Chapter 2

Plane Movements

Plane movements are transformations of the plane that do not change the lengths of segments and, as a consequence, preserve all parameters of geometric figures, such as areas, angles etc.

We begin this chapter with the discussion of some well-known problems of elementary geometry that allow a short solution using plane movements. All these problems share the same underlying idea: change the position of certain parts of the given geometrical configuration in such a way that the hidden relations between the elements become transparent.

We then proceed to a detailed discussion of the composition of movements which will provide experimental material for the introduction of transformation groups in the next chapter.

2.1 Parallel translations

Definition 4 *A parallel translation (or simply a translation) is a transformation of the plane that sends every point A into the point A' such that $\overrightarrow{AA'}$ is equal to a given constant vector $\mathbf{v}$. This transformation is denoted by $T_{\mathbf{v}}$.*

Problem 12 *Two villages A and B are located across the river from each other. The sides of the river are rectilinear and parallel to each other. Where should one build the bridge MN so that the distance $AMNB$ be as small as possible? The bridge must be perpendicular to the sides of the river.*

Solution. If there were no river, the shortest path joining A and B would be a straight line. Let us try to get rid of the river by moving one of its sides towards the other perpendicularly until both sides coincide (Fig. 2.1). Let B' be the new position of the point B. The lengths of $AMB'B$ and $AMNB$ are equal. The position of point B' does not depend on the choice of the place for the bridge. Hence we only have to minimize the distance AMB', which is can be done simply by making AMB' a straight line.

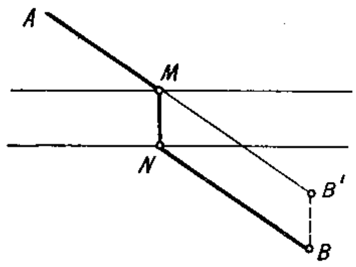

Figure 2.1: Bridge over a river

Exercise 26. Construct the shortest path that connects two points A and B separated by two rivers (Fig. 2.2). Both bridges must be perpendicular to the sides of the rivers.

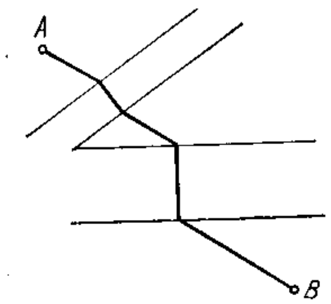

Figure 2.2: Two bridges over two rivers

Problem 13 *Inscribe a given vector into a given circle (i.e. construct a chord of a given circle which is equal and parallel to a given segment).*

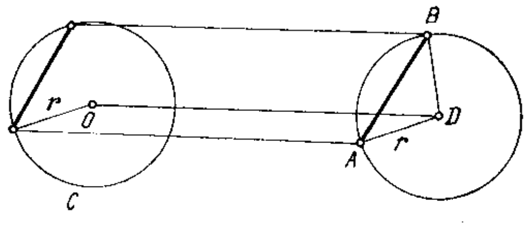

Figure 2.3: Inscribing a vector into a circle

Solution.

Let AB be the given vector and C the given circle with centre O and radius r (Fig. 2.3). We have to move AB keeping it parallel to itself towards C so that it gets inscribed into the circle. In fact, it is much easier to perform the reverse operation: move the circle in the opposite direction so that in the new position it will pass through both endpoints of the vector, A and B. To do so, we construct the triangle ABD such that $AD = BD = r$. The point D is the centre of the moved circle. Now if we translate the points A and B by the vector $\overrightarrow{DO}$, we will obtained the segment inscribed into the initial circle.

Here are two more problems which can be solved using parallel translation.

Exercise 27. Inscribe a given vector into a given triangle, i.e. find a segment whose endpoints lie on the sides of the given triangle and which is equal and parallel to a given segment.

Exercise 28. Construct a trapezium if the lengths of its parallel sides and diagonals are known.

2.2 Reflections

Definition 5 *Let l be a line in the plane. The reflection with respect to l is a transformation of the plane that sends every point A into the point A' such that l is the perpendicular bisector of the segment AA'. This transformation is denoted by S_l and also called axial symmetry with axis l.*

Problem 14 *Two points A and B are on one side of the straight line l. Find the point $M \in l$ such that the length of the broken line AMB is minimal. If you prefer 'real life' problems, you may imagine a person with an empty bucket at point A, a fire at point B and a straightline river l.*

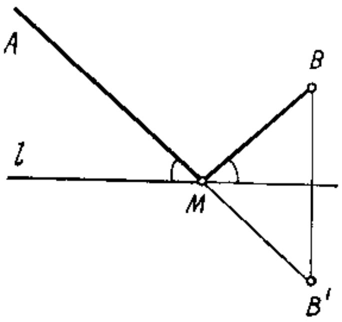

Figure 2.4: Shortest path

Solution. If both points A and B were situated on different sides of the line l, the solution would be a straight line AB. Let us try to reduce our problem to this case by reflecting the given point B in the line l (see Fig. 2.4). If B' is the image of B, then the lines AMB and AMB' have equal lengths for any arbitrary position of the point $M \in l$. To minimize this distance, we draw the straight line AB' and set M to be the intersection point of this line with l. Note that in this case the angles formed by either of the two lines AM and BM with l, are the same, which agrees with the well-known law of optics.

Exercise 29. Inside an angle XOY, two points, A and B, are given. Among all broken lines $AMNB$ where $M \in XO$, $N \in YO$ find the line of minimal length[1].

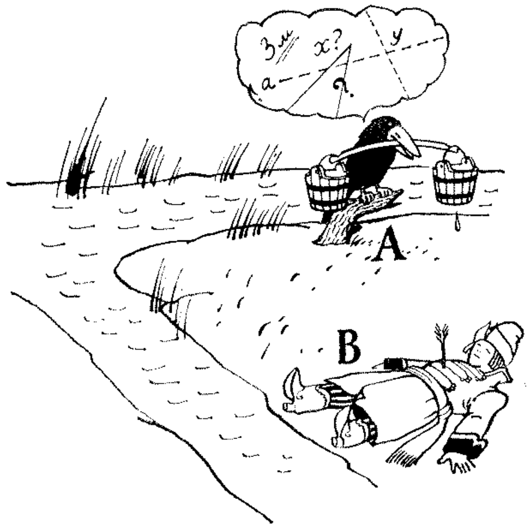

Figure 2.5: Two rivers of Exercise 29

We proceed with one more problem related to shortest paths.

