On numbers, formulas and figures, or: What do mathematicians do in the solitude of their studies?

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I will talk about my understanding of mathematics and say a few words about the daily work of a typical researcher in pure mathematics.

1 Mathematical riots

In the common opinion mathematics is something very boring. The reason is that by its nature mathematics deals with abstract notions and one normally has to have a certain level of special training to get excited by the beauty of mathematical facts. Yet, there have been quite a few occasions when news about mathematical discoveries reached the general ear and even produced public disturbances.

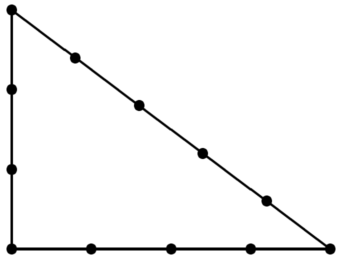
A recent example of this occurred in June 1993 when Andrew Wiles announced that he proved Fermat's Last Theorem.

This theorem is famous, because its statement is elementary, but no one could prove it during 360 years now. The statement is:

> If n is an integer number greater than 2, then there are no three positive integers x, y and z such that
>
> $$x^n + y^n = z^n.$$
>
> In other words, there are no integer numbers such that $x^3 + y^3 = z^3$, or $x^4 + y^4 = z^4$, or $x^5 + y^5 = z^5$, ...

This was written by the French mathematician Fermat in the margin of a mathematical book he was reading sometime around 1635. He said he found a marvellous proof of this fact, but there was not enough space to write it down in the margin.

Note that if $n = 2$, then there are numbers x, y and z such that $x^2 + y^2 = z^2$, the simplest example being $3^2 + 4^2 = 5^2$. By the way, this equality means that the triangle with sides 3, 4, 5 has a right angle — a fact used by ancient Egyptians when they measured their fields after floods coming from Nile river.



Egyptian rope

Fermat announced his theorem in 17th century, but did not publish any proof — probably, because he found an error in his argument. In 18th century all mathematicians tried to prove it, in 19th century many mathematicians tried to prove it, and in 20th century only some enthusiasts are continuing these efforts. From time to time there appear announcements that somebody has finally proved it, but then either the author himself or somebody else always found an error or a gap in the proof.

One such story happened in 1988. I remember when, during a meeting of Gelfand's seminar in Moscow University, it was announced that Japanese mathematician Yoichi Miyaoka proved Fermat's theorem and sent letters about that to many mathematicians all over the world. At the next meeting of that seminar, after 1 week, Gelfand said he received another letter from Miyaoka with a refutation of his proof.

Now let us return to the events of last summer. In a series of 3 lectures at Cambridge University, Andrew Wiles from Princeton told about his proof of the Taniyama-Shimura conjecture which implies Fermat's Last Theorem.

When the news reached America, student riots began in the streets and squares of some university cities. Here is what "Chicago Tribune" wrote on June 29, 1993:

> ...there was rioting and vandalism last week during the celebration. A few bookstores had windows smashed and shelves stripped, and vacant lots glowed with burning piles of old dissertations.
>
> ..."Math hooligans are the worst," said a Chicago Police Department spokesman. "But the city learned from the Bieberbach riots. We were ready for them this time."

You see that American policemen got to know some mathematics! "Bieberbach" riots occurred in 1984 after Louis DeBranges finally proved the Bieberbach

Conjecture from complex calculus. Eight years earlier, there was a similar incident related to the solution of the famous Four-Color Problem by Haken and Appel. At that time, mounted police throughout Hyde Park had to keep crowds of university students from tipping over cars on the midway.

Experts consider Wiles' effort to be the most serious attempt at Fermat's theorem. But mathematical riots did not help much: in December 1993 Andrew Wiles posted an e-mail message informing the world that he found a gap in his proof which he could not yet fill. The story of Fermat's last theorem continues!

2 Mathematics: duality of reason and vision

Japanese word "suugaku" literally means science about numbers. Some of the well-known Western encyclopaedias give nearly the same definition of mathematics, although the word for mathematics in all European languages comes from the ancient Greek verb meaning "to learn" — a very general understanding indeed!

Traditionally, since the time when mathematics separated itself from other sciences, it consisted of two major parts: arithmetic (the study of numbers) and geometry (the study of figures).

As time went on, new notions have been introduced and new subjects appeared within mathematics. Nevertheless, the fundamental duality between formula and figure, between algebraic manipulations and geometric imagination still constitutes the heart of mathematics.

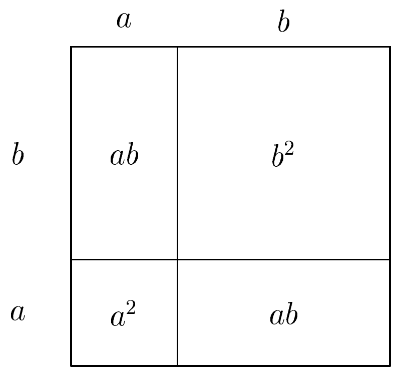
Mathematical thinking about spacial forms always includes rigorous (algebraic) reasoning. On the other hand, one can acquire intuition necessary to work with abstract algebraic objects, only through visual concepts of some kind.

$$\begin{tikzcd} \boxed{\text{FORMULA}} \arrow[r, shift right, "\text{vision}"'] & \boxed{\text{FIGURE}} \arrow[l, shift right, "\text{reason}"'] \end{tikzcd}$$

Interplay between algebra and geometry

Below are some examples of how geometric ideas can be applied to algebra and vice versa, algebraic ideas applied to geometry.

