Chapter 6

Other Types of Transformations

The main protagonists of the book — transformation groups — have so far appeared in the particular case of groups of plane movements. In the present chapter, we are going to discuss other types of plane transformations: affine and projective transformations, similitudes and inversions. All these transformations can be described by fractional linear functions either of two real or one complex argument.

6.1 Affine transformations

Affine transformations constitute an important class of plane transformations which is a natural generalization of movements. In fact, the group of plane movements $\mathcal{M}$ is a subgroup of the affine group $\mathrm{Aff}(2, \mathbb{R})$. The transition from movements to affine transformations is easily achieved in coordinates.

Problem 49 *Find a description of plane movements in Cartesian coordinates.*

Solution. Let Oxy be a system of Cartesian coordinates in the plane (below referred to as 'old'). The image of Oxy under the movement f is another Cartesian coordinate system $O_1x_1y_1$, below referred to as 'new'.

If a point A has coordinates (p, q) in the old coordinate system, then its image A' has the same coordinates (p, q) in the new system.

For an orientation preserving movement (translation or rotation), it is easy to find the coordinates (p', q') of point A' in the old system, using Figure 6.1:

$$\left\{ \begin{array}{lcl} p' & = & p\cos\alpha - q\sin\alpha + p_0, \\ q' & = & p\sin\alpha + q\cos\alpha + q_0, \end{array} \right. \tag{6.1}$$

where α is the angle between the rays Ox and O_1x_1.

For an orientation reversing transformation we get similar formulas with q changed to $-q$:

$$\left\{ \begin{array}{lcl} p' & = & p\cos\alpha + q\sin\alpha + p_0, \\ q' & = & p\sin\alpha - q\cos\alpha + q_0, \end{array} \right. \tag{6.2}$$

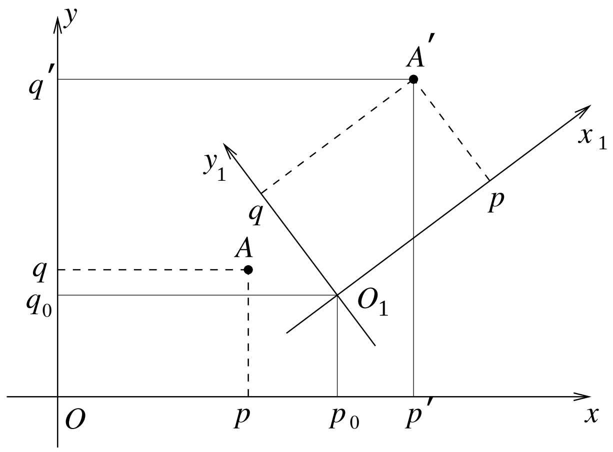


Figure 6.1: Movement in coordinates

Affine transformations are given by a formula similar to 6.1 and 6.2, where the coefficients may be arbitrary numbers, not necessarily sines and cosines.

Definition 31 *An affine transformation of the plane is a transformation that takes a point* (x, y) *to the point* (x', y') *according to the equations*

$$\left\{ \begin{array}{rcl} x' & = & ax + by + x_0, \\ y' & = & cx + dy + y_0, \end{array} \right. \tag{6.3}$$

Formulas 6.3 make sense for any values of the coefficients a, b, c, d. However, if we want to obtain a genuine (one-to-one) transformation of the plane, we must suppose that $ad - bc \neq 0$ (we remind that $ad - bc$ is the determinant of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$). Indeed, basic vectors $(1, 0)$ and $(0, 1)$ are taken, by the transformation 6.3, into vectors (a, c) and (b, d), and the area of the parallelogram constructed on these two vectors is equal to $ad - bc$.

Under an affine transformation, parallel lines go into parallel lines, but the angles are not necessarily preserved: a square may become an arbitrary parallelogram. Figure 6.2 shows an example of an affine transformation with the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ equal to $\begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix}$.

