Chapter 1

Algebra of points

In this chapter we are going to introduce algebraic operations, addition and multiplication, in the set of points in the plane. This will allow us to apply algebra to geometry and geometry to algebra.

1.1 Checkered plane

Consider a plane with a regular square grid, i.e. two sets of parallel equal-distanced lines, perpendicular to each other. We will be interested in the polygons with all vertices in the nodes of the grid, like the isosceles triangle or the square shown in Fig. 1.1.

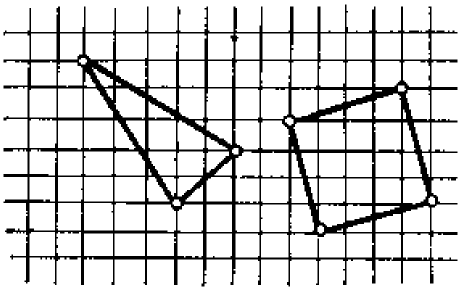

Figure 1.1: Polygons in the checkered plane

Problem 1 *Prove that a regular polygon different from a square cannot have all its vertices in the nodes of a square grid.*

> **Solution.** Suppose, on the contrary, that such a polygon $A_1A_2\ldots A_n$ exists. Let O be its centre. For every triple of consecutive vertices $A_{k-1}A_kA_{k+1}$ find a point B_k which is the fourth vertex of the parallelogram $A_{k-1}A_kA_{k+1}B_k$. The whole construction of Figure 1.2 goes into itself under the reflection with axis OA_k and under the rotation through $360/n$ degrees around the point O. Therefore every point B_k lies on the corresponding line OA_k, and $B_1B_2\ldots B_k$ is a regular polygon.

If $n > 6$, then this polygon is smaller than the initial one. Indeed, in this case the angle $\alpha = \frac{n-2}{n}180°$ is greater than the angle $\beta = \frac{2}{n}360°$, hence the point B_k belongs to the segment OA_k. It is a crucial observation that all the points B_1, B_2,..., B_n lie again in the nodes of the square grid.

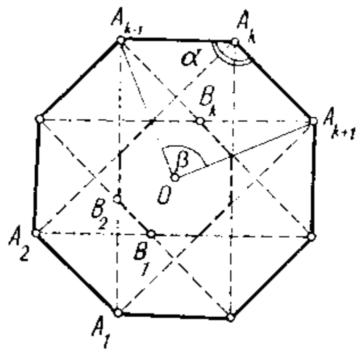

Figure 1.2: Regular polygon

Repeating the same procedure for the polygon $B_1B_2 \dots B_n$ instead of $A_1A_2 \dots A_n$, we will arrive at a third polygon $C_1C_2 \dots C_n$ whose vertices have the following properties:

- they coincide with some nodes of the grid,
- C_k belongs to the segment OA_k and lies closer to O than B_k.

Since there are only finitely many integer points on the segment OA_k, after several iterations of this procedure we will arrive at a contradiction.

The same argument remains valid also in the case of a regular pentagon, the only difference being that now the point B_k lies on the line passing through O and A_k outside of the segment OA_k.

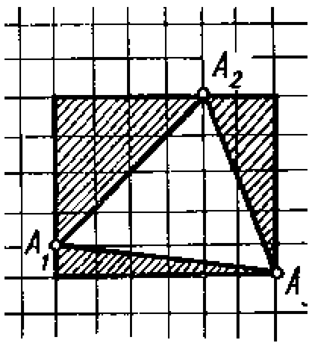

Figure 1.3: Is there such a regular triangle?

If $n = 3$ or 6, the argument fails (why?), and we will give a different proof of our assertion. Note first of all that the three vertices of a regular hexagon form a regular triangle, thus it is sufficient only to consider the case $n = 3$. Suppose that a regular triangle has all its vertices in the nodes of the checkered plane (Fig. 1.3). Then, by Pythagoras' theorem, the square of the side of this triangle must be an integer (we assume that the grid is 1 by 1), hence its area $S = a^2\sqrt{3}/4$ is an irrational number. On the other hand, the triangle $A_1A_2A_3$ can be obtained from a rectangular with integer sides by removing three right triangles as shown in Fig. 1.3, thus its area must be rational — in fact, either m or $m + \frac{1}{2}$ where m is a whole number.

Exercise 1. Suppose that the sides of the squares making the grid are 1. Is there a right triangle with all vertices in the nodes such that all its sides have integer lengths and no side is parallel to the lines of the grid?

1.2 Point addition

Our solution of Problem 1 was based on the following nice property of the integer grid: if three vertices of a parallelogram are in the nodes, then the fourth vertex also gets into a node. The usual mathematical wording for this phenomenon is: *the set of all nodes is closed with respect to the operation under study.* We will now give an exact definition of this operation.

Given three points in the plane, say M, N and P, there are three different ways to add one more point so that the triangle MNP becomes a parallelogram. One way is to connect P with the midpoint K of MN and choose the point L on the line PK which is symmetric to P with respect to K (Fig. 1.4).

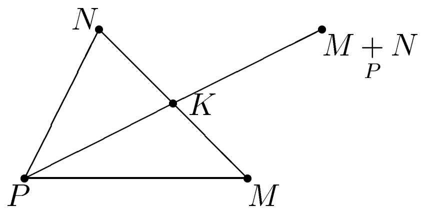

Figure 1.4: Point addition

Definition 1 *We will call the point L thus constructed the sum of the points M and N over the pole P and we will write $L = M \underset{P}{+} N$, which should be read aloud as "M plus N over P". When the pole is fixed, we may omit it from the notation and simply write $L = M + N$.*

This definition holds for an arbitrary triple of points in the plane. If the M, N and P belong to one straight line, then the parallelogram $MPNL$ degenerates into a line segment. If all of them coincide, then it degenerates even more and becomes a point.

Now we can give an exact statement for the property of the integer grid that was used in Problem 1: *the sum of any two nodes of the grid over any other node is always a node.*

Let us now forget about the grid and study the properties of addition for arbitrary points.

Exercise 2. Given two triangles ABC and DEF and a point P, denote by Φ the set of all points $M \underset{P}{+} N$ where M is an interior point of $\triangle ABC$ and N an interior point of $\triangle DEF$.

a) Prove that Φ is a polygon. How many sides may it have?

b) Prove that its perimeter is the sum of perimeters of the two given triangles.

Point addition is closely related to vector addition: $L = M \underset{P}{+} N$ is equivalent to $\overrightarrow{PL} = \overrightarrow{PM} + \overrightarrow{PN}$, and enjoys similar properties:

1° Associative law

$$(A \underset{P}{+} B) \underset{P}{+} C = A \underset{P}{+} (B \underset{P}{+} C)$$

holds for any arbitrary points A, B, C over any pole P.

2° We always have

$$P \underset{P}{+} A = A,$$

i.e. point P behaves as a neutral element with respect to the operation $\underset{P}{+}$.

