Chapter 8

Answers, Hints and Solutions to Exercises

1. Example: triangle with vertices $(0, 0)$, $(12, 9)$, $(24, -7)$.

2. The problem can be solved either by a direct construction or using the result of exercise 6. Answer: polygon Φ may have 3, 4, 5 or 6 vertices.

3. A necessary and sufficient condition is $k - l = 1$. To prove this, rewrite the given expression as $A_1 + A_2 - B_1 + A_3 - B_2 + \cdots$.

4. Problem 2 shows that the given assertion is equivalent to *Euler's theorem* which says that $\overrightarrow{MH} = 2\overrightarrow{OM}$ where M is the median intersection point, O is the outcentre (the centre of the circumscribed circle) and H is the orthocentre (the intersection point of the altitudes) of the triangle. To prove this theorem, construct the triangle $A_1B_1C_1$ which is twice bigger than ABC and with sides parallel to those of ABC (see Fig. 8.1) and remark that the perpendicular bisectors of triangle ABC coincide with the altitudes of triangle $A_1B_1C_1$.

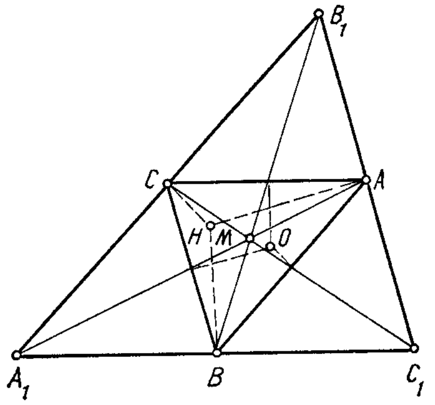

Figure 8.1: Euler's theorem

5. The necessary and sufficient condition is $\alpha+\beta+\cdots+\omega=1$. The proof follows from vector equality $\alpha\overrightarrow{PA}+\beta\overrightarrow{PB}+\cdots+\omega\overrightarrow{PZ}=\alpha\overrightarrow{QA}+\beta\overrightarrow{QB}+\cdots+\omega\overrightarrow{QZ}+(\alpha+\beta+\cdots+\omega)\overrightarrow{PQ}$.

6. The set of all points $a_1A_1+a_2A_2+\cdots+a_nA_n$ where all coefficients a_i are nonnegative and $a_1+a_2+\cdots+a_n=1$.

7. Express all the points under consideration through the four vertices of the quadrangle.

8. Express these points through the vertices of the hexagon.

9. Express all the points under consideration through the four vertices of the quadrangle.

10. Take the intersection point of the two medians for the pole and use the result of Problem 2.

11. Answer: the diagonal is divided in the ration 1:6. Hint: take one vertex of the parallelogram for the pole, two others for the basic points and express the intersection point of the given line and the diagonal in two different ways.

12. (a) $y=b$, (b) $x=a$, (c) $ay=bx$.

13. The union of the three medians.

14. Let K be the pole and A, B, the basic points. Express the coordinates of the points D, E, F through the coordinates (a,b) of the point C.

15. See the solution to the previous exercise.

16. To both questions the answer is negative.

17. Expand the point $K(0,1)$ through the basis (E,A) (see Definition 3) and prove that $K^2=-E$. Therefore, this multiplication coincides with the multiplication of complex numbers (see p. 24). Using the trigonometric representation of complex numbers, we obtain the following answer:

	E	A	B	C	D
E	E	A	B	C	D
A	A	B	C	D	E
B	B	C	D	E	A
C	C	D	E	A	B
D	D	E	A	B	C

18. (a) 0, (b) $\pm(1-2i)$, (c) 1 (note that the cube of the given number is -1).

19. (a) Circle of radius 5 centred at point 3. (b) Perpendicular bisector of the segment $[-4,2i]$. (c) Circle (use the identity $|p|^2+|q|^2=(|p+q|^2+|p-q|^2)/2$ for complex numbers p and q).