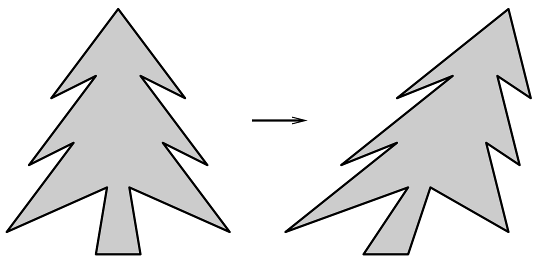


Figure 6.2: An affine transformation

Definition 32 *The group of affine transformations of the plane, denoted by* $\mathrm{Aff}(2,\mathbb{R})$*, consists of all affine transformations 6.3 with* $ad - bc \neq 0$*.*

The group of affine transformations acts transitively on the plane. The stable subgroup of the origin O is the *group of linear transformations* $\mathrm{GL}(2,\mathbb{R})$.

Definition 33 *A linear transformation of the plane is a transformation given in Cartesian coordinates by the equations*

$$\begin{cases} x' &= ax + by, \\ y' &= cx + dy, \end{cases} \tag{6.4}$$

The group $\mathrm{GL}(2,\mathbb{R})$ *consists of all such transformations with* $ad - bc \neq 0$*.*

Exercise 132. Let $\mathbb{R}^2$ denote the group of plane translations. Prove the isomorphism $\mathrm{Aff}(2,\mathbb{R})/\mathbb{R}^2 \cong \mathrm{GL}(2,\mathbb{R})$.

The fundamental property of affine transformations is that they preserve the ratio of points on straight lines (see p. 14): if a point C divides a segment AB in the ratio $k : l$, then its image C' will divide the corresponding segment $A'B'$ in the same ratio $k : l$. In fact, one can prove that $\mathrm{Aff}(2,\mathbb{R})$ *coincides* with the set of all transformations of the plane that take straight lines into straight lines and preserve the ratio of points on every line.

Affine transformations are useful for the solution of geometric problems where the statement is invariant under affine transformations, but the solution is easier for some special case of the construction.

To take a simple example, consider the property of the medians that we talked about in Chapter 1: *the three medians in any triangle meet in one point and this point divides each of them in the ratio* $2 : 1$ (see Exercise 10). By an affine transformation, the given triangle can be reduced to a regular one, for which the assertion is evident.

Exercise 133. Find another solution of Problem 4 (page 16), using affine transformations.

The notions of linear and affine transformations make sense in one-dimensional case, too. Linear transformations of the line have the form $x \mapsto ax$, affine transformations are described by the formula $x \mapsto ax + b$. The corresponding groups are distinguished by the condition $a \neq 0$ and denoted by $\mathrm{GL}(1,\mathbb{R})$ and $\mathrm{Aff}(1,\mathbb{R})$, respectively. Instead of real numbers $\mathbb{R}$ we can consider residues over a prime number, thus arriving to finite groups. On can also use complex numbers instead of real: the corresponding groups will come up later in this chapter (see section 6.3).

Exercise 134. (a) How many elements are there in the group $G = \mathrm{GL}(2,\mathbb{Z}_2)$? Among the groups that we considered earlier, find a group isomorphic to G. (b) The same questions for the group $\mathrm{Aff}(1,\mathbb{Z}_3)$.

