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COMBINATORICS OF SYMMETRIC GROUPS

Lecture 3: Schur functions

Schur polynomials s_λ, also called S-functions, constitute one more basis in the space of symmetric functions, in many respects even more remarkable that the bases e, h and p studied in the previous lecture. The direct definition of s_λ is given in terms of skew-symmetric functions, about which we will say a few general words.

Consider once again the action of the permutation group S_n in the space $S^k(\mathbb{Z}^n)$ of degree k homogeneous polynomials in n integer variables. Let A_n^k denote the subgroup of antisymmetric elements with respect to this action:

$$A_n^k = \{f \in S^k(\mathbb{Z}^n) \mid \sigma(f) = \mathrm{sgn}(\sigma) f \ \forall \sigma \in S_n\},$$

where $\mathrm{sgn}(\sigma)$ is the sign of the permutation σ (1 or -1, depending on whether σ is even or odd).

The dimensions of the spaces A_n^k for small k and n are given in the following table.

$n\backslash k$	1	2	3	4	5	6	7	8	9	10	11	12
1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	1	2	2	3	3	4	4	5	5	6	6
3	0	0	1	1	2	3	4	5	7	8	10	12
4	0	0	0	0	0	1	1	2	3	5	6	9
5	0	0	0	0	0	0	0	0	0	1	1	2

Note that, unlike the case of symmetric functions, the dimensions of A_n^k do not stabilize, so there is no sensible notion of an antisymmetric function in an infinite number of variables. The study of A_n^k can, however, be reduced to the study of Λ^k be means of the following observations:

- The product of a symmetric and antisymmetric functions gives an antisymmetric function: $\Lambda_n^k \otimes A_n^l \to A_n^{k+l}$.
- Any antisymmetric polynomial in n variables is a multiple, in the polynomial ring, of the "minimal" antisymmetric polynomial $\Delta_n = \prod_{1\le i<j\le n}(x_i - x_j)$.

It follows that the mapping $\Lambda_n \to A_n \colon f \mapsto \Delta_n f$ gives a module isomorphism between Λ_n and A_n. This isomorphism is homogeneous, increasing the degree by $\deg \Delta_n = n(n-1)/2$, so that $\Lambda_n^k \cong A_n^{k+n(n-1)/2}$.

Below, we give several definitions of the Schur functions. The proof of their equivalence is not discussed.

First (algebraic) definition of Schur polynomials.

To any sequence $\lambda = (\lambda_1, \dots, \lambda_n)$ of length n there corresponds an antisymmetric polynomial

$$a_\lambda = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma)\, x_1^{\lambda_{\sigma(1)}} \dots x_n^{\lambda_{\sigma(n)}} = \det[x_i^{\lambda_j}]_{1\le i,j\le n}.$$

This expression is nonzero if and only if all numbers λ_j are different. In this case, up to a sign, we can assume that $\lambda_1 > \lambda_2 > \dots > \lambda_n$. We thus see that the polynomials a_λ, where λ are partitions of k into no more than n different parts, form a basis of A_n^k. Note that a partition into different parts can be represented as the sum of an arbitrary partition and the minimal partition into different parts $\delta = [n-1, n-2, \dots, 1, 0]$.

The Schur polynomial in n variables indexed by the partition λ is defined by

$$s_\lambda^{(n)} = \frac{a_{\lambda+\delta}^{(n)}}{a_\delta^{(n)}} = \frac{\det[x_i^{\lambda_j+n-j}]_{1\le i,j\le n}}{\det[x_i^{n-j}]_{1\le i,j\le n}} = \frac{\begin{vmatrix} x_1^{\lambda_1+n-1} & x_2^{\lambda_1+n-1} & \dots & x_n^{\lambda_1+n-1} \\ x_1^{\lambda_2+n-2} & x_2^{\lambda_2+n-2} & \dots & x_n^{\lambda_2+n-2} \\ \vdots & \vdots & & \vdots \\ x_1^{\lambda_n} & x_2^{\lambda_n} & \dots & x_n^{\lambda_n} \end{vmatrix}}{\begin{vmatrix} x_1^{n-1} & x_2^{n-1} & \dots & x_n^{n-1} \\ \vdots & \vdots & & \vdots \\ x_1 & x_2 & \dots & x_n \\ 1 & 1 & \dots & 1 \end{vmatrix}} \tag{1}$$

Note that $a_\delta^{(n)} = \Delta_n = \prod_{1\le i<j\le n}(x_i - x_j)$.