20. The sum in the left hand part of the inequality is equal to the length of a certain broken line connecting the points 0 and $5 + 5i$ of the complex plane. To see this, observe that the sum of the numbers $x_1 + (1 - x_2)i$, $(1 - x_3) + x_2 i$, ..., $x_9 + (1 - x_{10})i$, $(1 - x_1) + x_{10} i$ is $5 + 5i$.

21. 0°, 90°, 180°, 270°, 45°, 330°.

22. (a) See Fig. 8.2.
(b) For example, $(r - 2)(r - 2 - |\sin 3\phi|) = 0$ or, in Cartesian coordinates,

$$(\sqrt{x^2 + y^2} - 2)(\sqrt{x^2 + y^2} - 2 - \frac{|3x^2 y - y^3|}{(x^2 + y^2)^{3/2}}) = 0.$$

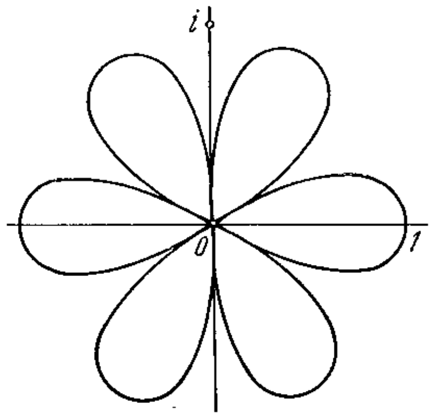

Figure 8.2: The curve of Exercise 22(a)

23. Prove that $z = \cos\alpha \pm i \sin\alpha$.

24. Answer: $2n$. Proof: similar to Problem 9.

25. If the point z lies outside of the polygon, then the arguments of all differences $z - c_i$ belong to one segment of width 180°. Therefore, the arguments of all numbers $1/(z - c_i)$ also lie in one segment of the same width, and these numbers cannot sum up to 0.

26. Use two translations.

27. One possible solution is to draw the lines parallel to the sides of the given triangle, through the endpoints of the given vector, thus *circumscribing* the triangle around the vector, and then make a parallel translation that the triangle to its initial position.

28. Let the trapezium be $ABCD$ with parallel sides AB and CD. Translate the point B by vector $\overrightarrow{DC}$ to point B' and consider the triangle ACB'.

29. Use two reflections in the sides of the given angle.

30. The point, symmetric to a vertex of the triangle with respect to a bisector, lies on the opposite side of the triangle (or its prolongation).

31. Roll the angle by successive reflections four times and unfold the reflections of the ray. The phenomenon occurs for all angles $180°/n$ with integer n.

32. After the rotation through $90°$ the vectors $\overrightarrow{MA_i}$ become the consecutive sides of the given polygon.

33. Use rotations through $60°$.

34. Let K be the result of rotation of the point M around the centre of the square which takes vertex A into B. Show that $AK \perp BM$, $BK \perp CM$, $CK \perp DM$, $DK \perp AM$.

35. Use the point symmetry with respect to the intersection point of the circles.

36. The first player should put the first coin in the centre of the table, then use point symmetry.

37. Similar to the proof of formula 2.2 (p. 41). Use a rotation around the intersection point of the given line with axis x.

38. Introduce a suitable complex structure in the plane.

39. Answer: identity. Proof: use formula 2.7 (p. 45).

40. Given three numbers that correspond to the points A, B, C, by formula 2.12 find the numbers that correspond to M, N and P.

41. It is easy to check that $R_M^d \circ R_N^d \circ R_P^d \circ R_Q^d = \mathrm{id}$. Rewrite this condition in terms of complex numbers.

42. (a) $R_B \circ R_A = T_{2\overrightarrow{AB}}$, (b) $S_l \circ R_A$ is a glide reflection with axis AK and vector $2\overrightarrow{AK}$, where K is the base of the perpendicular drawn from the point A to the line l.

43. Consider the composition of five point symmetries in the centres of the consecutive sides of the pentagon. Prove that it is a point symmetry with respect to one of the vertices of the pentagon.

44. Find the axis of the glide symmetry which is the composition of the three given reflections. Consider the line that passes through the given point parallel to this axis.