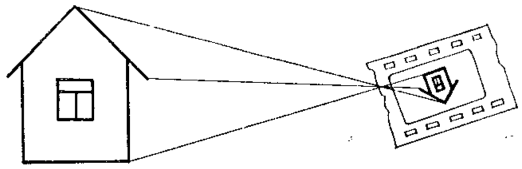


Figure 6.3: Photography as projection

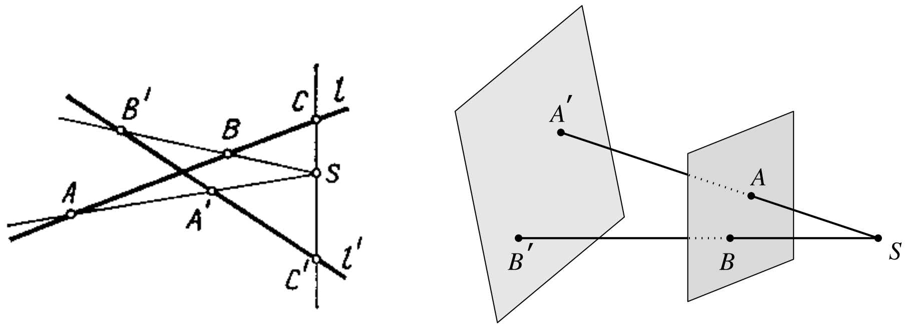
6.2 Projective transformations

The notion of projective transformations comes from daily life.

From the mathematical point of view, photography, as well as still life drawing, is a *perspective transformation*, or a *central projection*. Photography takes every point A of the given object into the point A' where the line AO (O being the optical centre of the camera) meets the plane of the film (Fig. 6.3). To make a drawing of nature, the artist does essentially the same thing, with the difference that the plane of the canvass is placed *between* the object and the eye.

It is clear that under such transformations straight lines go into straight lines, hence it is possible to study the perspective transformations of a line, too.

Definition 34 *Suppose that l and l' are two lines in the plane and S a fixed point in the same plane (Fig. 6.4a). A perspective transformation is a mapping $p : l \to l'$ that takes every point $A \in l$ into the intersection point A' of the lines SA and l'. Given two planes Π_1, Π_2 and a point S in space, one can define a perspective transformation $p : \Pi_1 \to \Pi_2$ in a similar way (see Fig. 6.4b).*



a *b*

Figure 6.4: Perspective transformation of lines (a) and planes (b)

Definition 35 *A projective transformation of a line or a plane into itself is a composition of several perspective transformations where auxiliary lines or planes are used.*

We begin the discussion of the properties of projective transformations with the following question. Suppose that we take a picture of a series of equidistant objects arranged along a line (e. g. trees on a road). It is clear that the images of these points on the picture do not need to be equidistant. It is also clear that they cannot be arbitrary and there must exist a certain invariant which is preserved by the projective transformations. Such an invariant must depend on more than three points, because the distance between two points can change and also the mutual relation of the three points can change arbitrarily: for example, in Fig. 6.4a the point B lies between A and C, but its image B' is no longer between the respective images A' and C'.

The remarkable fact is that a certain function of four points, called their *cross ratio*, or *anharmonic ratio*, does not change under projective transformations of the line.

Definition 36 *The cross ratio of points A, B, C and D is defined as follows:*

$$(A, B; C, D) = \frac{AC}{BC} : \frac{AD}{BD},$$

where the lengths of the lines, AC, BC, AD, BD are considered as signed numbers, positive or negative depending on the orientation of the given pair of points.

Theorem 13 *The cross ratio of four points is preserved under projective transformations.*

Proof It is enough to prove the invariance under perspective transformations.

To do so, let us express the areas of the triangles that you can see in Fig. 6.5, using two different formulas:

$$\begin{aligned}
S_{\triangle SAC} &= \frac{1}{2}h \cdot AC = \frac{1}{2}SA \cdot SC \sin\angle ASC, \\
S_{\triangle SBC} &= \frac{1}{2}h \cdot BC = \frac{1}{2}SB \cdot SC \sin\angle BSC, \\
S_{\triangle SAD} &= \frac{1}{2}h \cdot AD = \frac{1}{2}SA \cdot SD \sin\angle ASD, \\
S_{\triangle SBD} &= \frac{1}{2}h \cdot BD = \frac{1}{2}SB \cdot SD \sin\angle BSD.
\end{aligned}$$

Now,

$$\begin{aligned}
\frac{AC}{BC} : \frac{AD}{BD} &= \frac{S_{\triangle SAC}}{S_{\triangle SBC}} : \frac{S_{\triangle SAD}}{S_{\triangle SBD}} \\
&= \frac{SA \cdot SC \sin\angle ASC \cdot SB \cdot SD \sin\angle BSD}{SB \cdot SC \sin\angle BSC \cdot SA \cdot SD \sin\angle ASD} \\
&= \frac{\sin\angle ASC}{\sin\angle BSC} : \frac{\sin\angle ASD}{\sin\angle BSD}.
\end{aligned}$$

We see that the cross ratio of four points is expressed through the angles at which the corresponding segments are viewed from the centre S. These angles do not change under the perspective transformation, and the theorem is proved.

