forms a semiring. The only operation which is missing for this semiring to be a ring, is subtraction. The extension of the semiring Spe to a ring Virt can be done in a fashion quite similar to the classical construction of the ring of integers $\mathbb{Z}$ out of the semiring of natural numbers $\mathbb{N}$. The construction itself is easy enough, the difficult (and specifically combinatorial) part is the correct extension of all operations on combinatorial species from Spe to Virt.

Definition. A *virtual species* is an element of the quotient set

$$\mathrm{Virt} = (\mathrm{Spe} \times \mathrm{Spe})/\sim\ ,$$

where the equivalence relation $\sim$ is defined by

$$(F_1, G_1) \sim (F_2, G_2) \quad \Leftrightarrow F_1 + G_2 \cong F_2 + G_1.$$

The equivalence class of the pair (F, G) is then denoted by $F - G$.

Theorem. The relation $\sim$ thus introduced is an equivalence relation.

The theorem easily follows from the following lemma which looks trivial, but requires a rather unexpected trick to be proved.

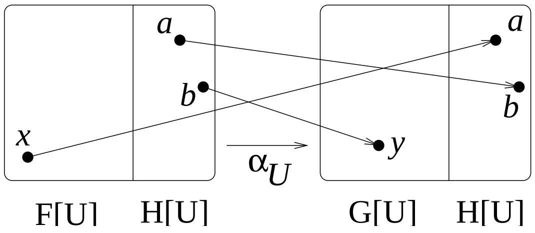
Lemma. Addition of combinatorial species satisfies the cancellation law: if $F + H \cong G + H$, then $F \cong G$.

Proof. By definition, the isomorphism $\alpha : F + H \to G + H$ is a family of one-to-one mappings $\alpha_U : (F+H)[U] \to (G+H)[U]$ which behaves naturally with respect to the finite set U. Given such a family, we must construct an isomorphism $\beta : F \to G$, i.e. a family of one-to-one mappings $\beta_U : F[U] \to G[U]$. The difficulty is that in general the mappings α_U do not have the property that $\alpha_U(F[U]) \subset G[U]$. A suitable definition of β is:

$$\beta_U(x) = \alpha_U^k(x),$$

where k is the smallest positive integer such that $\alpha_U^k(x) \in G[U]$. Note that the mapping α_U^k is not globally defined, but the formula does make sense! For example, in the following picture for the element $x \in F[U]$ we have: $\alpha_U(x) = a \in H[U]$, therefore the mapping α_U is defined on a and we set $\alpha_U(a) = b \in H[U]$ and finally $\alpha_U(b) = y \in G[U]$. Thus, in this example, $\beta_U(x) = y$.

Since the construction of β from α is natural, it is easy to check that a bijection $\sigma : U \to V$ will lead to a commutative diagram.



Definition. Addition and multiplication of virtual species are defined by the rules

$$\begin{aligned}(F_1 - G_1) + (F_2 - G_2) &= (F_1 + F_2) - (G_1 + G_2), \\ (F_1 - G_1) \cdot (F_2 - G_2) &= (F_1 \cdot F_2 + G_1 \cdot G_2) - (F_1 \cdot G_2 + F_2 \cdot G_1).\end{aligned}$$

Exercise. Prove that these operations turn the set Virt into a commutative ring with zero $0 = (0 - 0)$ and unity $1 = (1 - 0)$.

Definition. The generating series of the virtual species $\Phi = F - G$ are defined by

$$\begin{aligned}\Phi(x) &= F(x) - G(x), \\ \widetilde{\Phi}(x) &= \widetilde{F}(x) - \widetilde{G}(x), \\ Z_\Phi(x_1, x_2, ...) &= Z_F(x_1, x_2, ...) - Z_G(x_1, x_2, ...).\end{aligned}$$

Exercise. Prove that the right hand sides of the previous definition do not depend on the specific choice of the pair F, G that represent a given virtual species Φ.

Exercise. Prove that each of the three generating series is multiplicative with respect to the product of virtual species.

It turns out that among many pairs representing a virtual species it is possible to choose one canonical pair.

Definition. Let F and G be two species. We say that F is a subspecies of G, if

1. for every finite set U we have: $F[U] \subset G[U]$,
2. for very bijection $\sigma : U \to V$, we have: $F[\sigma] = G[\sigma]|_{F[U]}$.

Definition. Two species are said to be *unrelated*, if they do not contain isomorphic subspecies different from 0.

Theorem. Any virtual species Φ can be uniquely represented in the form

$$\Phi = \Phi^+ - \Phi^-,$$

where the (ordinary) species Φ^+ and Φ^- are unrelated.

A *proof* that relies on the construction of molecular decomposition, will be given later.

Definition. The species Φ^+ and Φ^- are called the *positive* and the *negative* part of Φ, respectively.

Example. The species C of oriented cycles and the species R of rooted trees are related: they contain a common subspecies of singletons X. It turns out that X is the maximal common subspecies of C and R, therefore the reduced form for Φ is

$$\Phi = (C - X) - (R - X),$$

where

$$\begin{aligned}\Phi^+ &= C - X = \text{ cycles on at least two points} \\ \Phi^- &= R - T = \text{ rooted trees on at least two points}\end{aligned}$$