45. Use the result of Exercise 37.

46. No, because each shot changes the orientation of the set of three pucks.

47. Use the definition of the argument.

48. (a) $R_A \circ S_l = S_l \circ R_A$, (b) $R_A \circ R_B = R_B \circ R_C$.

49. (a) The line l is the perpendicular bisector of the segment AB, (b) The lines l, m and n meet in one point.

50. The set of all points $(5-6k+12l, 3+12k-6l)$, $(7-6k+12l, 2+12k-6l)$, $(3-6k+12l, 1+12k-6l)$, $(4-6k+12l, -1+12k-6l)$, $(2-6k+12l, 9+12k-6l)$, $(3-6k+12l, 10+12k-6l)$, where k and l are arbitrary integers.

51. (a) yes, (b) yes, (c) no.

52. Take an arbitrary element $g \in G$. By property (2), $g^{-1} \in G$. By property (1), $g \circ g^{-1} = \mathrm{id} \in G$.

53. Use Fig. 3.1b and the fact that a movement is completely determined by the images of three non-collinear points.

54. All these groups are different.

55. Use the result of problem 21.

56. (a) yes, for example, a rectangle, (b) no. If R_A and R_B are symmetries of a certain figure, then R_C with $C = R_A(B)$ is also a symmetry.

57. A circle, in particular a point, or the union of a set of concentric circles.

58. All possible conjugations in the group of plane movements are displayed in the following table which, on the intersection of a row labelled g and a column labelled f, contains the element $f \circ g \circ f^{-1}$.

	$T_{\mathbf{b}}$	R_B^β	S_m	$U_m^{\mathbf{b}}$
$T_{\mathbf{a}}$	$T_{\mathbf{a}}$	$T_{R_B^\beta(\mathbf{a})}$	$T_{S_m(\mathbf{a})}$	$T_{S_m(\mathbf{a})}$
R_A^α	$R^\alpha_{T_{\mathbf{b}}(A)}$	$R^\alpha_{T_{\mathbf{b}}(A)}$	$R^{-\alpha}_{S_M(A)}$	$R^{-\alpha}_{U_m^{\mathbf{b}}(A)}$
S_l	$S_{T_{\mathbf{b}}(l)}$	$S_{R_B^\beta(l)}$	$S_{S_m(l)}$	$S_{U_m^{\mathbf{b}}(l)}$
$U_l^{\mathbf{a}}$	$U^{\mathbf{a}}_{T_{\mathbf{b}}(l)}$	$U^{R^\beta(\mathbf{a})}_{R_B^\beta(l)}$	$U^{S_m(\mathbf{a})}_{S_m(l)}$	$U^{S_m(\mathbf{a})}_{U_m^{\mathbf{b}}(l)}$

59. The multiplication table for the group D_3 is shown on page 70.

60. The identity transformation commutes with everything. All rotations commute between themselves. Apart from these, in the group D_n with odd n there are no more commuting pairs. In the group D_n with even n we also have commuting pairs of reflections in mutually perpendicular lines.

61. Verify the equality $(f^{-1})^k \circ f^k = \mathrm{id}$.

62. The order of f^k is equal to $n/\,\mathrm{GCD}(n,k)$.

63. (a)

2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	2	2	4	2	6	4	6	4	10	4	12	6	8

(b) $\varphi(m) = m(1 - 1/p_1) \cdots (1 - 1/p_k)$, where p_1, ..., p_k are all prime divisors of m.

64. No. Some straight lines go into themselves under this movement.

65. The composition of n rotations R_A^α yields an identity if and only if the total angle of rotation, $n\alpha$, is equal to a multiple of 360°.

66. The group generated by A and B is C_5, a cyclic group of order 5. A is a rotation by an angle $\alpha = \pm 72°$ or $\pm 144°$, B is the rotation around the same centre by the angle -2α.

67. No. Any word is equivalent to a word of the form $EI^kA^lU^m$, where k, l, m are integers between 0 and 6.

67. One can take either a suitable pair of reflections or a pair consisting of a reflection and a suitable rotation. For example, two reflections S_1 and S_2 whose axes are adjacent satisfy the defining relations $S_1^2 = S_2^2 = (S_1 \circ S_2)^n = \mathrm{id}$.