It is important that, although the numerator and the denominator in (1) depend on n and make no sense as $n \to \infty$, the ratio, $s_\lambda^{(n)}$, does define an element of the algebra Λ, because the family $\{s_\lambda^{(n)}\}_{n\ge k}$ agrees with the homomorphisms $\rho_{n.m} : \Lambda_n \to \Lambda_n$ (see Lecture 2):

$$\rho_{n+1,n}(s_\lambda^{(n+1)}) = s_\lambda^{(n)},$$

if $n \ge k$. Indeed, since $|\lambda| = k$, we have $\lambda_{n+1} = 0$, hence

$$s_\lambda^{(n+1)} = \frac{\begin{vmatrix} x_1^{\lambda_1+n} & x_2^{\lambda_1+n} & \dots & x_n^{\lambda_1+n} & x_{n+1}^{\lambda_1+n} \\ x_1^{\lambda_2+n-1} & x_2^{\lambda_2+n-1} & \dots & x_n^{\lambda_2+n-1} & x_{n+1}^{\lambda_2+n-1} \\ \vdots & \vdots & & \vdots & \vdots \\ x_1^{\lambda_n+1} & x_2^{\lambda_n+1} & \dots & x_n^{\lambda_n+1} & x_{n+1}^{\lambda_n+1} \\ 1 & 1 & \dots & 1 & 1 \end{vmatrix}}{\begin{vmatrix} x_1^{n} & x_2^{n} & \dots & x_n^{n} & x_{n+1}^{n} \\ x_1^{n-1} & x_2^{n-1} & \dots & x_n^{n-1} & x_{n+1}^{n-1} \\ \vdots & \vdots & & \vdots & \vdots \\ x_1 & x_2 & \dots & x_n & x_{n+1} \\ 1 & 1 & \dots & 1 & 1 \end{vmatrix}}$$

The homomorphism $\rho_{n+1,n}$ consists in setting x_{n+1} to 0. After this, the determinants expand over the last column and the ratio simplifies to $s_\lambda^{(n)}$.

Examples. According to the previous analysis, to compute the polynomial s_λ, it is enough to take $|\lambda|$ indeterminates. (It might seem that $l(\lambda)$ variables are also sufficient, but this is not true in general, because the mapping $\Lambda_k^{n+1} \to \Lambda_k^n$ is guaranteed to be an isomorphism only if $n \ge k$.)

$$s_{1,1,1} = \frac{\begin{vmatrix} x_1^3 & x_2^3 & x_3^3 \\ x_1^2 & x_2^2 & x_3^2 \\ x_1 & x_2 & x_3 \end{vmatrix}}{\begin{vmatrix} x_1^2 & x_2^2 & x_3^2 \\ x_1 & x_2 & x_3 \\ 1 & 1 & 1 \end{vmatrix}} = x_1 x_2 x_3 = e_3 = m_{1,1,1},$$

$$s_{2,1} = \frac{\begin{vmatrix} x_1^4 & x_2^4 & x_3^4 \\ x_1^2 & x_2^2 & x_3^2 \\ 1 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} x_1^2 & x_2^2 & x_3^2 \\ x_1 & x_2 & x_3 \\ 1 & 1 & 1 \end{vmatrix}} = (x_1 + x_2)(x_1 + x_3)(x_2 + x_3) = m_{2,1} + 2m_{1,1,1},$$

$$s_3 = \frac{\begin{vmatrix} x_1^5 & x_2^5 & x_3^5 \\ x_1 & x_2 & x_3 \\ 1 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} x_1^2 & x_2^2 & x_3^2 \\ x_1 & x_2 & x_3 \\ 1 & 1 & 1 \end{vmatrix}} = h_{1,1,1} = m_3 + m_{2,1} + m_{1,1,1}.$$

Second (geometric) definition of Schur polynomials.

In the ring of symmetric functions Λ there is a remarkable inner product, defined in terms of duality as follows.

Definition. Let $\{u_\lambda\}$ and $\{v_\lambda\}$ be two bases in $\Lambda \otimes \mathbb{Q}$ numbered by partitions. Then $\{u_\lambda\}$ is dual to $\{v_\lambda\}$ if and only if

$$\prod_{i,j} (1 - x_i y_j)^{-1} = \sum_\lambda u_\lambda(x) v_\lambda(y).$$

Theorem.

- Bases h and m are dual: $\langle h_\lambda, m_\mu \rangle = \delta_{\lambda,\mu}$.
- Basis p is orthogonal: $\langle p_\lambda, p_\mu \rangle = \delta_{\lambda.\mu} \|\lambda\|$, where $\|\lambda\| = \prod_i i^{b_i} b_i!$, if b_i is the number of parts in λ that are equal to i.
- Basis s is orthonormal: $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda.\mu}$.