Problem 15 *Into a given acute triangle inscribe a triangle of minimal perimeter.*

Solution. Let UVW be an arbitrary triangle inscribed into the given triangle ABC. Let K and L be the symmetric images of the point U with respect to the lines AB and BC (see Fig. 2.6). The paths $UVWU$ and $KVWL$ have equal lengths. To minimize this length among all triangles UVW with a fixed vertex U, we have to choose V and W so that $KVWL$ becomes a straight line, i.e. to set $V = M$ and $W = N$. Now among all triangles $\triangle UMN$ that correspond to different

[1]Figure 2.5 refers to a Russian folk tale where a raven has to bring two kinds of water, the 'dead' water and the 'live' water, to revive the prince.

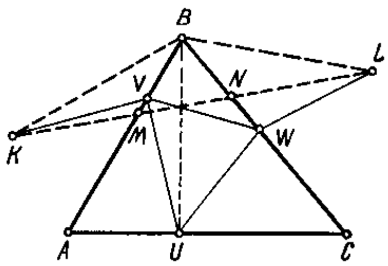

Figure 2.6: Inscribed triangle of minimal perimeter

positions of the point U, we will choose the one with the minimal perimeter, it will give the solution to the problem. We have to find the position of U for which the segment KL is shortest.

Note that $\triangle BKL$ is an isosceles triangle with $BK = BU = BL$. Its angle at vertex B does not depend on the position of the point U: $\angle KBL = 2\angle ABC$. Therefore, to minimize the length of the side KL we have to make sure that the side BK is as small as possible. Since $BK = BU$, this minimum is attained when U is the base point of the altitude drawn in the triangle ABC from the vertex B: $BU \perp AC$.

Because of the symmetry between the three points U, V and W, we conclude that V and W in the minimal triangle UVW are also basepoints of the corresponding altitudes of the triangle ABC.

Exercise 30. Construct a triangle, if one of its vertices and the three lines that contain its bisectors, are given.

Exercise 31. A ray of light enters an angle of 45° formed by two mirrors. Prove that after several reflections the ray will exit the angle moving along a line parallel to its initial trajectory. Are there other values of the angle with the same property?

2.3 Rotations

Definition 6 *Let O be a point in the plane and φ a real number, understood as an angle. The rotation around O through angle φ is a transformation of the plane that sends every point A into the point A' such that $|OA| = |OA'|$ and $\angle AOA' = \varphi$, where the angle is counted with sign, counterclockwise direction being considered as positive. This transformation is denoted by R_O^φ.*

Look at Figure 1.17. It is evident that the sum of all vertices of a regular polygon with an even number of vertices over its centre P is equal to P (see p.9 for the definition of the point addition). Indeed, the set of vertices splits into pairs of mutually opposite points. It is not so easy to prove the same property for the polygon with an odd number of vertices. If you try to directly compute the coordinates of all the vectors, you will have to deal with rather complicated trigonometric expressions. However, the problem looks difficult only as long as the plane does not move.

Problem 16 *Prove that the sum of vertices of a regular polygon over its centre P coincides with P.*

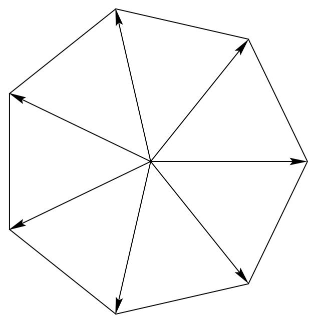

Figure 2.7: Sum of vertices of a polygon

Solution. Let n be the number of vertices. Under the rotation through $360/n$ degrees around P the given polygon goes into itself. Therefore, the sum of vertices remains unchanged. But in the plane there is only one point that goes into itself under a rotation: it is the centre of the rotation.

Exercise 32. A point M lies inside a convex polygon. Perpendiculars are drawn from M to all sides of the polygon and, on each of these half-lines, a point A_i is taken whose distance from M equals the length of the corresponding side. Prove that the sum of all these points over M is zero.

Problem 17 *Construct a regular triangle, if the distances of its vertices from a given point D are a, b and c.*

Solution. Everyone knows how to construct a triangle by the lengths of its sides. Unfortunately, the three segments a, b, c in Figure 2.8 do not form a triangle. Let us rotate the plane by $60°$ around the point C. The point B goes into A and D goes into D'. A rotation preserves distances, therefore the lengths of the sides of the triangle $\triangle ADD'$ are a, b and c. We will construct this triangle first, then find the point C ($\triangle CDD'$ is regular), then find the point B.

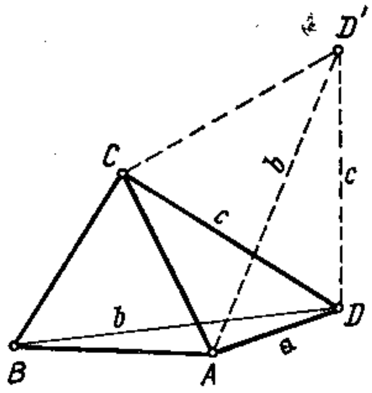

Figure 2.8: Constructing a regular triangle

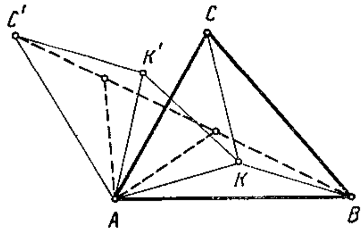

Figure 2.9: Minimize the sum of distances

Exercise 33. Construct a regular triangle whose vertices lie on three given parallel lines, one on each.

Problem 18 *Inside a given triangle find the point the sum of whose distances from the vertices is minimal.*

Solution. Let K be an arbitrary point inside the triangle ABC. Rotate the points C and K around A counterclockwise through $60°$ and denote their new positions by C' and K' (see Figure 2.9). The sum of the three distances in question, $AK+BK+CK$, equals the length of the broken line $C'K'KB$. It is minimal, if K and K' lie on the straight line BC'. Thus, the optimal position for K is the point K_0 on BC' such that the angle AK_0C' is $60°$ or, in other words, $\angle AK_0B = 120°$. By symmetry, we also have $\angle BK_0C = \angle CK_0A = 120°$.

Note that our analysis as well as the answer given hold only for triangles whose angles are smaller than $120°$. We leave it to the reader to guess the answer in the opposite case.