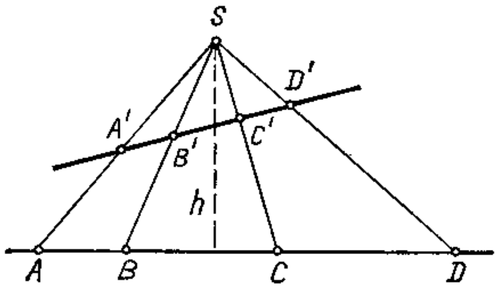


Figure 6.5: Derivation of the cross ratio

Problem 50 *Let A, B, C, D be four sequential equidistant trees on a straightline road and A', B', C', D' be their images on a photo. Suppose that the distance $A'B'$ is 6 cm and the distance $B'C'$ is 2 cm. What is the distance $C'D'$?*

Solution. We have:

$$\frac{AC}{BC} : \frac{AD}{BD} = \frac{2}{1} : \frac{3}{2} = \frac{4}{3}.$$

Denoting $C'D'$ by x, we have:

$$\frac{A'C'}{B'C'} : \frac{A'D'}{B'D'} = \frac{8}{2} : \frac{x+8}{x+2},$$

and from the equation

$$\frac{8(x+2)}{2(x+8)} = \frac{4}{3}$$

we find: $x = 1$.

A similar argument can be used to derive the general formula that expresses projective transformations in coordinates. Suppose that x is the coordinate of a variable point M on the line l and x' the coordinate of its image $M' \in l'$. Fix three different points A, B, C on l, denote by a, b, c their coordinates and let a', b', c' be the coordinates of their images A', B', C'. Then the relation

$$(A, B; C, M) = (A', B'; C', M')$$

can be rewritten as

$$\frac{c-a}{c-b} : \frac{x-a}{x-b} = \frac{c'-a'}{c'-b'} : \frac{x'-a'}{x'-b'}.$$

From this equation we can express x' in terms of x. The result looks like

$$x' = \frac{mx+n}{px+q}, \tag{6.5}$$

where m, n, p, q are certain constants depending on a, b, c, a', b', c'. Functions of this kind are referred to as *fractional linear functions.*

Note that in this formula the expression $mq - np$ (determinant of the matrix $\begin{pmatrix} m & n \\ p & q \end{pmatrix}$ must be different from 0. Otherwise the pair (m, n) would be proportional to the pair (p, q), and the fraction will give one and the value for all x.

Now suppose that m, n, p, q are four real numbers such that $mq - np \neq 0$. Is it true that the formula

$$f : x \mapsto \frac{mx + n}{px + q}$$

defines a one-to-one mapping of the real line $\mathbb{R}$ onto itself? The answer is negative: in fact, the point $x = -q/p$ (if $p \neq 0$) has no image under f.

Exercise 135. Indicate the real number that has no *inverse* image under this mapping.

We have already encountered these difficulties in one particular case (see 106). The way out is to introduce one more point ∞ (infinity) and extend the action of the projective transformation to the set $\bar{\mathbb{R}} = \mathbb{R} \cup \infty$ using the rules:

- $\dfrac{a}{0} = \infty$ for any $a \neq 0$,
- $\dfrac{m \cdot \infty + n}{p \cdot \infty + q} = \begin{cases} \frac{m}{p}, & \text{if } p \neq 0, \\ \infty, & \text{if } p = 0. \end{cases}$

Fractional linear functions with $mq - np \neq 0$ define one-to-one transformations of the extended line $\overline{\mathbb{R}}$.

Exercise 136. Check that the set of all transformations given by formula 6.5 with $mq-np \neq 0$ forms a group.