3° Over a given pole P, every point A has an *opposite* point, i.e. a point A' such that

$$A \underset{P}{+} A' = P.$$

In fact, one can simply take the point A' which is symmetric to A with respect to P.

4° Commutative law

$$A \underset{P}{+} B = B \underset{P}{+} A$$

holds for any arbitrary points.

The order in which these 4 items appear in our list, is not accidental — in fact, more fundamental rules come first. You will understand it better when you read this book until Chapter 4.

Rules 2° − 4° are obvious and do not require any proof. To check rule 1°, we first construct the points $M = A \underset{P}{+} B$ and $N = B \underset{P}{+} C$ (see Fig. 1.5). The segments AM and CN are both equal and parallel to the segment PB, hence the midpoints of MC and AN coincide, which, by the definition of point addition, ensures that $M \underset{P}{+} C = A \underset{P}{+} N$.

Using property 3°, we can define the *difference* of two points over a given pole: $B \underset{P}{-} A = B \underset{P}{+} A'$, where $A' = \underset{P}{-} A$ is the point opposite to A. The point $B \underset{P}{-} A$ is the unique solution to the equation $A \underset{P}{+} X = B$.

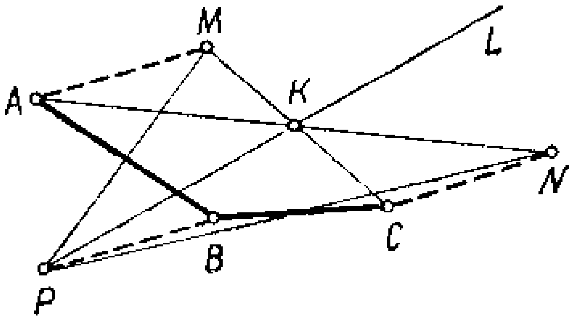

Figure 1.5: Associativity of point addition

If all operations are carried out over the same pole, then addition and subtraction of points satisfies the same rules as the usual operations on numbers, for example, $A-(B-C+D)=A-B+C-D$.

Problem 2 *Find the sum* $A \underset{M}{+} B \underset{M}{+} C$ *where* M *is the intersection point of the medians in a triangle* ABC.

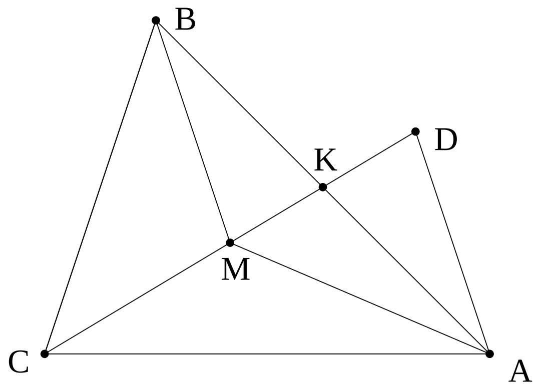

Figure 1.6: Sum of vertices of a triangle

Solution. Recall that each median is divided by their common intersection point M in the ratio $2:1$, therefore, in Figure 1.6, we have $CM=2MK$. The point $D=A \underset{M}{+} B$ lies on the prolongation of the median CK, and $DK=KM=\frac{1}{2}MC$. Therefore, $DM=MC$ and $D \underset{M}{+} C = M$.

It is interesting to observe that the intersection point of the medians is the only point which satisfies $A \underset{M}{+} B \underset{M}{+} C = M$. To prove this, let us first derive the rules of passing from one pole to another in the formulas involving point addition:

$$A \underset{Q}{+} B = A \underset{P}{+} B \underset{P}{-} Q, \tag{1.1}$$
$$A \underset{Q}{-} B = A \underset{P}{-} B \underset{P}{+} Q. \tag{1.2}$$

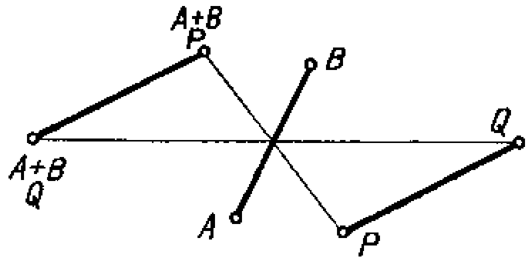

Figure 1.7: Change of base point

The first equality can be rewritten as $(A \underset{Q}{+} B) \underset{P}{+} Q = A \underset{P}{+} B$ and its validity is easily seen from Figure 1.7. To prove the second one, we will check that the point $A \underset{P}{-} B \underset{P}{+} Q$ is a solution to the equation $B \underset{Q}{+} X = A$. Indeed, using the formula we have just proved, we get

$$B \underset{Q}{+} (A \underset{P}{-} B \underset{P}{+} Q) = B \underset{P}{+} (A \underset{P}{-} B \underset{P}{+} Q) \underset{P}{-} Q = A.$$

Note that the point P does not appear in the left hand parts of equalities 1.1, hence the right hand parts do not depend on its choice. This observation is generalized in the following exercise.

Exercise 3. Investigate the conditions under which the expression

$$A_1 \underset{P}{+} \cdots \underset{P}{+} A_2 \underset{P}{+} \cdots \underset{P}{+} A_k \underset{P}{-} B_1 \underset{P}{-} B_2 \underset{P}{-} \cdots \underset{P}{-} B_l,$$

does not depend on the choice of the pole P.

Continuing the discussion of Problem 2, suppose that a point N has the same property as the median intersection point M, i.e. $A \underset{N}{+} B \underset{N}{+} C = N$. We can subtract the pole without violating the equation, hence $A \underset{N}{+} B \underset{N}{+} C \underset{N}{-} N \underset{N}{-} N = N$. A reader who has done Exercise 3 knows that the left hand side of this relation does not depend on the choice of N. In particular, substituting M in place of N, we get $A \underset{M}{+} B \underset{M}{+} C \underset{M}{-} N \underset{M}{-} N = N$, therefore $M \underset{M}{-} N \underset{M}{-} N = N$, $N \underset{M}{+} N \underset{M}{+} N = N$ and finally $N = M$. This means that the median intersection point M is the unique point with the property proved in Problem 2.

Exercise 4. Prove that $A \underset{O}{+} B \underset{O}{+} C = H$, where O is the centre of the circle circumscribed around the triangle ABC and H is the intersection point of its three altitudes.

1.3 Multiplying points by numbers

Over a given pole P, a point A can be *multiplied* by a real number α yielding a new point $B = \alpha_P A$.

Definition 2 *The product of a point A by a real number α over the pole P is the point B that lies on the line PA at the distance $|\alpha||PA|$ from the pole P and on the same side of P as A, if $\alpha > 0$, or on the other side, if $\alpha < 0$.*

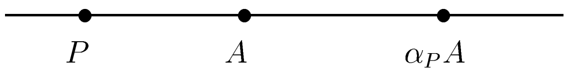

Figure 1.8: Multiplication of points by numbers

In other words, this operation means that you stretch the vector $\overrightarrow{PA}$, keeping its initial point P fixed, as if with a pin: $\overrightarrow{PB} = \alpha\overrightarrow{PA}$.