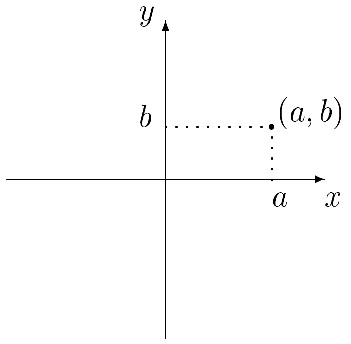
- Geometric proof of the formula $(a+b)^2 = a^2 + 2ab + b^2$:



This argument goes back to ancient Greeks and is typical for a branch of Greek mathematics which was called *geometric algebra*.

- Cartesian coordinates.

  The idea is to associate a point in the plane with a pair of numbers (its coordinates) according to the picture:



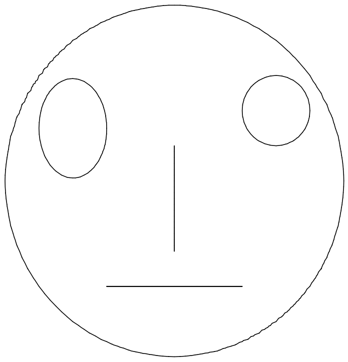
  Using this trick, one can rewrite any geometrical theorem as an algebraic equation and prove it by routine manipulations with formulas.

Applying the same idea in the opposite direction, one can translate algebraic equations into the language of geometrical figures.

For example, equation

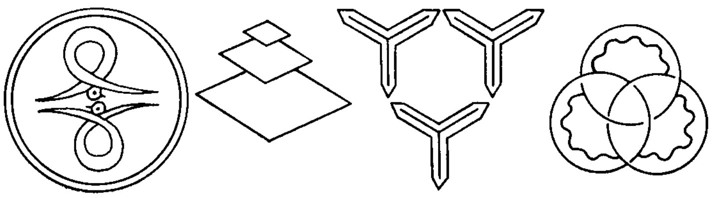
$$\left(x^2+y^2-25\right)\left(|x-2|+|x+2|-4+(y+3)^2\right)\left(x^2+|y-1|+|y+2|-3\right)$$
$$\left((x+3)^2+\frac{(y-3/2)^2}{2}-1\right)\left((x-3)^2+(y-2)^2-1\right)=0$$

defines the following curve:



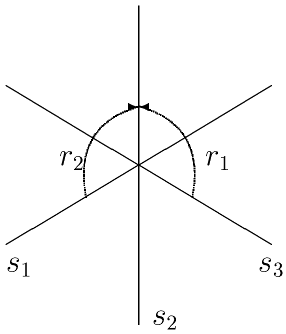
3 Measuring symmetry

In this picture, you see the family insignias of four different ancient Japanese families:



Japanese mons

From a mathematical point of view, they are interesting primarily because of different types of symmetry they have. For example, the third "mon" in the picture above has the group of symmetry consisting of 6 transformations: identical transformation e, two rotations r_1 and r_2 and three reflections s_1, s_2 and s_3:



If you perform two symmetry transformations one after the other, you obtain a new symmetry transformation according to the following "multiplication table":

	e	r_1	r_2	s_1	s_2	s_3
e	e	r_1	r_2	s_1	s_2	s_3
r_1	r_1	r_2	e	s_3	s_1	s_2
r_2	r_2	e	r_1	s_2	s_3	s_1
s_1	s_1	s_2	s_3	e	r_1	r_2
s_2	s_2	s_3	s_1	r_2	e	r_1
s_3	s_3	s_1	s_2	r_1	r_2	e

Algebraic properties of this "multiplication" are different from those of the ordinary numbers, for example, the commutative law, $xy = yx$, is not valid.

4 What do mathematicians usually do?

A short answer to this question is: they compute. And this is true indeed, if correct meaning is given to the word "compute".

In the old times, doing computations meant doing calculations with numbers. There were famous mathematicians who were known to be very strong in calculations; one of them was Leonard Euler ("Euler died and stopped computing", was the official announcement about his death).

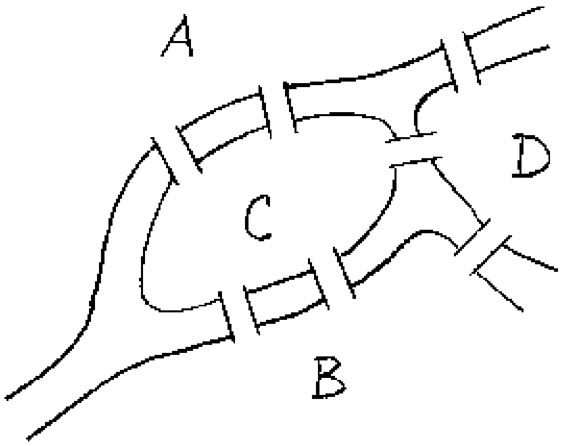
Here is the story of one of his famous computations.