This group is called *the group of projective transformations of the (extended) real line* and denoted by $\mathrm{PGL}(1, \mathbb{R})$.

Exercise 137. Check directly, using formula 6.5, that the cross ratio is an invariant of the projective group acting on the set of quadruples of points.

The argument that led us to formula 6.5 shows that any triple of distinct points can be taken into any other such triple by a suitable projective transformation. This means that the action of the projective group on the set of triples has no nontrivial invariants.

We have already several times considered the group generated by two projective transformations $x \mapsto 1/x$ and $x \mapsto 1 - x$ (see Exercise 74, Problem 43 etc). This is not the only finite subgroup in the group of projective transformations.

Exercise 138. Prove that the two transformations $x \mapsto 1/x$ and $x \mapsto (x-1)/(x+1)$ generate a group of eight elements, isomorphic to D_4.

Exercise 139. Find all projective transformations of the line that have finite order.

That's all about projective transformations of the line. Now a few words about the plane.

The set of all projective transformations of the plane is a group, denoted by $\mathrm{PGL}(2,\mathbb{R})$. It contains the set of all affine transformations $\mathrm{Aff}(2,\mathbb{R})$ as a subgroup.

Exercise 140. Is $\mathrm{Aff}(2,\mathbb{R})$ a normal subgroup of $\mathrm{PGL}(2,\mathbb{R})$?

Arguments similar to those that we used for projective transformations of the line, imply the following theorems.

Theorem 14 *Projective transformations of the plane are those and only those transformations which are described by formulas*

$$\begin{cases} x' &= \dfrac{a_1x+b_1y+c_1}{a_0x+b_0y+c_0}, \\ y' &= \dfrac{a_2x+b_2y+c_2}{a_0x+b_0y+c_0} \end{cases} \tag{6.6}$$

in a Cartesian (or affine) coordinate system.

Theorem 15 *The group* $\mathrm{PGL}(2,\mathbb{R})$ *acts transitively on the set consisting of all quadruples of points no three of which are collinear.*

The last theorem may turn quite useful for solving some problems in elementary geometry. If a problem involves nothing but the incidence between points and lines, then we can make a projective transformation that takes any given quadrangle into another quadrangle, for which the solution might be easier. Remember, however, that projective transformations, in general, not only change the angles, distances and areas, but also ratios of segments on a line and ratios of areas of different figures. They only preserve straight lines and cross ratio.

Here is an example of such application.

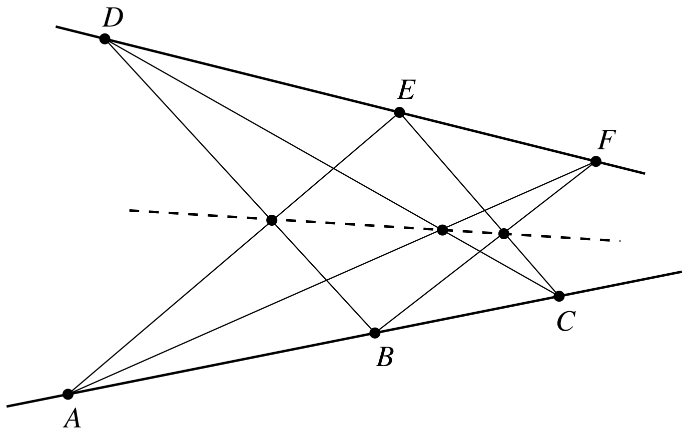


Figure 6.6: Pappus's theorem

Exercise 141. Prove the theorem of Pappus (see Figure 6.6): *if the three points A, B, C are collinear and the three points D, E, F are collinear, then the intersection points $AE\cap BD$, $AF\cap CD$ and $BF\cap CE$ are also collinear.*

6.3 Similitudes

Definition 37 *A similitude is a plane transformation that changes all distances by one and the same positive factor.*

Like affine transformations, similitudes constitute a class of plane transformations which is wider than the class of movements. It is clear from Definition 37 that the set of all similitudes is a transformation group.