Exercise 34. M is an arbitrary point inside a square $ABCD$. Draw four lines which pass through A, B, C and D and are perpendicular to BM, CM, DM and AM, respectively. Prove that these four lines pass through a common point.

The rotation through 180° is also referred to as *half turn*, or *central symmetry*. Speaking about central symmetries, we will often leave 180° out of notations, writing R_A instead of $R_A^{180°}$. Here are two problems where this kind of movements is used.

Exercise 35. Through the intersection point of two circles draw a line on which these circles cut equal chords.

Exercise 36. There is a round table and an unlimited number of equal round coins. Two players take turns at placing the coins on the table in such a way that they do not touch each other. What is the winning strategy for the first player?

2.4 Functions of a complex variable

We return once again to Problem 16 (see page 38). Apart from the geometric solution given above, this problem also has an algebraic solution. To explain it, we introduce a *complex structure* in the plane. More precisely, we choose a one-to-one correspondence between complex numbers and points in the plane in such a way that 0 corresponds to the centre of the polygon and 1 corresponds to one of its vertices. If ζ is a vertex adjacent with 1 (in the counterclockwise direction) then the remaining vertices are $\zeta^2, \ldots, \zeta^{n-1}$. We are interested in $x = 1 + \zeta + \zeta^2 + \ldots + \zeta^{n-1}$. Since $\zeta^n = 1$, we have $x\zeta = \zeta + \zeta^2 + \ldots + \zeta^{n-1} + 1 = x$ which implies that $x = 0$, because $\zeta \neq 1$.

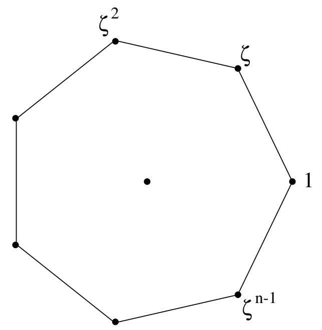

Figure 2.10: Regular polygon with complex vertices

Note that this algebraic proof is essentially the same as the geometric proof given above. More exactly, it is nothing but the translation of the geometric argument into algebraic language. Indeed, the new proof is based on the fact that the only number which satisfies the equation $\zeta x = x$, is $x = 0$. But what happens with a complex number when it is multiplied by ζ? According to the general rule, its modulus remains the same, because $|\zeta| = 1$, and its argument increases by $360°/n$. In the geometric terms, this means that the corresponding point rotates through $360/n$ degrees around the centre of the polygon.

In general, if points are viewed as complex numbers, then transformations of the plane, and in particular, plane movements, should be understood as *functions of a complex variable* $w = f(z)$, where z denotes an arbitrary point and w its image. For example, a rotation around 0 is represented by the function $w = \alpha z$, where $|\alpha| = 1$ (we have in this case $\alpha = \cos\varphi + i\sin\varphi$ where φ is the angle of rotation). It is likewise evident that the formula for a parallel translation is

$$w = z + a, \tag{2.1}$$

where a is a certain complex number.

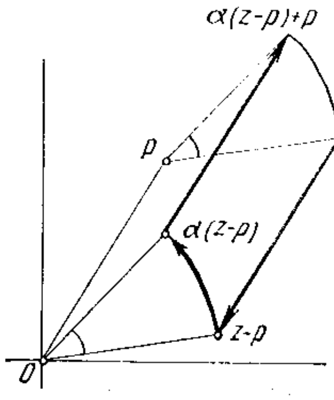

Figure 2.11: Plane movement in complex coordinates

Now let us derive the formula for the rotation of the complex plane around an arbitrary point p. Figure 2.11 shows that the rotation of the point z around p through angle φ can be split into three steps:

1. translation $z \mapsto z - p$;
2. rotation around the origin $z - p \mapsto \alpha(z - p)$;
3. inverse translation $\alpha(z - p) \mapsto \alpha(z - p) + p$.

The rotation around p through angle φ is thus described by the function

$$w = \alpha z + (1 - \alpha)p, \tag{2.2}$$

where $\alpha = \cos\varphi + i\sin\varphi$.

Parallel translations and rotations are referred to as *proper movements*. This expression is accounted for by the fact that one does not have to leave the plane in order to physically effectuate one of these transformations, whereas a reflection in a line requires a rotation of the plane in the surrounding three-space.

Theorem 3 *The set of proper movements of the plane coincides with the set of all transformations described by the functions of a complex variable*

$$w = \alpha z + m, \tag{2.3}$$

where α and m are complex numbers and $|\alpha| = 1$.

Proof Formulas 2.1 and 2.2 imply that any proper movement of the plane is described by a linear functions of type 2.3.

We will prove that the converse also holds, i.e. that every function 2.3 defines a proper movement. Indeed, if $\alpha = 1$, then Eq. 2.3 becomes Eq. 2.1 and we deal with a parallel translation. If $\alpha \neq 1$, then 2.3 can be rewritten as

$$w = \alpha z + m = \alpha(z - \frac{m}{1-\alpha}) + \frac{m}{1-\alpha}$$

which is the expression of the rotation around $p = m/(1-\alpha)$ through the angle φ such that $\cos\varphi + i\sin\varphi = \alpha$.

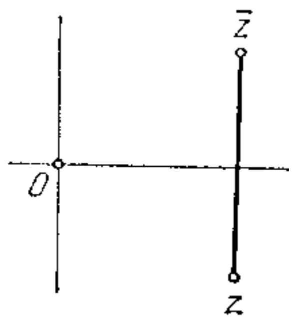

Figure 2.12: Complex conjugation

To find a similar description for the improper movements, for example reflections, apart from addition and multiplication of complex numbers, we need one more operation: *complex conjugation*. The conjugate of the number $z = x+iy$ is defined as $\bar{z} = x-iy$. Geometrically, conjugation corresponds to the reflection in the real axis (Fig. 2.12). Recall that we have already used conjugation to derive the formula for the quotient of two complex numbers (see p. 24).

Exercise 37. Prove the following formulas for the reflection in the line $y = kx + b$:

$$w = \bar{z} + 2bi, \quad \text{if} \quad k = 0, \tag{2.4}$$

$$w = \frac{1-k^2+2ki}{1+k^2}(\bar{z} + \frac{b}{k}) - \frac{b}{k}, \quad \text{if} \quad k \neq 0, \tag{2.5}$$

(note that $(1-k^2+2ki)/(1+k^2) = \alpha^2$, where $\alpha = \cos\varphi + i\sin\varphi$ and φ is the angle between the given line and axis x).