In particular,

1. any point multiplied by zero over P gives P,

2. P multiplied by any real number gives P.

It is easy to see that multiplication of points by numbers has these properties:

5° $1_P A = A$.

6° $\alpha_P(\beta_P A) = (\alpha\beta)_P A$.

7° $(\alpha + \beta)_P A = \alpha_P A \underset{P}{+} \beta_P A$.

8° $\alpha_P(A \underset{P}{+} B) = \alpha_P A \underset{P}{+} \alpha_P B$.

To multiply a point by a *natural* number n is the same thing as to add up n equal points: $n_P A = A \underset{P}{+} A \underset{P}{+} \ldots \underset{P}{+} A$ (A repeated n times). Using this fact, you can check that the point $\frac{1}{2}{}_P A$ is the (unique) solution to equation $X \underset{P}{+} X = A$.

Consider a *linear combination over the pole P*, i.e. the sum of several points with arbitrary coefficients

$$\alpha_P A \underset{P}{+} \beta_P B + \ldots \underset{P}{+} \omega_P Z = S. \tag{1.3}$$

In general, the resulting point S depends on the choice of the pole P. When $\alpha, \beta, \ldots, \omega$ are integer numbers, we have seen in Exercise 3 that there are some occasions when the result does not depend on P. This may also happen in the more general situation, when the coefficients are not integer. For example, the point $M = \frac{1}{2}A + \frac{1}{2}B$ is always the middle point of the segment AB, wherever you put the pole.

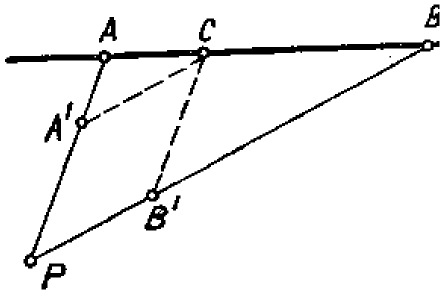

Figure 1.9: Point of a segment expressed through endpoints

Exercise 5. (A generalization of Exercise 3). Find a necessary and sufficient condition on the coefficients α, β, ..., ω which guarantee that the linear combination S (1.3) does not depend on the choice of the point P.

Using point addition and multiplication by numbers, it is possible to express any point of the segment AB through its endpoints. Indeed, suppose that the point C divides the segment AB in the ratio $k : l$ (by definition, this means that $l \cdot \overrightarrow{AC} = k \cdot \overrightarrow{CB}$). Choose an arbitrary point P outside of the line AB; we will use it as the pole in all subsequent operations on points. Through the point C we draw two lines, parallel to PB and PA, which meet PA and PB in points A' and B', respectively (see Figure 1.9). Then

$$\frac{PA'}{PA} = \frac{BC}{BA} = \frac{l}{k+l}$$
$$\frac{PB'}{PB} = \frac{AC}{AB} = \frac{k}{k+l}$$

Put $\frac{l}{k+l} = \alpha$, $\frac{k}{k+l} = \beta$. Then $C = A' + B' = \alpha A + \beta B$, and the sum of the two coefficients α and β is 1.

The converse is also true: if α and β are arbitrary nonnegative numbers such that $\alpha + \beta = 1$, then the point $C = \alpha A + \beta B$ belongs to the segment AB. Moreover, if one of the numbers α and β in the formula for C is negative, but the sum of the two is still 1, then the point C lies on the straight line AB, but outside of the segment AB. By changing Figure 1.9 appropriately, you can verify that relations $\alpha = \frac{l}{k+l}$, $\beta = \frac{k}{k+l}$ remain valid, albeit the ratio $k : l$ is now negative.

Thus, the straight line AB is the set of all points $\alpha A + (1 - \alpha)B$, where α is an arbitrary real number, while the segment AB is its subset specified by the restriction $0 \leq \alpha \leq 1$. Note again that this description does not depend on the choice of the base point (pole).

Exercise 6. Find a similar description of the set of all inner points of a convex polygon with vertices A_1, A_2, ..., A_n.

After doing Exercise 6, you can go back and tackle Exercise 2 once again, using the new technique.

Exercise 7. A middle line of a quadrilateral is the line joining the midpoints of two opposite sides. Any quadrilateral has two middle lines. Prove that these two lines, as well as the segment joining the midpoints of the two diagonals, meet in one point, and this point divides each of them in half and half (see Fig. 1.10).

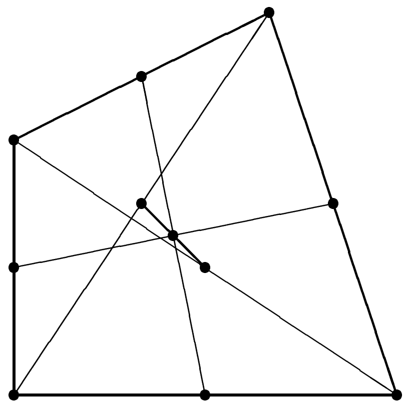

Figure 1.10: Quadrilateral of exercise 7

1.4 Centre of gravity

In Problem 2, the median intersection point M of a triangle ABC was described implicitly as the (unique!) solution to the equation $A \underset{M}{+} B \underset{M}{+} C = M$. We can now express M explicitly through A, B and C. Indeed, multiplying both parts of the equation by $\frac{1}{3}$, we get $\frac{1}{3}{}_{M}(A \underset{M}{+} B \underset{M}{+} C) = M$. According to the answer of Exercise 5, the left hand side of this equality does not depend on the choice of the pole, hence we can write

$$M = \frac{1}{3}(A + B + C).$$

In a similar way, the point referred to in Exercise 7 can be expressed through the vertices of the quadrilateral as

$$M = \frac{1}{4}(A + B + C + D).$$

In general, the arithmetic mean of several points is called the *centre of gravity* (or *centre of mass*) of the system consisting of these points: $M = \frac{1}{n}(A_1 + A_2 + \ldots + A_n)$ (over an arbitrary pole). Thus, the centre of gravity of a triangle (or, more exactly, of the set of its vertices) is the median intersection point, while the centre of gravity of the set of vertices of a quadrilateral is the intersection of its two middle lines.

We proceed to some examples where geometrical problems related to the centre of gravity are solved using operations on points.