The simplest type of similitudes, different from movements, is provided by *homotheties.*

Definition 38 *A homothety H_A^k with centre A and coefficient $k \neq 0$ is the transformation that takes every point M into the point M' such that $\overrightarrow{AM'} = k \cdot \overrightarrow{AM}$ (see Figure 6.7).*

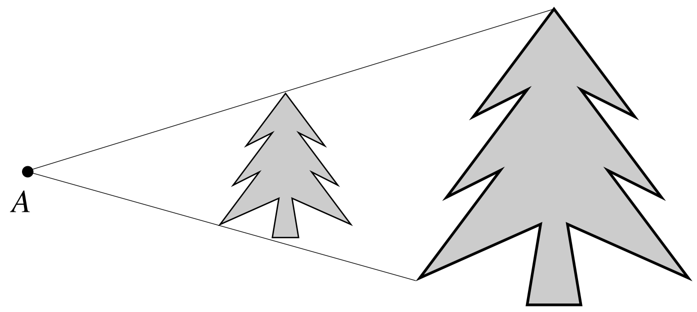


Figure 6.7: Homothety

The set of all homotheties of the plane does not constitute a group, but the set of all homotheties with a fixed centre does. Using a complex coordinate z, such transformations can be described by the formula $z \mapsto kz$, where k is a non-zero real number. This group is thus isomorphic to $\mathbb{R}^\star$, the multiplicative group of non-zero real numbers.

Exercise 142. Prove the isomorphism $\mathrm{GL}(2,\mathbb{R})/\mathbb{R}^\star \cong \mathrm{PGL}(1,\mathbb{R})$.

Below are some examples showing the usage of homotheties in elementary geometry.

Problem 51 *In a given triangle ABC inscribe a square in such a way that two of its vertices should belong to one side of the triangle and the remaining two vertices should lie on the other two sides.*

Solution. It is very easy to construct a square with three vertices satisfying the requirements of the problem (square $KLMN$ in Fig. 6.8).

Any homothety centred at vertex A preserves these properties, and it remains to find the coefficient k so that H_A^k would map the point N into the point E belonging to the side BC. The whole construction is clear from the figure.

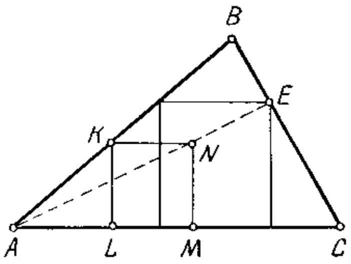


Figure 6.8: Inscribing a square into a triangle

Exercise 143. Into a given triangle, inscribe a triangle whose sides are parallel to the three given lines.

One more useful property of the homothety is that it preserves the direction of straight lines: the image of a line l is always a line parallel to l. We will use this fact in the following problem.

Problem 52 *Several circles are inscribed into one circular segment (Fig. 6.9).*

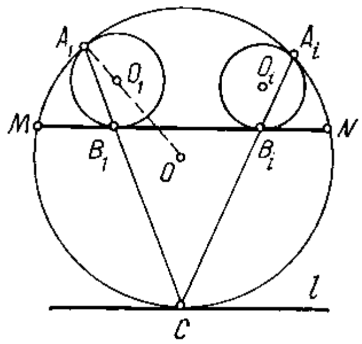


Figure 6.9: Circles inscribed into a circular segment

Let A_i and B_i be the tangency points of the i-th inscribed circle with the arc and the chord, respectively. Prove that all lines A_iB_i pass through a common point.

Solution. We will prove that every line A_iB_i passes through the point C which is the tangency point of the line l parallel to the chord MN and tangent to the big circle. Consider the homothety h_1 with centre A_1 and coefficient $k = OA_1 : O_1A_1$. It transforms the small circle S_1 (centred at O_1) into the big circle S. Therefore, the image of the line MN which is tangent to S_1, is the line l, tangent to S and parallel to MN. The point B_1, which is the common point of MN and S_1, goes under h_1 into the point C, common to l and S. The same argument can be repeat for each small circle S_i. This completes the proof.