In the following example we use the algebra of complex numbers to solve a geometrical problem.

Problem 19 *A pirate is hunting for a hidden treasure. According to the letter he has got, he has to go to the Treasure Island, find two trees A and B, a rock C (Figure 2.14) and dig* *for the buried treasure in the point K which is the middle point of the segment DE, where D is obtained by rotating C around A clockwise through $90°$, and E is obtained by rotating C around B counterclockwise through $90°$. When the pirate arrived at this place, he found that the trees A and B are there, but the rock C disappeared. Is it still possible to recover the position of point K?*

Figure 2.13: A pirate

Solution. Let us introduce a complex structure in the plane, i.e. associate the points with complex numbers, in such a way that A corresponds to 0, while B and C correspond to numbers b and c (see Figure 2.14).

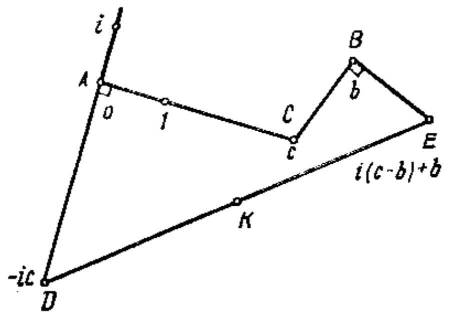

Figure 2.14: Where is the treasure?

Then, by formula 2.2, points D and E are represented by numbers $-ic$ and $i(c-b)+b$, therefore, point K is $\dfrac{1-i}{2}b$. As this expression does not involve c, we see that the position of the hidden place does not depend on the choice of the point C. We also see that K is the vertex of the isosceles right triangle with hypotenuse AB and as such can be found by our treasure hunter.

Exercise 38. Two sides of a triangle are rotated through 90° around their common vertex in opposite directions. Prove that the line joining the new endpoints is perpendicular to the median of the triangle.

2.5 Composition of movements

Given two movements of the plane, f and g, one can construct a third movement $g \circ f$, the *composition*, or the *product* of the given two, by performing first f, then g.

Definition 7 *The composition $f \circ g$ of two movements f and g is defined by the relation*

$$(f \circ g)(x) = f(g(x))$$

for any point x.

The transformation $f \circ g$ thus defined is really a movement, because it evidently preserves the distances between the points. In this section, we will study the composition of special types of movements: translations, reflections and rotations.

Problem 20 *Find the composition of two reflections.*

Solution. Let us denote by S_l the reflection in the line l. Suppose that two lines, l and m, are given, and we have to find the composition $S_m \circ S_l$. Let A' be the image of an arbitrary point A under the movement S_l, and A'', the image of A' under S_m.

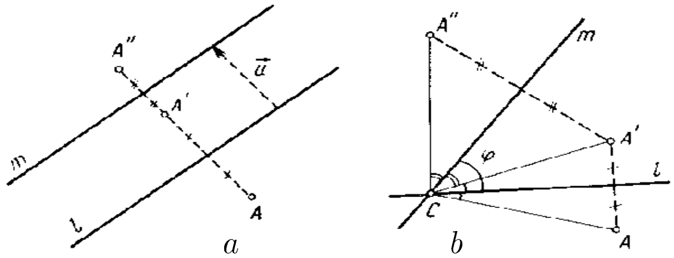

Figure 2.15: Product of two reflections

We first consider the case when the two lines l and m are parallel to each other (Fig. 2.14a). Then all the three points A, A', A'' lie on one line, perpendicular to l and m, and the distance between the points A and A'' is twice the distance between the lines l and m, independent of the position of A. Therefore, the composition of the two reflections S_l and S_m has the same effect as the translation by the vector $2\mathbf{u}$ where $\mathbf{u}$ is the vector perpendicular to l and m, of length equal to the distance between the two lines and directed from l to m:

$$S_m \circ S_l = T_{2\mathbf{u}}. \tag{2.6}$$

Now suppose that the lines l and m meet in a certain point C (Fig. 2.14b). If φ is the angle between l and m, then, as you can see from the figure, $\angle ACA'' = 2\varphi$. Note also that all the three points A, A' and A'' are at the same distance from C. It follows that

$$S_m \circ S_l = R_C^{2\varphi}, \tag{2.7}$$

where $R_C^{2\varphi}$ denotes the rotation around C through the angle 2φ (clockwise if $\varphi < 0$ and counterclockwise if $\varphi > 0$).

The reader may wish to consider other locations for the point A in the plane, different from that of Fig. 2.15b, and make sure that formulas 2.6 and 2.7 are always true. Mind that the angle φ should be measured from line l to line m, i.e., for example, $\varphi = \pi/4$ means that the line m can be obtained from line l by a positive (counterclockwise) rotation through 45°.

Formula 2.7 implies, by the way, that the composition of two movements in general depends on the order in which they are taken: thus, $S_l \circ S_m$ is a movement inverse to $S_m \circ S_l$.

Exercise 39. Let l, m and n be three lines meeting at one point. Find the movement $(S_n \circ S_m \circ S_l)^2 = S_n \circ S_m \circ S_l \circ S_n \circ S_m \circ S_l$. We suggest that the reader first experiment by applying the given composition to an arbitrary point of the plane, and then prove the result using the formulas we have established.

Formulas 2.6 and 2.7, read from right to left, show how to decompose a translation or a rotation into a product of reflections. This decomposition is not unique, and the freedom we have in the choice of the axes of reflection may prove quite useful for the solution of a specific problem.