Problem 3 *Suppose that A, B and C are three collinear points, while E and F are arbitrary points in the plane. Prove that the median intersection points of the triangles AEF, BEF, CEF are collinear.*

Solution. Median intersection points are arithmetic means of the vertices:

$$\begin{aligned}\frac{1}{3}(A+E+F) &= K,\\ \frac{1}{3}(B+E+F) &= L,\\ \frac{1}{3}(C+E+F) &= M.\end{aligned}$$

By assumption, point C lies on the line AB, thus $C = \alpha A + (1-\alpha)B$. Hence $\alpha K + (1-\alpha)L = \frac{\alpha}{3}(A+E+F) = \frac{1-\alpha}{3}(B+E+F) = \frac{1}{3}(C+E+F) = M$, which implies that the point M belongs to the line KL.

Exercise 8. Let A, B, C, D, E, F be the middle points of the consecutive edges of a hexagon. Prove that the centres of gravity of the triangles ACE and BDF coincide.

Exercise 9. In a quadrilateral $ABCD$, the point E is the midpoint of the side AB and K the midpoint of the side CD. Prove that the midpoints of four segments AK, CE, BK and ED form a parallelogram.

Problem 4 *Prove that the middle line of a quadrilateral (see Exercise 7) passes through the intersection point of its diagonals if and only if this quadrilateral is a trapezium, i.e. has two parallel sides.*

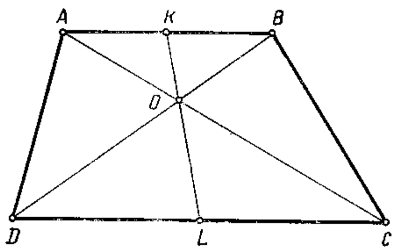

Figure 1.11: Trapezium

Solution. We choose the intersection point O of the diagonals as the pole (see Figure 1.11). Then $C = \alpha A$, $D = \beta B$ for appropriate numbers α and β, and for the middle points K and L we can write $K = \frac{1}{2}(A+B)$, $L = \frac{1}{2}(\alpha A + \beta B)$.

If $AB \parallel CD$, then the triangles OBA and ODC are similar, hence $\alpha = \beta$, $L = \alpha K$ and the points K, L, O are collinear.

Suppose, on the other hand, that we do not know whether AB is parallel to CD, but we do know that K, L and O lie on the same line. Then, using the point operations over the pole O, for a suitable real number γ we have $L = \gamma K$. Substituting the previous expressions for K and L, we get $\alpha A + \beta B = \gamma A + \gamma B$, or $(\alpha - \gamma)A = (\gamma - \beta)B$. But the points $(\alpha - \gamma)A$ and $(\gamma - \beta)B$ lie on different lines OA and OB, and if they coincide, this means nothing else but that they coincide with O. Thus, $\alpha - \gamma = \gamma - \beta = 0$, $\alpha = \beta$, triangles OAB and ODC are similar, and $AB \parallel CD$.

Exercise 10. Using point addition and multiplication by numbers, find an independent proof of the fact that the medians of a triangle are divided in proportion 2 : 1 by their intersection point (note that we have used this fact before, in Problem 2).

Exercise 11. A line cuts 1/3 of one side of a parallelogram and 1/4 of the adjacent side in such a way that the smaller parts have a common vertex (Fig.1.12). In what ratio does this line divide the diagonal of the parallelogram?

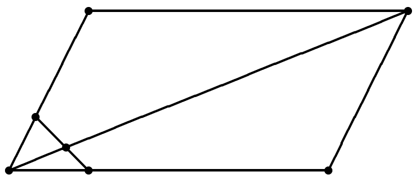

Figure 1.12: Cutting the diagonal

1.5 Coordinates

In the discussion of Problem 4, we have used the following important fact: *if two points M, N are not collinear with the pole, then the equality $\alpha M + \beta N = \gamma M + \delta N$ is possible only if $\alpha = \beta$ and $\gamma = \delta$.* In fact, the given equality can be rewritten as $(\alpha - \gamma)M = (\gamma - \delta)N$ which implies $\alpha = \gamma$ and $\beta = \delta$.

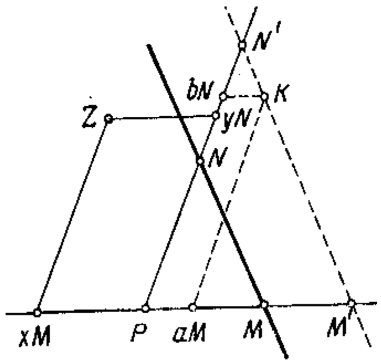

Figure 1.13: Affine coordinates

Choose a pole P and two points M, N that are not collinear with P. Then any point Z of the plane can be expressed as $Z = xM + yN$ for suitable real numbers x and y (Figure 1.13).

Definition 3 *A system of affine coordinates in the plane is an ordered set of three non-collinear points* $\{P, M, N\}$. *The first point* P *is referred to as the pole, or the origin, while the set* $\{M, N\}$ *is referred to as the basis of the given coordinate system. The coordinates of a point* Z *in the coordinate system* $\{P, M, N\}$ *are the coefficients* $\{x, y\}$ *in the expansion* $Z = x_P M + y_P N$.

The above argument shows that the coordinates x and y are *uniquely* determined by the point Z, thus we obtain a one-to-one correspondence between the points of a plane and pairs of real numbers.

If $\angle MNP$ is a right angle and both PM and PN are unit segments, then what we get is the usual Cartesian coordinates. In general, such coordinates are referred to as *affine coordinates*.

When two points are added, their coordinates add up:

$$(aM + bN) + (cM + dN) = (a + b)M + (c + d)N.$$

When a point is multiplied by a number, its coordinates get multiplied by the same number:

$$c(aM + bN) = (ca)M + (cb)N.$$

The correspondence between points of the plane and pairs of real numbers can be used as a dictionary which serves to translate geometrical propositions into the language of algebra and vice versa. Any geometrical figure is the set of all points whose coordinates satisfy a certain relation. For example, we know that a point belongs to the line MN if and only if has the expression $xM + yN$ where $x + y = 1$. In this sense, $x + y = 1$ is the *equation of the straight line* MN.

Problem 5 *Find the equation of the straight line which is parallel to* MN *and passes through the given point* K *with coordinates* a, b.

> **Solution.** Let M' and N' be the intersection points of this line with PM and PN, respectively. Since $M'N' \parallel MN$, we have $M' = tM$, $N' = tN$ for an appropriate number t (see Figure 1.13). Any point Z of the line $M'N'$ is equal to $\alpha M' + \beta N'$ where $\alpha + \beta = 1$, i.e. $Z = \alpha tM + \beta tN$ and $\alpha t + \beta t = t$. Thus, the coordinates $x = \alpha t$, $y = \beta t$ of an arbitrary point $Z \in MN$ satisfy the relation $x + y = t$ where the value of t is yet unknown. To find it, note that the point K lies on the line under study, hence its coordinates a, b satisfy the equation of this line: $a + b = t$ is true. We have found that $t = a + b$ and the answer to the exercise is: $x + y = a + b$.

Exercise 12. Write the equation of the straight line that contains a given point $K(a, b)$ and

(a) is parallel to PM,

(b) is parallel to PN,

(c) passes through P.