Exercise 144. Given two concentric circles, construct a line which intersects them in the four consecutive points A, B, C, D so that the following relation holds for the lengths of the segments that are cut by the circles: $AB = 2BC = CD$.

Exercise 145. Given a triangle, prove that the three lines, each of which passes through the midpoint of a side parallel to the bisector of the opposite angle, meet in one point.

Exercise 146. Prove that for any triangle ABC there exists a circle that contains the midpoints of the sides, the feet of altitudes and the midpoints of the segments KA, KB, KC, where K is the intersection point of the altitudes. (This circle is referred to as the *circle of 9 points*, or *Euler's circle*.)

The group of plane similitudes is not exhausted by the set of all homotheties.

Definition 39 *A spiral similarity is defined as the composition of a homothety and a rotation with the same centre (see Figure 6.10).*

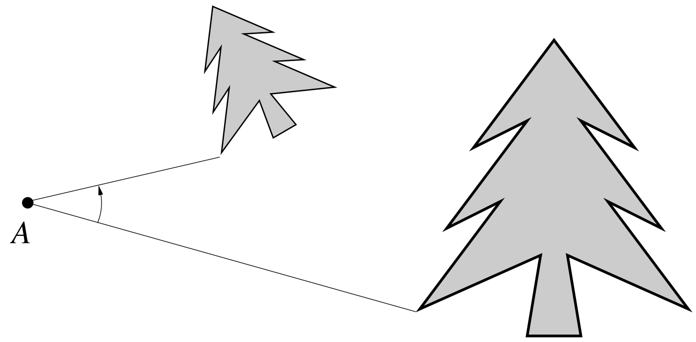


Figure 6.10: Spiral similarity

In this book, we have already encountered such transformations when studying complex numbers: we saw that multiplication by a number a is equivalent to a homothety with coefficient $|a|$ and subsequent rotation through the angle $\arg a$ around the origin (see 2.4). Let us consider some geometrical applications of spiral similarities.

Problem 53 *Given an arbitrary triangle ABC, draw two triangles ABP and BQC, lying outside of $\triangle ABC$, having right angles at vertices P, Q and equal angles β at the vertex B (see Fig. 6.11).*

Find the angles of $\triangle PQK$ where K is the middle point of the side AC.

Solution. Consider two spiral similarities: $F_P = H_P^k \circ R_P^d$, $F_Q = H_Q^{1/k} \circ R_Q^d$, where $d = 90°$ and $k = PB : PA = QB : QC$. It is clear that $F_P(A) = B$ and $F_Q(B) = C$, hence $(F_Q \circ F_P)(A) = C$. When two spiral similarities are performed one after another their coefficients get multiplied, and their rotation angles add up (we will explain this a little later). Therefore, the composition $F = F_Q \circ F_P$ must be a rotation through 180°. Since $F(A) = C$, the centre of rotation is the point K

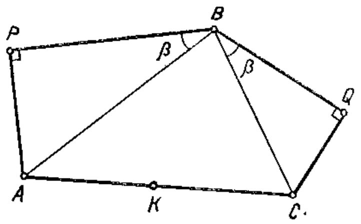


Figure 6.11: Right triangles built on the sides of triangle ABC

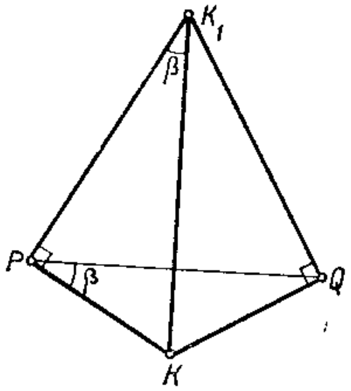


Figure 6.12: Product of two spiral similarities

and thus $F(K) = K$. Let $F_P(K) = K_1$, then $F_Q(K_1) = K$. Both right triangles KPK_1 and QKK_1 have the same angle β at the vertex K_1, hence they are equal (Fig. 6.12).

It follows that $PQ \perp KK_1$ and $\angle KPQ = \angle KQP = \beta$.