Problem 21 *Find the composition of two rotations.*

Solution. If the centres of both rotations coincide, then the answer is obvious:

$$R_A^{\varphi} \circ R_A^{\psi} = R_A^{\varphi+\psi}. \tag{2.8}$$

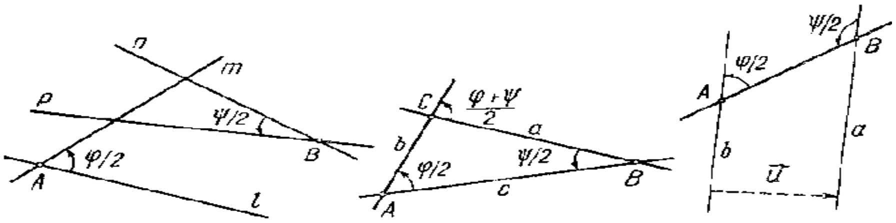

Figure 2.16: Product of two rotations

Now consider two rotations R_A^{φ} and R_B^{ψ} with different centres. To find their composition, we will represent each rotation as the product of two reflections and then use the formulas that we already know. We have: $R_A^\varphi = S_m \circ S_l$, where the lines l and m form the angle $\varphi/2$ at the point A, and $R_B^\psi = S_p \circ S_n$, where the lines n and p form the angle $\psi/2$ at the point B (see Fig. 2.16a). Then $R_A^\varphi \circ R_B^\psi = S_m \circ S_l \circ S_p \circ S_n$. This expression simplifies to $S_m \circ S_n$, when the two lines l and p coincide, because in this case $S_l \circ S_p = \mathrm{id}$ is the *identity* transformation, i.e. the transformation which takes every point into itself.

After this analysis, we start anew from Figure 2.16b. We denote by c the line joining A and B, then, rotating c around A through the angle $\varphi/2$ and around B through $-\psi/2$, we obtain the lines b and a. If the lines b and c have a common point, we denote it by C and in this case we can write:

$$R_A^\varphi \circ R_B^\psi = S_b \circ S_c \circ S_c \circ S_a = S_b \circ S_a = R_C^{\varphi+\psi}.$$

or, setting $\alpha = \varphi/2, \beta = \psi/2, \gamma = \pi - \alpha - \beta$,

$$R_A^{2\alpha} \circ R_B^{2\beta} = R_C^{-2\gamma}, \tag{2.9}$$

where C is the third vertex of the triangle with two vertices A and B and angles at these vertices equal to α and β; γ is the angle of this triangle at C.

After both parts of equation 2.9 are multiplied by $R_C^{2\gamma}$ on the right, it takes a more symmetric form

$$R_A^{2\alpha} \circ R_B^{2\beta} \circ R_C^{2\gamma} = \mathrm{id}\,. \tag{2.10}$$

The converse is also true: if the three points A, B, C and three angles α, β, γ between 0 and 180° satisfy equation 2.10, then α, β and γ are equal to the angles of the triangle ABC.

Equality 2.10 can be checked directly. Since $2\alpha + 2\beta + 2\gamma = 360°$, the composition $R_A^{2\alpha} \circ R_B^{2\beta} \circ R_C^{2\gamma}$ is a parallel translation. To prove that it is identical, it suffices to check that it has one fixed point. But Figure 2.17 shows that the point A remains fixed under the successive mappings $R_C^{2\gamma}$, $R_B^{2\beta}$, $R_A^{2\alpha}$.

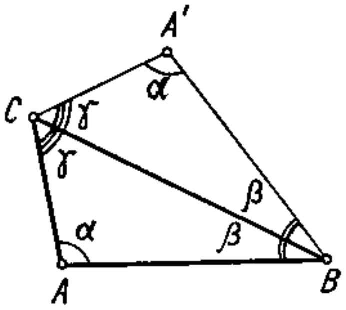

Figure 2.17: Composition of three rotations

Should the lines a and b be parallel (this happens when $\varphi+\psi$ is a multiple of 2π), then

$$R_A^{2\alpha} \circ R_B^{2\beta} = T_{2\mathbf{u}}, \tag{2.11}$$

where $\mathbf{u}$ is defined in Figure 2.16c.

Using complex numbers, one can derive an algebraic formula for the composition of two rotations. We take a complex number z and apply successively first the rotation R_B^{ψ}, then the rotation R_A^{φ}. According to formula 2.2, we can write:

$$\begin{aligned} R_B^{\psi}(z) &= q(z-b)+b, \\ R_A^{\varphi}(w) &= p(w-a)+a, \end{aligned}$$

where $p=\cos\varphi+i\sin\varphi$, $q=\cos\psi+i\sin\psi$. Now we substitute $R_B^{\psi}(z)$ instead of w and try to rewrite the result in a similar form:

$$\begin{aligned} (R_A^{\varphi}\circ R_B^{\psi})(z) &= p(q(z-b)+b-a)+a \\ &= pq(z-\frac{a-pa+pb-pqb}{1-pq})+\frac{a-pa+pb-pqb}{1-pq}. \end{aligned}$$

Note that $pq=\cos(\varphi+\psi)+i\sin(\varphi+\psi)$. Therefore, the result obtained means that

$$R_A^{\varphi}\circ R_B^{\psi} = R_C^{\varphi+\psi},$$

where the point C corresponds to the complex number

$$c=\frac{a-pa+pb-pqb}{1-pq}. \tag{2.12}$$

We see that geometric and algebraic arguments lead to two different formulas for the composition of rotations. We can benefit from this fact by deriving the following corollary:

If two vertices A and B of a triangle ABC correspond to complex numbers a and b, and the angles at these vertices are $\varphi/2$ and $\psi/2$, then the third vertex, as a complex number, is determined by formula 2.12.

We pass to examples where the composition of movements and the formulas we have found are used.

Problem 22 *Three regular triangles are built on the sides of an arbitrary triangle ABC (Fig. 2.18). Prove that their centres M, N, P form a regular triangle.*[2]

Solution. Triangles AMB, BNC and CPA are isosceles with obtuse angles of $120°$. Consider the composition of three rotations $F=R_P^{120°}\circ R_N^{120°}\circ R_M^{120°}$. Formulas 2.9 and 2.10 show that F is either a rotation or a parallel translation. Since the

[2]This problem is known as *the problem of Napoléon*, although the famous French general is not its author.

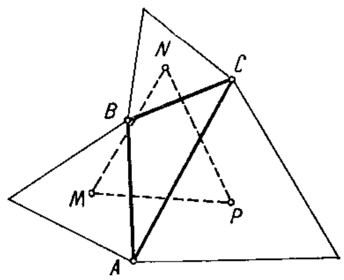

Figure 2.18: Problem of Napoléon

sum of the three angles of rotation is 360°, F must be a parallel translation. Let us trace how the point A is moved by F. It is clear that $R_M^{120°}(A) = B$, $R_N^{120°}(B) = C$, $R_P^{120°}(C) = A$ and thus $F(A) = A$. It follows that F is a translation by zero vector, i.e. the identity:

$$R_P^{120°} \circ R_N^{120°} \circ R_M^{120°} = \mathrm{id}$$

Comparing this to formula 2.10, we conclude that M is the third vertex of the triangle having two vertices at N and P and angles 60° and 60° at these vertices, i.e. a regular triangle.