Problem 6 *Suppose that in a certain triangular region of the plane the laws of optics are such that a ray of light which goes parallel to one side of the triangle and hits the second side, after reflection assumes the direction of the third side of the triangle. Prove that a person standing inside of this triangle and directs the beam of his flashlight parallel to one of the sides of the triangle, is in fact sending the light on his own back.*

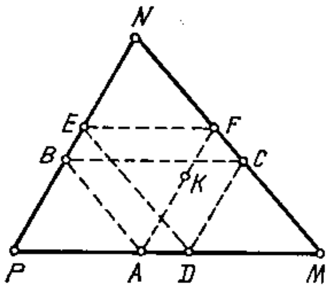

Figure 1.14: Zigzag inside a triangle

Solution. Let one vertex of the triangle, P, be the pole and two others, M and N, be the two basic points of a coordinate system (Figure 1.14).

Suppose that the person with the flashlight stands in the point $K(a,b)$ and the beam of his flashlight goes parallel to PN and meets the side PM in the point A. The coordinates of the point A are $(a,0)$, because, on one hand, $KA \parallel PN$ and hence the first coordinate of A is equal to the first coordinate of K (see Exercise 12), on the other hand, point A lies on PM and hence its second coordinate is 0.

The next segment of the beam, AB, is parallel to MN. According to Problem 5, the equation of the line AB is $x+y=a$, because a is the sum of coordinates of the point A. Since the point B lies on PN, it has $x=0$, therefore its second coordinate must be equal to a.

Proceeding in the same way, we successively find the coordinates of all points where the beam meets the sides of the triangle: $C(1-a,a)$, $D(1-a,0)$, $E(0,1-a)$, $F(a,1-a)$. The line FK is parallel to PN, this is why the beam does return to the initial point K — from the opposite direction.

A vigilant reader may have noticed a flaw in the previous argument: in fact, it may happen that the beam returns to the point K before it makes the complete tour of $ABCDEF$ — and hits the flashlighter in a side, not in the back.

Exercise 13. Describe the set of all points K in the triangle MNP for which the trajectory of the flashlight beam consists of only three segments, not six.

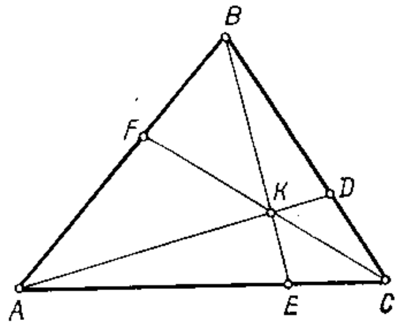

Figure 1.15: Lines in a triangle meeting in one point

Exercise 14. A point K is inside the triangle MNP. Straight lines AK, BK, CK meet the sides BC, CA, AB in the points D, E, F, respectively (Figure 1.15). Prove that $KD/AD + KE/BE + KF/CF = 1$.

Exercise 15. Given three points D, E, F on the sides of the triangle ABC (Figure 1.15), prove that the lines AD, BE, CF pass through one point if and only if $AF \cdot BD \cdot CE = FB \cdot DC \cdot EA$ (theorem of Ceva).

1.6 Point multiplication

We have learned how to multiply a point in the plane by a real number. Now recall that real numbers can be represented as points lying on a line. Let us insert this line into the plane so that its origin (zero point) coincide with the pole P which is used to define the point addition and multiplication of points by numbers. The unit point of the real line will be E (see Figure 1.16a).

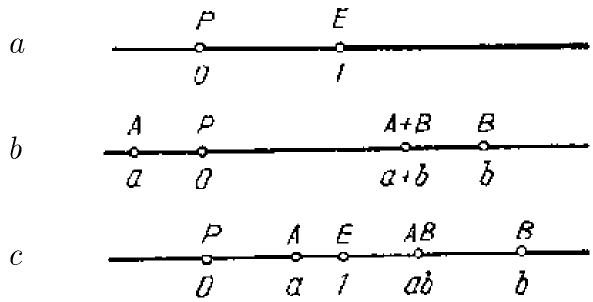

Figure 1.16: Algebraic operations in the line

Our definition of point addition agrees with the usual addition of real numbers in the sense that if the points A and B correspond to numbers a and b, then the sum $A + B$ (over the pole P) corresponds to the number $a + b$ (Figure 1.16b).

Moreover, our multiplication of points by numbers, restricted to the real line, also agrees with the usual product of numbers in the sense that if $A \leftrightarrow a$, $B \leftrightarrow b$, then both points $a_P B$ and $b_P A$ correspond to the number ab. It is natural to call this point the *product* of the two points A and B and denote it by AB.

The next step we want to make is to extend this definition to the entire plane. We want to find a rule to assign a new point AB to any pair of arbitrary points A, B in such a way that this *point multiplication* satisfies the usual rules of multiplication:

9° Associativity

$$(AB)C = A(BC).$$

10° Commutativity

$$AB = BA.$$

11° Distributive law with respect to point addition with the same pole

$$A(B + C) = AB + AC.$$

We also require that the new operation agree with the previously defined multiplication of points by numbers, i.e. that for any point Z in the plane and any point A on the real line that corresponds to the number a we should have $AZ = a_P Z$. In particular, this means that the unit point E of the real line must play the role of the number 1 for all points Z of the plane in the sense that $EZ = 1_P Z = Z$.

It is not immediately clear whether it is possible to introduce such an operation for the points of the plane. We will see, however, that in fact there are many ways to do so, and they come in three essentially different types. But let us first do some exercises.

Problem 7 *Let $EABCDK$ be a regular hexagon with the centre at P (recall that E is the unit point) and suppose that $A^2 = B$ for a certain choice of point multiplication. Find all pairwise products of the vertices of the given hexagon.*

Solution. Expand all the vertices over the basis E, A taking P for the pole (Figure 1.17): $B = A - E$, $C = -E$, $D = -A$, $K = E - A$. We know the products of all pairs consisting of basic points: $E^2 = E$, $EA = A$, $A^2 = B$. Using the distributive law, we can find, for example, that $BK = (A - E)(E - A) = -A^2 + 2AE - E^2 = -B + 2A - E = A$. Other products can be found in the similar fashion, yielding the multiplication table:

	E	A	B	C	D	K
E	E	A	B	C	D	K
A	A	B	C	D	K	E
B	B	C	D	K	E	A
C	C	D	K	E	A	B
D	D	K	E	A	B	C
K	K	E	A	B	C	D

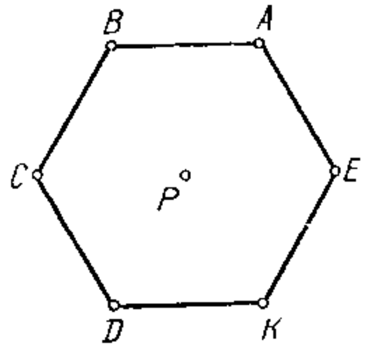

Figure 1.17: Multiplication of the vertices of a hexagon

Note that the set of 6 vertices of the hexagon turns out to be *closed* under the chosen rule of multiplication, i.e. the product of any two vertices is also a vertex.