Proof. The assertion follows from the identities:

- $$\prod_{i,j} (1 - x_i y_j)^{-1} = \sum_\lambda h_\lambda(x) m_\lambda(y).$$
- $$\prod_{i,j} (1 - x_i y_j)^{-1} = \sum_\lambda \|\lambda\|^{-1} p_\lambda(x) p_\lambda(y).$$

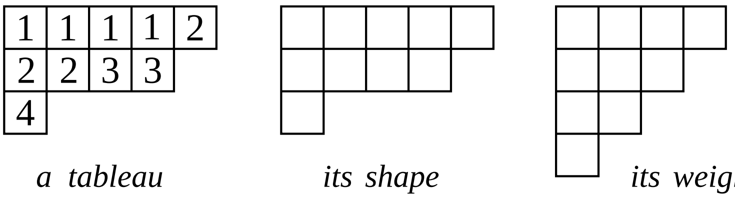
•
$$\prod_{i,j}(1-x_iy_j)^{-1}=\sum_\lambda s_\lambda(x)s_\lambda(y).$$

An orthonormal basis over $\mathbb{Z}$ is determined uniquely, up to a permutation and a change of signs. Therefore, this property of Schur functions can be considered almost as a definition.

Third (combinatorial) definition of Schur polynomials.

Definition. Let λ and μ be two partitions. A *tableau* of shape λ and weight μ is an arrangement of μ_1 numbers 1, μ_2 numbers 2, etc. over the cells of the Young diagram $[\lambda]$ in such a way that the numbers are increasing in each column and nondecreasing in each row.

Example.



a tableau *its shape* *its weight*

Definition. For a tableau T, we denote its shape by $\varphi(T)$ and its weight by $w(T)$. Set
$$s_\lambda=\sum_{\{T|\varphi(T)=\lambda\}}x^{w(T)}$$
(the sum being taken over *all* tableaux of shape λ). Here $x^{w(T)}$ is the monomial with exponent $w(T)$.

Example. To us compute $s_{2,1}$ it is enough to take 3 variables, x_1, x_2 and x_3. There are 8 tableaux of shape $[2,1]$ filled with numbers 1, 2 and 3:

$$\begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array}\quad\begin{array}{|c|c|}\hline 1&1\\\hline 3\\\cline{1-1}\end{array}\quad\begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array}\quad\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}$$

$$\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}\quad\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}\quad\begin{array}{|c|c|}\hline 2&2\\\hline 3\\\cline{1-1}\end{array}\quad\begin{array}{|c|c|}\hline 2&3\\\hline 3\\\cline{1-1}\end{array}$$

They give 8 monomials that sum up to $m_{2,1}+2m_{1,1,1}$.

Expression of s through h and e.

There are determinantal expressions of Schur polynomials through (a) complete symmetric functions h_n and (b) elementary symmetric functions e_n,

Theorem.

1.
$$s_\lambda=\det[h_{\lambda_i-i+j}]_{1\le i,j\le n},\quad\text{where}\quad n\ge l(\lambda),$$

2.
$$s_\lambda = \det[e_{\lambda'_i - i+j}]_{1\le i,j\le m}, \quad \text{where} \quad m \ge l(\lambda'),$$

(Here $h_0 = e_0 = 1$ and $h_i = e_i = 0$ for $i < 0$.)
The theorem says that s_λ is equal to the minor of the infinite matrix

$$\begin{pmatrix} h_0 & h_1 & h_2 & h_3 & \dots \\ 0 & h_0 & h_1 & h_2 & \dots \\ 0 & 0 & h_0 & h_1 & \dots \\ 0 & 0 & 0 & h_0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

formed by the first n rows and columns number λ_1, λ_2, etc. (similarly for e_i).

Transition matrices.

We now have five different bases in the linear space of symmetric functions: m, e, h, p and s. In Lecture 2, the transition matrices between the first four bases were computed for degree 3 symmetric polynomials. It turns out that the most fundamental transition matrix (for all bases except for p) is $K = M_{s.m}$. For one thing, its elements have a transparent combinatorial meaning. For another, all the other transition matrices have a simple expression through K.

Theorem. If $|\lambda| = k$, then

$$s_\lambda = \sum_{|\mu|=k} K_{\lambda,\mu} m_\mu,$$

where $K_{\lambda,\mu}$ is the *Kostka number* defined as *the number of tableaux of shape λ and weight μ.*

Proof. The theorem follows immediately from the combinatorial definition of the Schur functions.

Example.

$$\begin{pmatrix} s_3 \\ s_{2,1} \\ s_{1,1,1} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} m_3 \\ m_{2,1} \\ m_{1,1,1} \end{pmatrix}$$

Theorem. Transition matrices $M_{u,v}$ for bases $u, v = m, e, h, s$ are expressed through the Kostka matrix K, its inverse K^{-1}, its tranposed K' and the inverse of the transposed K^* as follows:

	m	e	h	s
m	1	$K^{-1}JK^*$	$K^{-1}K^*$	K^{-1}
e	$K'JK$	1	$K'JK^*$	$K'J$
h	$K'K$	$K'JK^*$	1	K'
s	K	JK^*	K^*	1

where J is the matrix with 1 on the auxiliary diagonal and 0 elsewhere.

Proof. The theorem follow from the properties of the scalar product $\langle\, , \rangle$, involution ω and the fact that ω is an isometry.

The transition matrix between p and s will be discussed in the next lecture.