Exercise 40. Find a solution of the previous problem based on computations with complex numbers.

Exercise 41. On the sides of an arbitrary quadrangle four squares are built. Prove that their centres form a quadrangle whose diagonals are mutually perpendicular and have equal length.

Exercise 42. Find the composition of

1. two central symmetries,
2. a central symmetry and a reflection.

Exercise 43. Construct a pentagon, given the midpoints of all its sides.

2.6 Glide reflections

We have studied three types of plane movements: translations, rotations and reflections. However, these three types do not cover all plane movements. For example, in Exercise 42, the product of a reflection and a central symmetry does not belong to any of these types.

Definition 8 *A glide reflection with axis l and vector v is a movement that consists in a reflection with respect to a line l and a translation by the vector v, which is supposed to be parallel to the line l (see Fig. 2.19).*

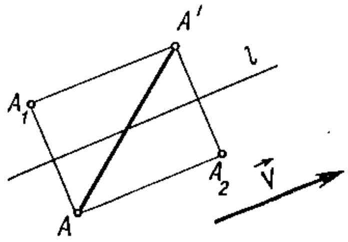

Figure 2.19: Glide reflection

Denoting the glide reflection by U_l^v, we can write the definition as $U_l^v = T_v \circ S_l = S_l \circ T_v$. The movements S_l and T_v commute, i.e. the two products taken in different order are indeed equal, because the figure $AA_1A'A_2$ is always a rectangle.

Glide reflections, like all other types of plane movements, can be successfully used for solving geometrical problems.

Problem 23 *Construct a line parallel to the side AC of a given triangle ABC and intersecting its sides AB and BC in points D and E such that $AD = BE$.*

Solution. The solution relies on the following two properties of glide reflections which immediately follow from Fig. 2.19:

1. the midpoint of a segment joining an arbitrary point with its image under a glide reflection always lies on the axis;
2. the axis of the glide reflection is preserved.

There is a glide reflection U which takes the half-line AB into the half-line BC. Its axis is the line NK where N is the midpoint of the segment AB while K belongs to BC and $BK = NB$. By the premises, $AD = BE$, hence $U(D) = E$ and the midpoint of DE must belong to the line NK. But, since $DE \parallel AC$, the midpoint of DE lies on the median BM. Therefore the three segments DE, BM and NK have a point in common, and the required construction can be effectuated in the following order. First we find the points N and K as mentioned above. Then we draw the median BM. Finally, we draw the line parallel to AC, through the intersection point of BM and NK. This is the desired line.

Exercise 44. A point and three straight lines are given. Draw a line l passing through the given point in such a way that its image under the three reflections with respect to the three given lines (in a prescribed order) would be parallel to l.

Exercise 45. Using complex numbers, find an algebraic formula for glide reflection.

2.7 Classification of movements

In the previous section, we have got acquainted with a new kind of plane movements. So far, we have encountered four types of plane movements: translations, rotations, reflections and glide reflections. A natural question arises: *are there any plane movements that do not belong to any of these four types?* The answer is given by the following theorem.

Theorem 4 *Any plane movement is either a translation, rotation, reflection or a glide reflection.*

Proof First of all, we note that a plane movement is completely defined by the images of three non-collinear points A, B, C. In fact, if A', B', C' are the images of these points, then for any point D there exists exactly one point D' whose distances from A', B', C' are equal to the distances of D from A, B, C.

The second useful observation is that for any two different points M and M' there is a reflection that carries M over to M'. In fact, this reflection is uniquely defined: its axis is the middle perpendicular to the segment MM'.

Using these two observations, we are going to decompose any plane movement as the product of several reflections. Note that we have already used this trick earlier: see the discussion of Problem 21.

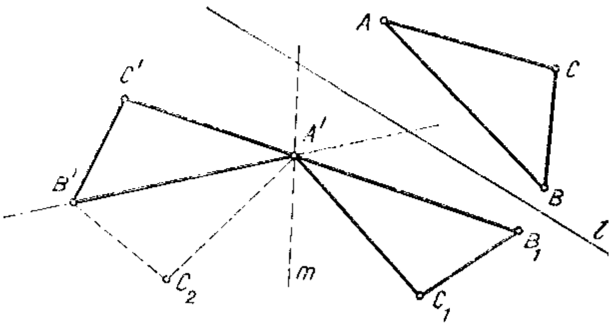

Figure 2.20: Decomposition of a plane movement into reflections

Let f be an arbitrary movement of the plane. Choose three non-collinear points A, B and C. Denote $f(A) = A'$, $f(B) = B'$, $f(C) = C'$. Suppose that A' is different from A, denote by S_l the reflection that takes A to A' and set $B_1 = S_l(B)$, $C_1 = S_l(C)$ (see Fig. 2.20). If B_1 is different from B', then we denote by S_m the reflection that takes B_1 to B', while preserving A', and set $C_2 = S_m(C_1)$. Finally, if $C_2 \neq C'$, we find a third reflection, S_n, which takes C_2 into C'. We thus see that in the worst case, when all the steps of the described procedure are necessary, f can be represented as the composition $S_n \circ S_m \circ S_l$. If some steps turn out to be unnecessary, we can represent f as one reflection or a composition of two reflections.

Now we will prove that the product of no more than three reflections is a movement belonging to one of the four types that we know. Indeed, one reflection is a reflection, and

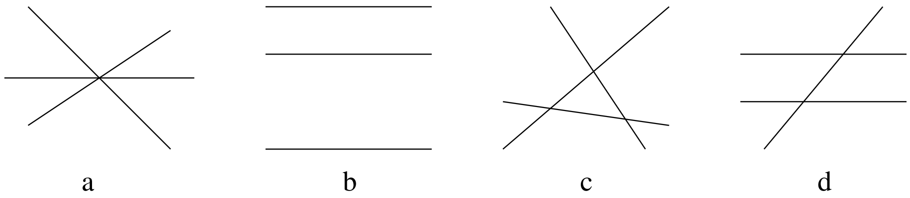

Figure 2.21: Three lines in the plane

that's it. Two reflections make either a rotation or a translation. The only nontrivial case is to analyze the product of three reflections $S_n \circ S_m \circ S_l$.