Exercise 16. Is the set of vertices of the same hexagon closed under multiplication, if (a) $A^2 = A$; (b) $A^2 = P$? Fill the corresponding multiplication tables.

Exercise 17. Find the multiplication table for the set of vertices of a regular pentagon $EABCD$ centred at the pole P, if A^2 is known to be equal to B.

After these examples, we will investigate the general case. Suppose that we are given a point multiplication rule that satisfies all the requirements stated above.

Besides the two already chosen points $P = 0$ and $E = 1$, pick an arbitrary point F not on the line PE. Then the pair (E, F) is a basis over P, and, as we saw in the discussion of Problem 7, point multiplication is completely defined, if we only know the square of F.

We have $F^2 = \alpha E + \beta F$ for suitable real numbers α and β. Let us try to find another point G such that the pair (E, G) is also a basis in the plane, but the square G^2 has a simpler expansion over this basis.

Let $G = F - \frac{\beta}{2}E$. Then the lines FG and PE are parallel, so that E and G constitute a basis, and

$$\begin{aligned} G^2 &= (F - \frac{\beta}{2}E)^2 \\ &= F^2 - \beta EF + \frac{\beta^2}{4}E^2 \\ &= (\alpha + \frac{\beta^2}{4})E. \end{aligned}$$

If you look closely at this relation, you will see that multiplication in the basis (E, G) looks simpler than in the initial basis (E, F), because the square of the second basic point is now just E with a certain coefficient — and not the combination of the two points, as before. To further simplify the multiplication rule, we will change G once again, depending on the sign of this coefficient.

1. $\alpha + \dfrac{\beta^2}{4} = 0$ (cf. Exercise 16b). In this case the product is given by the formulas

$$E^2 = E, \quad EG = G, \quad G^2 = 0;$$
$$(aE + bG)(cE + dG) = acE + (ad + bc)G.$$

2. $\alpha + \dfrac{\beta^2}{4} > 0$. Denoting $\dfrac{1}{\sqrt{\alpha + \beta^2/4}} G$ by H, in the basis (E, H) we will have the following rules of multiplication:

$$E^2 = E, \quad EH = H, \quad H^2 = E;$$
$$(aE + bH)(cE + dH) = (ac + bd)E + (ad + bc)H.$$

   (try to find such a point H among the vertices of the hexagon in Exercise 16b).

3. $\alpha + \dfrac{\beta^2}{4} < 0$. Set $I = \dfrac{1}{\sqrt{|\alpha + \beta^2/4|}} G$. Then

$$E^2 = E, \quad EI = I, \quad I^2 = -E;$$
$$(aE + bI)(cE + dI) = (ac - bd)E + (ad + bc)H.$$

It is easily verified that in each of the three cases our operation satisfies all the laws imposed on multiplication. The next question that naturally appears is whether this multiplication has an inverse operation of *division*, i.e. whether the equation $AZ = B$ can always be resolved in terms of Z, provided that $A \neq 0$.

In the first case let us try to divide E by G, i.e. find such a point $Z = xE + yG$ that $GZ = E$. According to the definition, $G(xE + yG) = xG$, which never equals E. Thus, division is in general impossible.

The same is true in the second case where, as you can check, H is not divisible by $E+H$.

We claim, however, that in the third case division by a non-zero point is always possible. Indeed, let $M = aE + bI$, $N = cE + dI$, where the coefficients c and d do not vanish simultaneously. We want to find the quotient M/N, that is, a point $Z = xE + yI$ such that $NZ = M$, or $(cE + dI)(xE + yI) = aE + bI$. When expanded, this equality becomes equivalent to the system of equations

$$\begin{aligned} cx - dy &= a, \\ dx + cy &= b, \end{aligned}$$

which has a unique solution $x = \dfrac{ac + bd}{c^2 + d^2}$, $y = \dfrac{bc - ad}{c^2 + d^2}$, provided that $c^2 + d^2 \neq 0$.

The result of our investigation can be stated as follows.

Theorem 2 *Multiplication of points in the plane can be introduced in three essentially different ways, depending on the existence of an element X with the property*

1. $X^2 = 0$,

2. $X^2 = 1$,

3. $X^2 = -1$.

Only in case (3) division by non-zero element is always possible.

Speaking more formally, there exist three different two-dimensional algebras over the field of real numbers and only one of them (case 3) is an algebra with division.

Note that the actual geometrical meaning of multiplication, say, in case (3), depends on the mutual position of points E and I with respect to the origin P. For example, let $PEAB$ be a square drawn on the segment PE. Where is the point A^2? This depends on the choice of I. If I coincides with A, then $A^2 = -E$. If I coincides with B, then

$$A^2 = (B+E)^2 = B^2 + 2BE + E^2 = -E + 2B + E = 2B.$$

Of course, other choices are also possible — giving other answers.

Among all these possibilities we now choose the one where $I = A$, i.e. I is obtained from E by a rotation through 90° in the positive direction (counterclockwise) — and study it in more detail in the next section.

1.7 Complex numbers

The points of the line PE are identified with real numbers. Now that we have introduced algebraic operations for the points of the plane, we can view the set of all points as a number system which is wider than real numbers. These numbers are called *complex numbers*. In the conventional notation for complex numbers, our pole P is denoted by 0, point E by 1, point I by i or $\sqrt{-1}$, and $a + bi$ is written instead of $aE + bI$. Here are, once again, definitions for algebraic operations on complex numbers in this standard notation:

$$\begin{aligned}
(a+bi)+(c+di) &= (a+c)+(b+d)i,\\
(a+bi)-(c+di) &= (a-c)+(b-d)i,\\
(a+bi)(c+di) &= (ac-bd)+(ad+bc)i,\\
\frac{a+bi}{c+di} &= \frac{ac+bd}{c^2+d^2}+\frac{bc-ad}{c^2+d^2}i.
\end{aligned}$$

To put it shortly, the operations are performed as if on polynomials in the "variable" i with the rule $i^2 = -1$ applied at all possibilities. To derive the formula for the quotient from this rule, both numerator and denominator should be multiplied by the same number $c - di$.

Two basic complex numbers 1 and i are referred to as the *real unit* and the *imaginary unit*, respectively.

Exercise 18. Perform the operations on complex numbers:

(a) $\dfrac{3-5i}{1+i} - i(3+4i) + \dfrac{1}{i}$,

(b) $\sqrt{3-4i}$,

(c) $(\frac{1}{2} - \frac{\sqrt{3}}{2}i)^{2004}$.