69. Start by proving that none of the movements F_i can be a rotation.

70. Under the assumptions of the exercise, prove the following two facts: (1) for any given element a there are two integers m and n such that $a^m = a^n$, (2) the law of cancellation holds: if $xy = xz$, then $y = z$.

71. (1) No: the numbers $\sqrt{2}$ and $-\sqrt{2}$ are irrational, but their sum is rational. (2) Yes. (3) Yes. (4) No: the inverse of 3/4 is not binary-rational. (5) For example, the set $\{\tan n \mid n \in \mathbb{Z}\}$.

72. $x = ba^{-1}$.

73. Check that this operation is associative.

74. $G = \{x, 1/(1-x), (x-1)/x, 1-x, 1/x, x/(x-1)\}$. Compositions of these functions are given in the table.

	x	$\frac{1}{1-x}$	$\frac{1-x}{x}$	$1-x$	$\frac{1}{x}$	$\frac{x}{x-1}$
x	x	$\frac{1}{1-x}$	$\frac{1-x}{x}$	$1-x$	$\frac{1}{x}$	$\frac{x}{x-1}$
$\frac{1}{1-x}$	$\frac{1}{1-x}$	$\frac{1-x}{x}$	x	$\frac{1}{x}$	$\frac{x}{x-1}$	$1-x$
$\frac{1-x}{x}$	$\frac{1-x}{x}$	x	$\frac{1}{1-x}$	$\frac{x}{x-1}$	$1-x$	$\frac{1}{x}$
$1-x$	$1-x$	$\frac{x}{x-1}$	$\frac{1}{x}$	x	$\frac{1-x}{x}$	$\frac{1}{1-x}$
$\frac{1}{x}$	$\frac{1}{x}$	$1-x$	$\frac{x}{x-1}$	$\frac{1}{1-x}$	x	$\frac{1-x}{x}$
$\frac{x}{x-1}$	$\frac{x}{x-1}$	$\frac{1}{x}$	$1-x$	$\frac{1-x}{x}$	$\frac{1}{1-x}$	x

75. For example, the sum of squares of all expressions found in Exercise 74.

76. Yes.

77. Elements S_a, S_b, S_c can be arbitrarily permuted. The total number of isomorphisms, including the identity, is 6.

78. A direct verification of the definition of the isomorphism.

79. Point E corresponds to the identical transformation, while the points A, B, C, D, K correspond to rotations by 60°, 120°, 180°, 240°, 300°, respectively.

80. The circle corresponds to the identical transformation, while the triangle and the square correspond to the remaining two elements of the group C_3. There are two different isomorphisms.

81. Write out the two multiplication tables. By permuting the rows and columns in one table, make it look like the second one, up to notation of the elements.

82. Yes. The correspondence $k \leftrightarrow 2k$ is an isomorphism.

83. The only pair of isomorphic groups is D_1 and C_2.

84. Considering the expression $\varphi(g \circ g^{-1})$, prove that $\varphi(g^{-1}) = h^{-1}$. It follows that $\varphi(g^{-n}) = \varphi((g^{-1})^n) = (h^{-1})^n = h^{-n}$.

85. (1) Number the vertices of the regular triangle. (2) Consider the action of this group on the extended real line and the permutations of the set $\{0, 1, \infty\}$ under this action.

86. Denoting the Napier logarithm by N, we will have $N(x_1x_2) = N(x_1) + N(x_2) - B$.

87. (a) A direct check of group axioms. (b) The inverse image of this group under the mapping $\varphi(x) = \tan x$ would be an additive group of real numbers containing some open interval. Prove that it coincides with the whole of $\mathbb{R}$.

88. (a) $x \star y = \sqrt[3]{x^3 + y^3}$, (b) this operation is the pullback of multiplication along the mapping $x \mapsto x - 1$.