Three lines in a plane can be arranged in one of the four essentially different patterns depicted in Fig. 2.21. We will show that in cases (a) and (b), the product is a reflection, and in cases (c) and (d), a glide reflection.

In case (a), the composition of the two last reflections $S_m \circ S_l$ is a rotation through an angle equal twice the angle between the lines m and l. We can choose another line l' passing through the same point, so that $S_m \circ S_l = S_n \circ S_{l'}$. Then:

$$S_n \circ S_m \circ S_l = S_n \circ S_n \circ S_{l'} = S_{l'}.$$

In case (b), a similar argument holds.

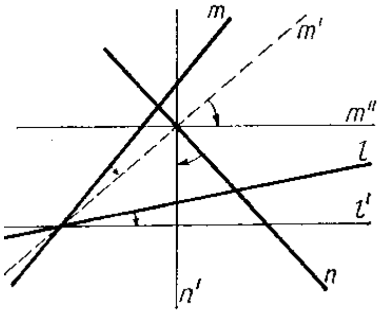

Figure 2.22: Adjusting two rotations

Now consider case (c). In the initial product of three reflections, we will make two changes. First of all, we change the product $S_m \circ S_l$ by an equal product $S_{m'} \circ S_{l'}$, where the line m' is chosen to be perpendicular to n (Fig. 2.22). We have $S_n \circ S_m \circ S_l = S_n \circ S_{m'} \circ S_{l'}$. Now we will change the product $S_n \circ S_{m'}$ by $S_{n'} \circ S_{m''}$, where n' is perpendicular to l'. We obtain:

$$f = S_n \circ S_m \circ S_l = S_n \circ S_{m'} \circ S_{l'} = S_{n'} \circ S_{m''} \circ S_{l'}.$$

Note that the lines l' and m'' are parallel, therefore the composition $S_{m''} \circ S_{l'}$ is a parallel translation in the direction of line n', and the whole movement is a glide reflection.

Finally, case (d) is reduced to case (c), because the composition $S_n \circ S_m \circ S_l$ remains the same, if two of the three lines (n and m or m and l) get rotated by the same angle.

Plane movements have a simple description in terms of complex functions. Theorem 3, proved above, says that translations and rotations correspond to functions $\alpha z + m$ with $|\alpha| = 1$.

Theorem 5 *The set of reflections and glide reflections of the plane coincides with the set of all transformations described by complex formulas*

$$w = \alpha \bar{z} + m, \tag{2.13}$$

where α and m are complex numbers and $|\alpha| = 1$.

Proof The fact that reflections and glide reflections are indeed described by such formulas follows directly from the result of Exercises 37 and 45.

To prove the second half of the theorem, note that the composition of transformation 2.13 with the standard reflection $z \mapsto \bar{z}$ is given by formula $z \mapsto \alpha z + m$, which, by theorem 3, is either a translation or a rotation.

2.8 Orientation

We have learned that there are four types of plane movements: translations, rotations, reflections and glide reflections. The movements belonging to the first two types can be represented as the product of an even number (two) of reflections; they are referred to as *proper* movements. The remaining two types are products of an odd number (one or three) of reflections; they are referred to as *improper* movements, because one has to exit the plane in order to physically implement such a movement.

The distinction between the two kinds of plane movements can be best understood using the notion of *orientation*.

We say that the ordered triple of non-collinear points A, B, C is *positively oriented*, if this ordering agrees with a counterclockwise walk around the triangle ABC, or, in other words, if in the sequence $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CA}$ every next vector is a turn to the left with respect to the previous one. If the order is clockwise, the triple is said to be *negatively oriented*.

Figure 2.23: Three pucks

Exercise 46. Three pucks form a triangle in the plane. A hockey-player chooses a puck and sends it along a straight line so that it passes between the two remaining pucks. Is it possible that after 25 shots each of the three pucks returns to its initial position?

It is remarkable that any movement f of the plane either preserves or reverses the orientation of all triples: the orientation of $f(A)$, $f(B)$, $f(C)$ either coincides with that of A, B, C for all triples — or differs from it for all triples. More specifically, it is easy to see that proper movements (translations and rotations) preserve the orientation, while improper movements (reflections and glide reflections) reverse it.

As a consequence of this observation, we obtain the following fact: the composition of an odd number of reflections can never be an identical transformation.

The notion of orientation has a simple interpretation in terms of complex numbers.

Exercise 47. Prove that the triple (z_1, z_2, z_3) is positively oriented if and only if the argument of the complex number $(z_3 - z_1)/(z_2 - z_1)$ is between 0 and 180°.

2.9 Calculus of involutions

Definition 9 *A transformation f is called an involution, if it is not identical, but its square is identical: $f \neq \mathrm{id}$, $f^2 = f \circ f = \mathrm{id}$. This is the same as to say that f is inverse to itself: $f = f^{-1}$, that is $f(A) = B$ if and only if $f(B) = A$.*

There are two types of involutive movements of the plane:

- R_A — half turn around point A (see sec. 2.3).
- S_l — reflection in a line l (see sec. 2.2),

We see that involutive movements correspond to geometric elements of two kinds: points and lines. This correspondence is in fact one-to-one, because different points and different lines produce different involutions. Therefore, the passage from geometrical objects to involutions preserves all information, and every fact about points and lines can be reformulated in terms of the corresponding involutions.

Problem 24 *Find the property of a pair of reflections S_l, S_m which is equivalent to the fact that the lines l and m are mutually perpendicular.*

Solution. The composition $S_m \circ S_l$ is a translation, if $m \parallel l$, or a rotation through 2φ, if m and l intersect at an angle φ. Unless $m = l$, this composition can never be identical. Its square $(S_m \circ S_l)^2$ is either a translation (in the first case) or a rotation through 4φ (in the second case). Hence the lines m and l are perpendicular if and only if

$$(S_m \circ S_l)^2 = \mathrm{id}, \tag{2.14}$$

i.e. the product $S_m \circ S_l$ is be an involution. Note that this involution is a half turn around the intersection point of the two given lines.

If we multiply equation 2.14 by S_m on the left and by S_l on the right, then it becomes

$$S_m \circ S_l = S_l \circ S_m, \tag{2.15}$$

i.e. that the two involutions S_m and S_l *commute*. This is required condition for the two lines to be perpendicular.