Let $z = a + bi$. The distance between the points z and 0 is called the *modulus*, or *absolute value* of the complex number z and denoted by $|z|$. Since a and b are Cartesian coordinates of the point z, we have $|z| = \sqrt{a^2 + b^2}$. For example, the modulus of both $\cos t + i \sin t$ and $\frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2}i$ is 1 for any value of the real number t.

The distance between the two points represented by complex numbers z and w is $|z-w|$, because the four points 0, w, z, $z-w$ form a parallelogram (see Fig.1.18).

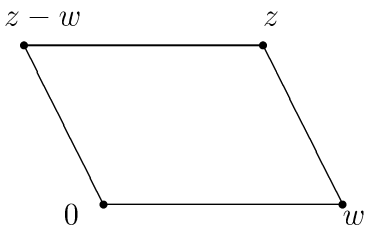

Figure 1.18: Complex parallelogram

Exercise 19. Find the set of all points z in the complex plane which satisfy:

(a) $|z+3| = 5$,

(b) $|z+4| = |z-2i|$,

(c) the sum of squares of distances from z to two fixed points is a given number.

The fact that $\sqrt{a^2+b^2}$ is the distance between two points provides a means to visualize certain purely algebraic problems.

Problem 8 *Prove the inequality*

$$\sqrt{a_1^2+b_1^2} + \sqrt{a_2^2+b_2^2} + \ldots + \sqrt{a_n^2+b_n^2}$$
$$\geq \sqrt{(a_1+a_2+\ldots+a_n)^2 + (b_1+b_2+\ldots+b_n)^2}.$$

Solution. Put $z_1 = a_1 + b_1 i$, $\ldots$, $z_n = a_n + b_n i$ and consider the broken line with vertices at 0, z_1, $z_1 + z_2$, $\ldots$, $z_1 + z_2 + \ldots + z_n$. The left hand side of the inequality is the total length of this line, while the right hand side is the distance between its endpoints.

Exercise 20. Prove the inequality

$$\sqrt{x_1^2+(1-x_2)^2}+\sqrt{x_2^2+(1-x_3)^2}+\ldots+\sqrt{x_{10}^2+(1-x_1)^2}>7.$$

for any real numbers x_1, ..., x_{10}.

The angle by which the half line 01 should be rotated counterclockwise in order for it to pass through the point z, is called the *argument* of the complex number z; it is denoted by $\arg z$. Here 0 and 1 are the points that correspond to the numbers 0 and 1.

Exercise 21. Find the arguments of the following complex numbers: 2, i, -3, $-2i$, $1+i$, $\sqrt{3}-i$.

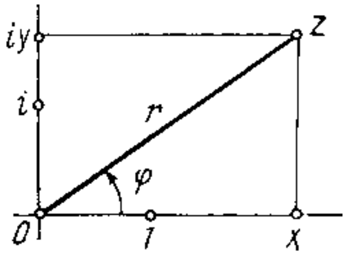

Figure 1.19: Polar coordinates

A complex number is completely defined, if one knows its modulus r and argument φ. Indeed, as you can see in Figure 1.19, $z=x+yi$, where $x=r\cos\varphi$, $y=r\sin\varphi$; thus

$$z=r(\cos\varphi+i\sin\varphi).$$

This expression is referred to as the *trigonometric form* of the complex number.

The correspondence $z\leftrightarrow(r,\varphi)$ between complex numbers and pairs of real numbers is not one to one. For one thing, the argument of the number 0 is undefined. On the other hand, the argument of any non-zero complex number, is only defined up to a whole number of complete rotations. Thus, one is free to choose 0, 2π, -2π, 4π, $\ldots$ as the argument of the number 1. Nevertheless, the pair (r,φ) is usually viewed as a pair of coordinates for the point z, called *polar coordinates*.

These coordinates are widely used in practice, e.g. in airport control centres: to determine the location of an aircraft, you first find the direction and then measure the distance.

The equations of some figures look much simpler when written in polar coordinates.

Exercise 22. (a) Plot the line given in polar coordinates by the equation $r=|\cos 3\varphi|$. (b) Find a polar equation which describes a flower with six petals similar to the one shown in Figure 1.20. Try to rewrite it in Cartesian coordinates.

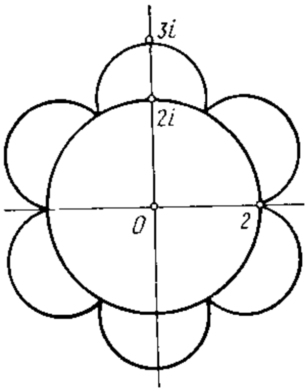

Figure 1.20: A flower in the complex plane

Multiplication of complex numbers looks simpler when written in terms of modulus and argument. In fact, the following two relations hold:

$$|zw| = |z||w|, \tag{1.4}$$
$$\arg(zw) = \arg z + \arg w. \tag{1.5}$$

The first one is a consequence of the remarkable identity $(ac - bd)^2 + (ad + bc)^2 = (a^2 + b^2)(c^2 + d^2)$. To prove the second one, let $z = r(\cos\varphi + i\sin\varphi)$, $w = s(\cos\psi + i\sin\psi)$. Then $zw = rs(\cos\varphi + i\sin\varphi)(\cos\psi + i\sin\psi) = rs((\cos\varphi\cos\psi - \sin\varphi\sin\psi) + i(\sin\varphi\cos\psi + \cos\varphi\sin\psi))$ which is simplified to

$$zw = rs(\cos(\varphi + \psi) + i\sin(\varphi + \psi)).$$

This proof is based on the well-known trigonometric formulas for the sine and cosine of the sum of two numbers. We will give another, more elegant, proof which only relies on elementary Euclidean geometry and, by the way, implies the trigonometric rules used above.

Consider two triangles with vertices 0, 1, z and 0, w, zw (Figure 1.21). Since $|w| : 1 = |zw| : |z| = |zw - w| : |z - 1|$, these two triangles are similar, so that their respective angles are equal. The equality of the two angles which are marked in Figure 1.21, proves that $\arg(zw) = \arg z + \arg w$.

OK. Multiplication of complex numbers means that their moduli get multiplied, while their arguments add up. Iterated as appropriate, this observation yields the formula for the power of a complex number in trigonometric notation:

$$[r(\cos\varphi + i\sin\varphi)]^n = r^n(\cos\varphi + i\sin\varphi).$$

Exercise 23. Prove that, if z is a complex number and a is real number such that $z + 1/z = 2\cos\alpha$, then $z^n + 1/z^n = 2\cos n\alpha$.

