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COMBINATORICS OF SYMMETRIC GROUPS

Lecture 2: Symmetric functions

In this lecture we discuss different bases in the space of symmetric polynomials.

Degree k homogeneous polynomials in n integer variables form a free abelian group $S^k(\mathbb{Z}^n)$ of dimension $\binom{n+k-1}{k}$, in which the symmetric group S_n acts by permuting the indeterminates. For example, the group $S^3(\mathbb{Z}^3)$ is 10-dimensional, spanned by monomials

$$x_1^3, x_2^3, x_3^3, x_1^2x_2, x_1^2x_3, x_2^2x_1, x_2^2x_3, x_3^2x_1, x_3^2x_1, x_1x_2x_3,$$

and the group S_3 acts on the basic elements by permuting the subscripts. You may notice that the set of basic elements splits into 3 orbits that correspond to the partitions of the number 3: [2], [2,1] and [1^3].

Let Λ_n^k denote the subgroup of $S^k(\mathbb{Z})$ consisting of all the elements invariant with respect to this action:

$$\Lambda_n^k = \{f \in S^k(\mathbb{Z}^n) \mid \sigma(f) = f \ \forall \sigma \in S_n\}.$$

The dimensions of the spaces Λ_n^k for small k and n are given in the following table.

$n \backslash k$	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	2	2	3	3	4
3	1	2	3	4	5	7
4	1	2	3	5	6	9
5	1	2	3	5	7	10
6	1	2	3	5	7	11

Observation. $\dim \Lambda_n^k$ is equal to the number of partitions of k into no more than n parts. In particular, if $n \geq k$, then $\dim \Lambda_n^k = p(k)$ — the number of partitions of k.

The reason why the numbers $\dim \Lambda_n^k$ stabilize as $n \to \infty$ lies in the following fact. *The homomorphisms* $\rho_{n,m}\colon \Lambda_n^k \to \Lambda_m^k$ *defined by* $f(x_1,\dots,x_n) \mapsto f(x_1,\dots,x_n,0,\dots,0)$ *are isomorphisms for* $n \geq m \geq k$. Therefore, we can correctly define the space of degree k homogeneous symmetric polynomials in an infinite number of variables as the inverse limit with respect to the family $\{\rho_{n,m}\}$:

$$\Lambda^k = \varprojlim \Lambda_n^k$$

and the ring of symmetric functions

$$\Lambda = \bigoplus_{k \geq 0} \Lambda^k.$$

Remark. $\Lambda \neq \varprojlim \Lambda_n$, because, according to our definition, Λ does not contain, e.g. the infinite product $\prod_{n \geq 1}(1 + x_n)$.

We will list several important bases in Λ^k (visualizable in Λ_n^k for any $n \geq k$) numbered by partitions of k.

1. *Monomial symmetric polynomials.* This is the most natural basis: its elements are the sums of monomials belonging to one orbits of S_n, each with coefficient 1. More exactly, if $\lambda \vdash k$ (λ is a partition of k) and $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multiindex, then we say that α is *of type* λ and write $\tau(\alpha) = \lambda$, if all nonzero components α_i, arranged in nonincreasing order, form the partition λ. The monomial function corresponding to λ is
$$m_\lambda = \sum_{\tau(\alpha)=\lambda} x_1^{\alpha_1} \cdot \ldots \cdot x_n^{\alpha_n}.$$
Example.
$$\begin{aligned} m_3 &= x_1^3 + x_2^3 + x_3^3, \\ m_{21} &= x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_1 + x_2^2 x_3 + x_3^2 x_1 + x_3^2 x_1, \\ m_{111} &= x_1 x_2 x_3. \end{aligned}$$

2. *Elementary symmetric polynomials.* If $\lambda = [\lambda_1, \ldots, \lambda_n]$, then we put $e_\lambda = e_{\lambda_1} \cdot \ldots \cdot e_{\lambda_n}$, where e_m with a scalar subscript has the usual meaning:
$$e_l = \sum_{i_1 < \ldots i_l} x_{i_1} \cdot \ldots \cdot x_{i_l}.$$
Example.
$$\begin{aligned} e_3 &= x_1 x_2 x_3, \\ e_{21} &= (x_1 x_2 + x_1 x_3 + x_2 x_3)(x_1 + x_2 + x_3), \\ e_{111} &= (x_1 + x_2 + x_3)^3, \end{aligned}$$

3. *Complete symmetric polynomials.* Like in the case of elementary symmetric polynomials, we put $h_\lambda = h_{\lambda_1} \cdot \ldots \cdot h_{\lambda_n}$, if $\lambda = [\lambda_1, \ldots, \lambda_n]$. The complete symmetric polynomial with a scalar subscript is defined as the sum of all monomials of a given degree:
$$h_l = \sum_{|\alpha|=l} x_1^{\alpha_1} \cdot \ldots \cdot x_n^{\alpha_n}.$$
Example.
$$\begin{aligned} h_3 &= x_1^3 + x_2^3 + x_3^3 + x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_1 + x_2^2 x_3 + x_3^2 x_1 + x_3^2 x_1 + x_1 x_2 x_3, \\ h_{21} &= (x_1^2 + x_2^2 + x_3^2 + x_1 x_2 + x_1 x_3 + x_2 x_3)(x_1 + x_2 + x_3), \\ h_{111} &= (x_1 + x_2 + x_3)^3, \end{aligned}$$

4. *Power sum symmetric polynomials.* Like before, we set $p_\lambda = p_{\lambda_1} \cdot \ldots \cdot p_{\lambda_n}$ for a partition $\lambda = [\lambda_1, \ldots, \lambda_n]$ and
$$p_l = \sum_{i=1}^{n} x_i^l$$

if the subscript is a single number.