89. The group D_3 has four proper subgroups: one of order 3 and three of order 2.

90. Yes, the number -1.

91. Prove that any subgroup of $\mathbb{Z}$ is generated by its smallest positive element.

92. Check the group axioms. The notion of the quotient group (section 5.2) provides an easier way to prove this fact.

93. No, because, for example, $\bar{2} \cdot \bar{3} = \bar{6} = \bar{0}$.

94. Yes. To prove this, write out the multiplication table.

95. Every solution of the equation $x^2 = 3y^2 + 8$ is also a solution of the equation $x^2 \equiv 3y^2 + 8 \pmod 3$, which is equivalent to $x^2 \equiv 2 \pmod 3$. However, this last equation has no solutions.

96. Answer: 81.

    The last two digits of a positive integer is the same thing as its residue modulo 100. The number 2003 is mutually prime with 100, because it is not divisible by 2 and 5. Since $\phi(100) = 100 \cdot (1 - 1/2) \cdot (1 - 1/5) = 40$ and $2004 \bmod 40 = 4$, by Fermat's little theorem we have: $2003^{2004} \equiv 2003^4 \equiv 3^4 = 81$.

97. A homomorphism from $\mathbb{Z}_m$ onto $\mathbb{Z}_n$ exists if and only if m is divisible by n. Under this assumption, one of the possible homomorphisms is given by the correspondence $\bar{a} \mapsto \bar{a}$ where the bar on the left means the residue class modulo m, while the bar on the right means the residue class modulo n.

98. One can get one of the following expressions: x, $1 - x$, $1/x$, $1/(1 - x)$, $1 - 1/x$, $x/(x - 1)$. Compare this with the result of Exercise 74.

99. Use the classification of plane movements (Theorem 4 in section 2.7).

100. Prove that $\angle OAC = \angle BEF$, denote this angle by α and consider the rotation of BE around E and the rotation of AC around A through α.

101. Both assertions follow from the fact that the determinant of the product of two matrices equals the product of the determinants of each matrix.

102. (a) Yes. (b) No.

103. The kernel consists of all odd functions, the image of all even functions.

104. Consider the homomorphism $\varphi(z) = z^n$.

105. Consider the homomorphism $\varphi(x) = \cos x + i \sin x$.

106. This is the dihedral group D_n.

107. If $aba = bab$, then the elements $x = ab$, $y = aba$ satisfy $x^2 = y^3$.
     If $x^2 = y^3$, then the elements $a = x^{-1}y$, $b = y^{-1}x^2$ satisfy $aba = bab$.

108. The set splits into 5 orbits.

109. $\{-1, 2, 1/2\}$.

110. $\{1/2 + i\sqrt{3}/2, 1/2 - i\sqrt{3}/2\}$.

111. In both cases the action is transitive. Every edge is preserved by 2 movements, while every vertex — by 3 movements.

112. (a) D, S, T, (b) D.

113. (a) 2, (b) 1, (c) 7.

114. $\frac{1}{m}\sum_{k=1}^{m} n^{\mathrm{GCD}(k,m)}$, where GCD stands for the greatest common divisor.

115. 60.

116. 16.

117. 30.

118. (a) 23, (b) 218.

119. $(\binom{15}{6} + 15\binom{7}{3} + 2\binom{5}{2})/30 = 185$.

120. The black beads split the set of all white beads into 4 parts. Let m be the number of white beads in the biggest part, and n, k the numbers of beads in the two adjacent parts such that $n \geq k$ (in case of ambiguity, any choice can be used). Then the triple (m, n, k) is a complete invariant of the necklace. (Note that this invariant is rather ugly, because of the ambiguity for the necklaces with 2+2+2+0 white beads and because the set of values it takes can only be described in a very complicated way.)

121. (a) The well known criteria for the equality of two triangles (angle and two adjacent sides, side and two adjacent angles, three sides) provide the examples of complete invariants.
(b) The ordered set consisting of the lengths of all sides AB, BC, CD, DA and one angle $\angle ABC$ is an invariant. This invariant is complete on the set of convex quadrilaterals.