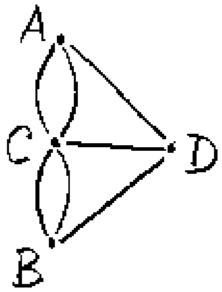
In the theory of geometric construction and elsewhere, the numbers $2^{2^1}+1 = 5$, $2^{2^2}+1 = 17$, $2^{2^3}+1 = 257$, $2^{2^4}+1 = 65537$, $2^{2^5}+1 = 4294967297$, etc. are very important, if they are *prime*, i.e. cannot be represented as a product of two integer numbers. (For example, 6 is not prime, because $6 = 2 \cdot 3$.)

In 17th century Pierre Fermat (the author of Fermat's theorem!) noticed that the first four numbers (5, 17, 257, 65537) are prime and conjectured that all other numbers in the series are also prime. This conjecture stood for about 100 years, until Euler found — *by a direct computation* — that the fifth number, 4294967297, is divisible by 641.

Another well-known thing related with the name of Euler is his problem about Königsberg Bridges: is it possible to walk across the map below so that to cross every bridge exactly once?

This picture shows both the problem and the key to its solution:

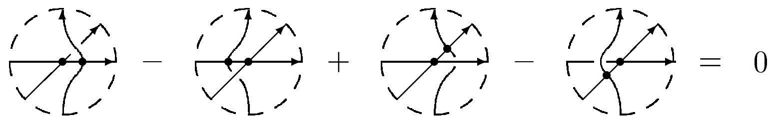




Euler's problem, simple as it is, gives a good idea of a typical thread of mathematical reasoning: first you reformulate the problem using a suitable language — in this case, the language of graph theory — and then find the important characteristics of the object thus obtained — here, the degrees of vertices — that enable you to prove the result.

This kind of reasoning is typical for algebraic topology, a branch of mathematics which began from Königsberg Bridges and nowadays is a field of intense research activity.

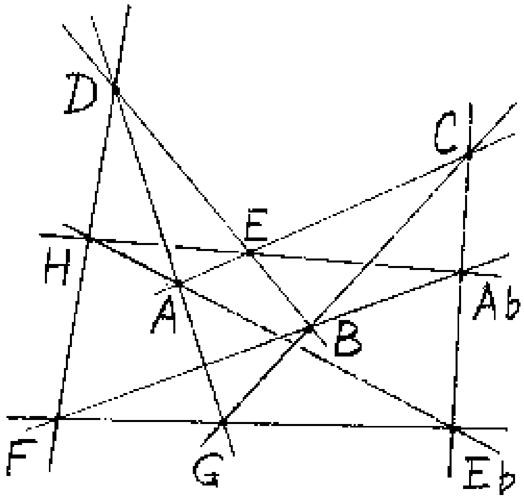
I cannot talk much about modern topology, so let me just show you one typical equation of the kind you can find in contemporary papers on knot invariants:

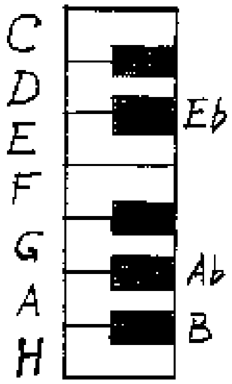


Mathematicians are no more satisfied with formulas made from numbers and letters, they put pictures there!

5 Listening to geometry

In the picture below you can see 10 lines and 10 points arranged in such a way that there are exactly 3 points on each line and exactly 3 lines passing through each point. This is the famous Desargues' configuration.





In 1991, a post-graduate student of Moscow University Alexei Zubov and myself taught this construction to the participants of a summer computer camp in Pereslavl-Zalessky, Russia.

In the evenings, we liked to play music together with our students, and the spontaneous atmosphere of a summer camp led us to the idea of writing music related to Desargues' configuration.

To each of the 10 points, we have assigned one of the 10 musical notes that we picked from the complete set of 12 notes. Every line binds three of these notes into a triad chord. In this way, we obtain 10 different chords. Of course, if you assign notes to the points randomly, you will most probably obtain some completely awful triads, like consisting of two half-tones, — so we wrote a computer program that searched for a satisfactory distribution of notes in the vertices of Desargues' configuration.

After we found a satisfactory distribution, we decided to write a piece of music based on the Desargues' harmony, i.e. on the set of 10 chords thus obtained. We did not restrict ourselves in the choice of melody, measure and instrumentation, the only thing that mattered was that the music should be consistent with the prescribed harmony: it had to use all of the 10 Desargues chords and nothing more.

Now, you can judge the result of this strange endeavour: let us listen to a piece of "mathematical" music.