Example.

$$\begin{aligned} p_3 &= x_1^3 + x_2^3 + x_3^3, \\ p_{21} &= (x_1^2 + x_2^2 + x_3^2)(x_1 + x_2 + x_3), \\ p_{111} &= (x_1 + x_2 + x_3)^3, \end{aligned}$$

It is convenient to set $m_0 = e_0 = h_0 = p_0 = 1$ and $m_l = e_l = h_l = p_l = 0$ for $l < 0$.

As a starter, we compute the transition matrices between these four bases in the space Λ_3^3 and give the result as a table which, on the intersection of a column labelled u and a row labelled v, contains the matrix $M_{u,v}$ such that $u = M_{u,v}v$, where u and v are two different bases written as columns in the reverse lexicographical order of partitions (for example, [3]>[2,1]>[1,1,1]).

	m	e	h	p
m	1	$\begin{pmatrix} 3 & -3 & 1 \\ -3 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 3 & -3 & 1 \\ -3 & 5 & -2 \\ 1 & -2 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1/3 & -1/2 & 1/6 \end{pmatrix}$
e	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 3 \\ 1 & 3 & 6 \end{pmatrix}$	1	$\begin{pmatrix} 1 & -2 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1/3 & -1/2 & 1/6 \\ 0 & -1/2 & 1/2 \\ 0 & 0 & 1 \end{pmatrix}$
h	$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{pmatrix}$	$\begin{pmatrix} 1 & -2 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$	1	$\begin{pmatrix} 1/3 & 1/2 & 1/6 \\ 0 & 1/2 & 1/2 \\ 0 & 0 & 1 \end{pmatrix}$
p	$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 3 & 6 \end{pmatrix}$	$\begin{pmatrix} 3 & -3 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 3 & -3 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{pmatrix}$	1

(Note that each matrix can be naturally viewed as an integer or rational function of a pair of partitions of the same weight k.)

Looking at this table, you may notice several interesting laws which turn out to be true for any k, not only for $k = 3$. We pass to the exact formulations.

Theorem 1. Let λ' denote the partition, conjugate to λ (i.e. the rows of λ' are columns of λ and vice versa or, in other words, the Young diagrams of λ and λ' are symmetric to each other with respect to the diagonal). Then

$$e_{\lambda'} = m_\lambda + \sum_{\mu<\lambda} a_{\lambda\mu} m_\mu$$

where $a_{\lambda\mu}$ are integer coefficients.

Proof. Let $\lambda = [\lambda_1, \lambda_2, \dots]$ and $\lambda' = [l_1, l_2, \dots]$. Then the highest term in $e_{\lambda'}$ is

$$(x_1 x_2 \dots x_{l_1})(x_1 x_2 \dots x_{l_2}) \dots = x_1^{\lambda_1} x_2^{\lambda_2} \dots$$

Corollary. The functions e_λ form a basis of the space Λ over $\mathbb{Z}$.

Theorem 2. The functions e_i and h_j are related by the equations

$$\sum_{i=0}^{n}(-1)^i e_i h_{n-i} = 0$$

for all $n \geq 1$.

Proof. The generating functions of the sequences e_r and h_r satisfy:

$$\begin{aligned} E(t) &= \sum_{r\geq 0} e_r t^r = \prod_{i\geq 1}(1 + x_i t), \\ H(t) &= \sum_{r\geq 0} h_r t^r = \prod_{i\geq 1}(1 - x_i t)^{-1}, \end{aligned}$$

whence we have

$$H(t)E(-t) = 1.$$

Corollary (1) The functions h_λ form a basis of the space Λ over $\mathbb{Z}$. (2) The automorphism $\omega\colon \Lambda \to \Lambda$ defined by $e_i \mapsto h_i$, is involutive. In coordinates, this means that the two transtition matrices are equal: $M_{e,h} = M_{h,e}$.

Theorem 3 (Newton's formulas)

$$\begin{aligned} nh_n &= \sum_{i=1}^{n} p_i h_{n-i}, \\ ne_n &= \sum_{i=1}^{n}(-1)^{i-1} p_i e_{n-i}. \end{aligned}$$

Proof. Set $P(t) = \sum_{r\geq 1} p_r t^{r-1}$. Then $P(t) = H'(t)/H(t)$ and $P(-t) = E'(t)/E(t)$.

Corollary. The functions p_λ form a basis of $\Lambda \otimes \mathbb{Q}$.