122. See the discussion of finite rotation groups on page 61.

123. Note that an infinite set of points on a circle cannot be discrete.

124. Angle of $360°/n$ for C_n or $180°/n$ for D_n with vertex at the common centre of rotations.

125. The condition $|kn - lm| = 1$ means that the area of the parallelogram is 1. Therefore, by the well-known Pick's formula, if P is such a parallelogram with one vertex at point A, then the points of the orbit of A that belong to P are only the vertices of P.

126. Use the table of conjugations (page 177).

127. Let O be an arbitrary point of the plane and S be the orbit of A under the action of the given group. Let A be an arbitrary point of S such that the segment OA does not contain other points of S. Let B be a point of S at the minimal distance from the line OA. Prove that the pair $\overrightarrow{OA}, \overrightarrow{OB}$ is the system of generators.

128. Group of order 18 with generators a, b, c and relations $a^2 = b^2 = c^2 = (ab)^3 = (bc)^3 = (ca)^3 = (abc)^2 = e$.

129. $p4m$, $p4g$, $p2$.

130.

C_1	$p1$
D_1	pm, pg, cm
C_2	$p2$
D_2	pmm, pmg, pgg, cmm
C_3	$p3$
D_3	$p31m$, $p3m1$
C_4	$p4$
D_4	$p4m$, $p4g$
C_6	$p6$
D_6	$p6m$

131. It is easy to understand that a number, if present, stands for the maximal order of a rotation, the symbol m (‘*mirror*’) is for a reflection, g for a glide reflection. The distinction between p and c is more subtle: the crystallographers’ implication is *primitive* or *centred* cell, but that does not seem to have any mathematical meaning other than the groups with a c may have a cell in the form of an arbitrary rhombus (see page 120), which leaves a layman wondering why no similar notation is used for rectangles or hexagons. Finally, the order of m and 1 in the notations $p3m1$ and $p31m$ looks completely enigmatic.

132. Define a homomorphism of $\mathrm{Aff}(2,\mathbb{R})$ onto $\mathrm{GL}(2,\mathbb{R})$ and use the first homomorphism theorem (page 102).

133. There is an affine transformation that takes the given trapezium to a trapezium with equal sides.

134. Both groups have order 6 and are isomorphic to the dihedral group D_3.

135. Point m/p, if $p \neq 0$.

136. Write out explicit formulas for the composition and the inverse transformation.

137. By a direct computation check that

$$\frac{x_3'-x_1'}{x_3'-x_2'} : \frac{x_4'-x_1'}{x_4'-x_2'} = \frac{x_3-x_1}{x_3-x_2} : \frac{x_4-x_1}{x_4-x_2},$$

if $x_i' = (mx_i+n)/(px_i+q)$.

138. Like in Exercise 74, the complete list of elements of this group can be obtained directly, taking the compositions of the two given functions until the elements begin to repeat.

To prove the isomorphism, find two generating elements of this group that satisfy the defining relations of the group D_n (see formula 3.1 on page 70).

139. Transformation $x \mapsto (mx+n)/(px+q)$ has finite order if and only if

$$\frac{m^2+q^2+2np}{2(mq-np)} = \cos\alpha,$$

where α is an angle measured by a rational number of degrees.

140. No. Using formula 6.6, find an example of a translation t and a projective transformation p such that ptp^{-1} is not affine.

141. By a projective transformation, the quadrangle $ACFD$ may be made into a square. Then we can introduce a Cartesian coordinate system adapted to this square and solve the problem by a direct computation. *Remark.* Of course, this is not a very beautiful solution. It only shows how to reduce the amount of computations needed to solve the problem by brute force. Indeed, in the original setting we had 6 basic points described in coordinates by a set of 12 real numbers related by two equations. After the transformation, the configuration is described by only two independent parameters.

142. Define a homomorphism of $\mathrm{GL}(2,\mathbb{R})$ onto $\mathrm{PGL}(1,\mathbb{R})$ and use the first homomorphism theorem (page 102).

143. It is easier to construct a triangle, circumscribed around a given triangle, with sides parallel to the three given lines.