The trigonometric power formula is very convenient in problems such as Exercise 18c which you might have already tried. Let us do it together once again. Denote $\dfrac{1}{2} - \dfrac{\sqrt{3}}{2}i$ by

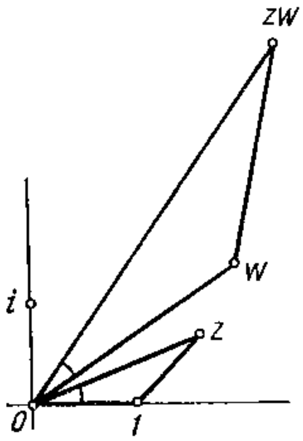

Figure 1.21: Product of complex numbers

ζ. Then $|\zeta| = 1$, $\arg\zeta = -\pi/6$, hence $|\zeta^{1998}| = 1^{1998} = 1$ and $\arg\zeta^{1998} = 1998 \cdot (-\pi/6) = -333 \cdot 2\pi$. This implies that $\zeta^{1998} = 1$.

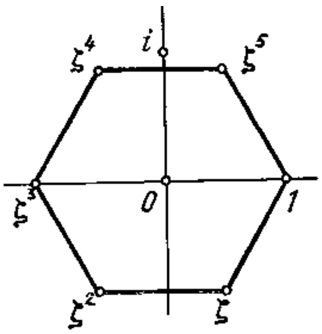

Figure 1.22: Complex roots of 1

Note that an integer power of the number ζ can occupy only one of the six positions in the plane — the vertices of the regular hexagon shown in Figure 1.22. Any of these six complex numbers is a power of ζ and plays the role of a sixth root of the number 1, because $(\zeta^k)^6 = (\zeta^6)^k = 1^k = 1$. In general, for any natural n, there are exactly n complex n-th roots of unity, arranged as the vertices of a regular polygon.

Problem 9 *Find the product of all diagonals and both sides that issue from one vertex of the regular n-gon inscribed into the circle of radius 1.*

Solution. Put the pole (number 0) in the centre of the polygon and the real unity (number 1) in the given vertex A_1. All the vertices are the roots of the equation $z^n - 1 = 0$, therefore all the vertices but A_1 also satisfy the equation $z^{n-1} + z^{n-2} + \ldots + z + 1 = 0$ obtained in dividing $z^n - 1$ by $z - 1$. Now compare

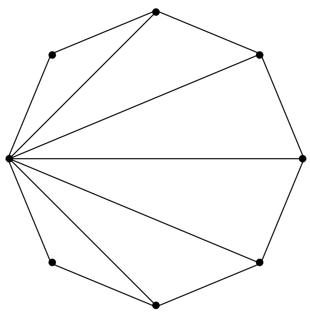

Figure 1.23: Sides and diagonals of a regular polygon

the two polynomials $z^{n-1}+z^{n-2}+\ldots+z+1=0$ and $(z-A_2)(z-A_3)\ldots(z-A_n)$. They are identically equal, because they have the same roots and equal leading coefficients. Hence, their values at $z=A_1$ are equal:

$$(A_1-A_2)(A_1-A_3)\ldots(A_1-A_n)=A_1^{n-1}+A_1^{n-2}+\ldots+A_1+1.$$

Recalling that $A_1=1$ by our choice, we obtain the answer:

$$|A_1-A_2||A_1-A_3|\ldots|A_1-A_n|=n.$$

Exercise 24. A regular polygon $A_1A_2\ldots A_n$ is inscribed into the circle of unit radius and A is an arbitrary point of this circle. Find the sum of squares of distances from A to all the vertices of the polygon.

Since division is an operation inverse to multiplication, it satisfies the formulas, inverse to 1.4:

$$\begin{aligned}\left|\frac{z}{w}\right| &= \frac{|z|}{|w|},\\ \arg\frac{z}{w} &= \arg z-\arg w\end{aligned}$$

The latter equality is interesting from the point of view of elementary geometry: it allows to express the magnitude of an angle through its vertex and two points belonging to its sides: $\varphi=\arg\dfrac{z_1-z_3}{z_2-z_3}$ (Figure 1.24). Here are two examples where this observation is applied: in the first one, we solve a geometric problem using the algebra of complex numbers, in the second one, vice versa, we solve an algebraic problem by a geometric method.

Problem 10 *Three squares are placed side by side as shown in Figure 1.25. Prove that the sum of $\angle KAH$, $\angle KDH$ and $\angle KFH$ is a right angle.*

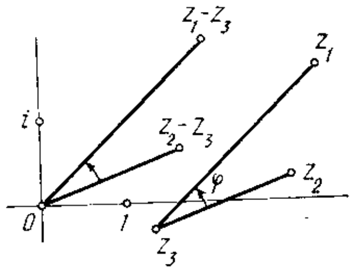

Figure 1.24: Angle expressed through complex numbers

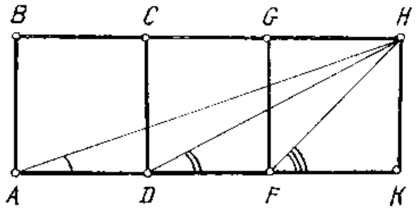

Figure 1.25: Sum of three angles

Solution. Evidently, $\angle KFH = \pi/4$, so we have to prove that $\angle KAH + \angle KDH = \pi/4$, too. Assuming that $A = 0$, $D = 1$ and $B = i$, we have: $F = 2$, $K = 3$, $H = 3 + i$. Therefore, $\angle DAH = \arg \frac{H-A}{D-A} = \arg(3 + i)$, $\angle FDH = \arg \frac{H-D}{F-D} = \arg(2 + i)$, whence $\angle DAH + \angle FDH = \arg(3 + i)(2 + i) = \arg(5 + 5i) = \pi/4$, which is just what was required.

Problem 11 *Prove that if z_1, z_2, z_3, z_4 are different complex numbers with equal absolute values, then*

$$\frac{z_1 - z_3}{z_2 - z_3} : \frac{z_1 - z_4}{z_2 - z_4}$$

is a real number.

Solution. The four given points lie on the same circle centred at 0. Points z_1 and z_2 split this circle into two arcs. The other two points z_3, z_4 can belong either to the same arc, or to different arcs. In the first case the angles $z_1z_3z_2$ and $z_1z_4z_2$ are equal, because they subtend the same arc. Therefore, $\arg \dfrac{z_1 - z_3}{z_2 - z_3} = \arg \dfrac{z_1 - z_4}{z_2 - z_4}$ and $\arg \dfrac{z_1 - z_3}{z_2 - z_3} : \dfrac{z_1 - z_4}{z_2 - z_4} = 0$, i.e. the number in question is real and positive. In the second case the two angles $z_1z_3z_4$ and $z_2z_4z_1$ have the same orientation and together make $180°$. Therefore, the number in question is real and negative.

The assertion of Problem 11 evidently generalizes to any set of four complex numbers that belong to an arbitrary circle or straight line in the plane. The converse is also true: if the given expression is real, then the four numbers must belong either to the same circle or to the same straight line.

Exercise 25. Let $c_1, c_2, \ldots, c_n$ be the vertices of a convex polygon. Prove that all complex roots of the equation

$$\frac{1}{z-c_1} + \frac{1}{z-c_2} + \ldots + \frac{1}{z-c_n} = 0$$

are inner points of this polygon.