This is an appropriate moment to discuss the notions of commutativity and associativity. Multiplication of movements is in general non-commutative. As we have just seen, two different reflections commute if and only if the corresponding lines are perpendicular. But the composition of movements, like that of any arbitrary transformations, always has the property of *associativity*.

Let us be given four sets and three mappings between them arranged according to the scheme

$$V \xrightarrow{f} W \xrightarrow{g} X \xrightarrow{h} Y.$$

Then one can form the following compositions: $g \circ f : V \to X$, $h \circ g : W \to Y$, $h \circ (g \circ f) : V \to Y$, $(h \circ g) \circ f : V \to Y$. Associativity means that the two double compositions $h \circ (g \circ f)$ and $(h \circ g) \circ f$ coincide.

To find a formal proof of this almost evident property, it is enough to understand the meaning of composition. Thus, the mapping $g \circ f$ is defined by the equation $(g \circ f)(v) = g(f(v))$ for an arbitrary element $v \in V$. In the following chain of equations this definition is used several times:

$$(h \circ (g \circ f))(v) = h((g \circ f)(v)) = h(g(f(v))) = (h \circ g)(f(v)) = ((h \circ g) \circ f)(v). \tag{2.16}$$

Since the values of $h \circ (g \circ f)$ and $(h \circ g) \circ f$ on any element are the same, these two mappings coincide.

The following analogy might be useful to better understand the meaning of associativity. Imagine that f, g and h are the actions of putting on the socks, boots and overshoes, respectively. Then the composition $(h \circ g) \circ f$ means that one first puts on the socks, then puts the boots inside of overshoes, and puts this object on the feet in socks. The other composition $h \circ (g \circ f)$ means that one first puts the socks inside of the boots, puts on this combination and then puts the overshoes on top. Evidently, the result in both cases is the same!

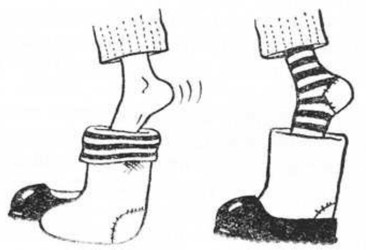

Figure 2.24: Associativity

The same analogy shows that the composition of operations in question is not commutative: to put on the socks, then the boots is not the same thing as to put on the boots, then the socks! However, commuting operations do exist, for example, putting on the sock on one foot and putting on the sock on another foot.

Using this analogy, it is easy to understand the formula for the operation, inverse to a composition of several operations. For example, if you put on the socks, then the boots, then the overshoes, then the inverse operation means that you first take off the overshoes, then the boots, then the socks:

$$(h \circ g \circ f)^{-1} = f^{-1} \circ g^{-1} \circ h^{-1}.$$

Now we return to the calculus of involutions in the plane.

Problem 25 *Express in terms of involutions the property of four points A, B, C, D to form a parallelogram.*

> **Solution.** According to Exercise 42, the composition $R_A \circ R_B$ is the translation by vector $2\,\overrightarrow{BA}$, while the composition $R_D \circ R_C$ is the translation by vector $2\,\overrightarrow{CD}$. The figure $ABCD$ (with this order of vertices!) is a parallelogram if and only if $\overrightarrow{BA} = \overrightarrow{CD}$ which is equivalent to the following condition for the four involutions:
>
> $$R_A \circ R_B = R_D \circ R_C.$$
>
> Multiplying both parts of this relation by appropriate involutions, we can rewrite it in two more equivalent forms: $R_A \circ R_B \circ R_C \circ R_D = \mathrm{id}$ and
>
> $$R_A \circ R_B \circ R_C = R_D. \qquad (2.17)$$
>
> The last equation may be viewed as a formula that expresses the fourth vertex of a parallelogram through the three given ones.

Exercise 48. Express the following geometrical facts as algebraic relations between the corresponding involutions: (a) point A belongs to the line l; (b) point A is the midpoint of the segment BC.

Exercise 49. Find the geometrical meaning of the following relations: (a) $R_A \circ S_l = S_l \circ R_B$; (b) $(S_n \circ S_m \circ S_l)^2 = \mathrm{id}$.

You can see that the algebra of involutions often provides a short and convenient way to write down facts about points and lines in the plane. Here is a more complicated example where this techniques is essential.

Problem 26 *Let M, N, P, Q be the centres of the four squares built on the sides of a quadrangle $ABCD$ (Fig. 2.25). What conditions should be imposed on $ABCD$ in order that $MNPQ$ be a square?*

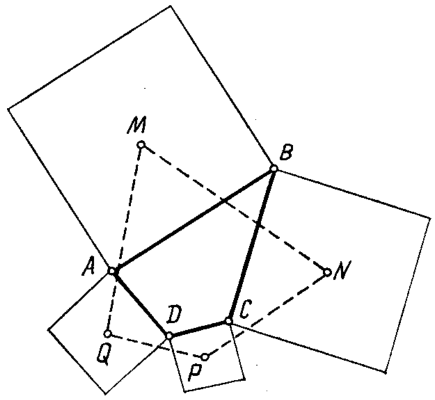

Figure 2.25: Squares on the sides of a quadrilateral

Solution. The reader knows that the diagonals of $MNPQ$ are always equal and mutually perpendicular (see Exercise 41). Therefore, quadrangle $MNPQ$ is a square if and only if it is a parallelogram. Using the result of Problem 25, we can write this as the following condition on the four involutions:

$$R_M \circ R_N \circ R_P \circ R_Q = \mathrm{id}\,. \tag{2.18}$$

By formula 2.9 we have:

$$\begin{aligned} R_M &= R_A^d \circ R_B^d, \\ R_N &= R_B^d \circ R_C^d, \\ R_P &= R_C^d \circ R_D^d, \\ R_Q &= R_D^d \circ R_A^d, \end{aligned}$$

where $d = 90°$ and we remind that if the angle of rotation is not specified, it is assumed to be $180°$. Upon substitution into (2.18), this gives:

$$R_A^d \circ R_B \circ R_C \circ R_D \circ R_A^d = \mathrm{id} \tag{2.19}$$

or, after the multiplication by R_A^{-d} both on the left and on the right,

$$R_B \circ R_C \circ R_D = R_A.$$

According to formula 2.17, this means that $ABCD$ is a parallelogram.

Hence, the necessary and sufficient condition for $MNPQ$ to form a square is that the initial quadrangle $ABCD$ be a parallelogram.