144. Use a homothety with centre on the outer circle and coefficient $3/4$.

145. Use a homothety with centre in the median intersection point and coefficient -2.

146. Consider the homothety with coefficient $1/2$ centred at the orthocentre of the triangle. Prove that the circumscribed circle of the given triangle goes into the desired circle under this transformation.

147. We know that any similitude with coefficient different from 1 has a unique fixed point. The problem is to prove that this point lies inside the smaller map. Suppose that it lies outside, draw a line through this point that intersects the smaller map and consider its intersection points with the boundaries of the maps.

148. E is the centre of homothety that takes A into B and C into D.

149. Find the images of segments MC and PN under the action of the spiral homotheties $F_A(\sqrt{2}, 45°)$ and $F_C(\sqrt{2}, 45°)$, respectively.

150. Apart from the inversions centred at 0, this group also contains all positive homotheties with the same centre and is isomorphic to the multiplicative group of non-zero real numbers $\mathbb{R}^\star$.

151, 152. Straight lines that do not pass through the centre of the inversion, go into circles that pass through the centre, and vice versa.

153. This fact is proved in Chapter 7, theorem 18 (see page 160).

154. The length of an orbit can be 2, 3, 6 or 12. See Fig. 6.16 and discussion below.

155, 156. Derive explicit formulas for the composition and the inverse transformation.

157. $(-3, 0)$.

159. (a) Yes, for example $y' = 0$. (b) No.

160. $y = 2x + C$, $y = \sin x + C$, $y = -1/x$.

161. A family of straight lines for equation (7.3) and a family of hyperbolas for equation (7.5).

162. Circles with centre in the origin. The corresponding differential equation is $y' = -x/y$. It is defined everywhere except for the line $y = 0$. There is no such differential equation defined in all the plane.

163. $y = 1/(C - x)$.

164. $v = x + y$, $u = x$.

165. $y = -x^4/3 + x^2 + Cx$.

166. $y = e^{x^2/4}\left(\int e^{x^2/2}dx + C\right)^{1/2}$.

167. $y = \tan ax + C$.

168. $y = \dfrac{1}{x^2 \tan(1/x + C)} - \dfrac{1}{x}$.

169. Find n such that $-(n+3)/(n+4) = -4$.

170. Note that $\dfrac{1}{q(n)+2} = \dfrac{1}{n+2} - 1$, whence $\dfrac{1}{q^k(n)+2} = \dfrac{1}{n+2} - k$.

171. See Problem 66.

172. (a) $\dfrac{dr}{d\phi} = \dfrac{(yy' + x)\sqrt{x^2+y^2}}{xy' - y}$.
(b) The solution is given implicitly by $\sqrt{x^2+y^2}(C - 2\arctan(y/x))^2 = 1$.

173. $g_t \circ g_s = g_{t+s} = g_{s+t} = g_s \circ g_t$.

174. Yes. Assuming that the centre is 0, the correct parametrization is given by $g_t(x, y) = (e^t x, e^t y)$.

175. $x_t = x \cos t - y \sin t$, $y_t = x \sin t + y \cos t$.

176. These formulas define the group of spiral homotheties. Its orbits are logarithmic spirals.

177. The orbits are straight lines $x + y = \text{const}$. Group property follows from the theorem of Viète.

178. $y = f(y/x)$, where f is an arbitrary function.

179. $y = \dfrac{xf(\xi) + y}{x - yf(\xi)}$, where $\xi = \arctan \dfrac{y}{x} - \dfrac{b}{a} \ln \sqrt{x^2 + y^2}$.

180. The function $x^2 + y^2$ is a universal invariant of this group action.

181. Function ξ from the answer to Exercise 179.

182. $y = Cx^2 - x$.

183. Yes.

184. Use the group of homotheties with centre at the intersection point of the lines $ax + by + c = 0$ and $a_1 x + b_1 y + c_1 = 0$.

185. Use the group $x_1 = e^{2t} x$, $y_1 = e^t y$. Answer: $y = \sqrt{Cx^2 - 2x}$.

186. $y = \dfrac{x + C - 1}{x + C} e^x$.

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