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COMBINATORICS

Derivative of combinatorial species

The rule of differentiation of exponential generating series

$$\Big(\sum_{k=0}^{\infty} f_n \frac{x^n}{n!}\Big)' = \sum_{k=1}^{\infty} f_n \frac{x^{n-1}}{(n-1)!} = \sum_{k=0}^{\infty} f_{n+1} \frac{x^n}{n!}$$

suggests the following definition for the derivative of combinatorial species.

Definition. Let F be an arbitrary combinatorial species. Its *derivative* F' is defined as follows. For any finite set U we put

$$F'[U] = F[U \cup \{*\}],$$

where $\{*\}$ is a point that does not belong to the set U (people knowledgeable in set theory would take $* = \{U\}$). For a bijection $\sigma : U \to V$ we set

$$F'[\sigma] = F[s^+],$$

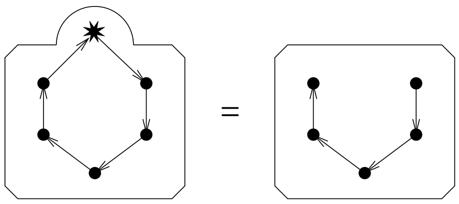
where $\sigma^+ : U \cup \{*\} \to V \cup \{*\}$ acts as σ on U and sends $*$ into $*$.

Examples.

1. The derivative of the species of cycle permutations is the speies of linear orders:

$$C' = L.$$

This fact is visualized by the following picture:



2. In a similar way, $L' = L^2$.

3. If Par is the species of partitions, then $\mathrm{Par}' = E \cdot \mathrm{Par}$.

Theorem. The generating series of the derivative is computed as follows:

$$\begin{aligned} F'(x) &= \frac{d}{dx} F(x), \\ \widetilde{F'}(x) &= \Big(\frac{\partial}{\partial x_1} Z_F\Big)(x, x^2, x^3, ...) \\ Z_{F'}(x_1, x_2, ...) &= \Big(\frac{\partial}{\partial x_1} Z_F\Big)(x_1, x_2, ...). \end{aligned}$$

The operation of differentiation is related to the other operations on combinatorial species in a variety of ways. Let us emphasize the following two identities (the Leibnitz rule and the chain rule for species):

$$\begin{aligned}(F \cdot G)' &= F' \cdot G + F \cdot G', \\ (F \circ G)' &= (F' \circ G) \cdot G'.\end{aligned}$$

Exercise. Let T be the species of trees, R the species of rooted trees and $F = E(R)$, the species of forests of rooted trees. Check that $T' = F$.