In the previous argument, we have used the fact that the composition of two spiral similarities is a spiral similarity whose coefficient is the product of the two coefficients, while the angle of rotation is the sum of the two rotation angles. This fact is evident, if the two transformations have a common centre. To prove it in full generality, we will use calculations with complex numbers, based on the following theorem.

Theorem 16 *A plane transformation is a similitude if and only if, in complex coordinate z, it can be written as either*

$$z \mapsto pz + a \tag{6.7}$$

or

$$z \mapsto p\bar{z} + a, \tag{6.8}$$

where p and a are arbitrary complex numbers, $p \neq 0$. The two cases 6.7 and 6.8 correspond to proper (i.e. orientation preserving) and improper (i.e. orientation reversing) transformations, respectively.

Proof Indeed, recall that we have already proved in 2.3 that proper movements of the plane correspond to the linear functions $w = pz + a$ with $|p| = 1$. Now, suppose that F is a proper similitude of the plane, i.e. a transformation that stretches all distances by a certain factor k and preserves the orientation. Let H be the homothety with coefficient k and centre 0. The composition $H^{-1} \circ F$ preserves the distances and the orientation, hence it is a proper movement and corresponds to a function $w = pz + a$ with $|p| = 1$. Then the transformation $F = H \circ (H^{-1} \circ F)$ can be written as $w = k(pz + a)$ which is an arbitrary linear function.

Vice versa, given a complex function $pz + a$ with arbitrary coefficients, we can verify that it stretches the distances between points by the factor $k = |p|$:

$$|(pz_1 + a) - (pz_2 + a)| = |p| \cdot |z_1 - z_2|.$$

The case of improper transformations is reduced to the case of proper transformations by the simple observation that the function 6.8 is the composition of function 6.7 and the standard reflection $z \mapsto \bar{z}$.

In the terminology and notations of section 6.1 Theorem 16 means that the group of proper similitudes is $\mathrm{Aff}(1, \mathbb{C})$ where $\mathbb{C}$ stands for complex numbers.

Using the description of similitudes by complex numbers, we can easily prove two important facts, related to each other:

- Any transformation of similitude $pz + a$ which is not a translation, i.e. satisfies $p \neq 1$, has a unique fixed point.
- Any transformation of similitude $pz + a$ which is not a translation, is a spiral similarity (in particular, a homothety).

Indeed, a fixed point is a number z_0 such that $pz_0 + a = z_0$. If $p \neq 1$, this equation has a unique solution $z_0 = a/(1 - p)$. The formula

$$pz + a = p\left(z - \frac{a}{1-p}\right) + \frac{a}{1-p}$$

shows that this transformation is in fact a spiral similarity with centre $a/(1 - p)$, stretching coefficient $|p|$ and rotation angle $\arg p$.

Now we can prove the fact used in Problem 53 above. The composition of two spiral similarities $w = pz + a$ and $u = qw + b$ corresponds to the function

$$u = q(pz + a) + b = pqz + (aq + b).$$

This is a spiral similarity with coefficient $|pq| = |p||q|$ and angle of rotation $\arg(pq) = \arg p + \arg q$.

Exercise 147. Two maps of the same country drawn to different scales on transparent paper, are put on the table in such a way that one of the maps completely covers the other. Prove that one can pierce both maps with a pin in a point that corresponds to the same place on both maps.

Exercise 148. Given four points A, B, C, D in the plane, such that $\overrightarrow{AB} \neq \overrightarrow{CD}$, prove that there exists a point E for which the two triangles ABE and CDE are similar.

Exercise 149. Points M, N and P are centres of the squares constructed on the sides AB, BC, CA of an arbitrary triangle ABC outside of it. Prove that the segments NP and CM are perpendicular and have equal lengths.

6.4 Inversions

Problem 54 *A circle S touches two circles S_1 and S_2 in the points A and B. Prove that the line AB passes through the centre of similitude of the circles S_1 and S_2.*

Solution. Let K be the intersection point of the lines AB and O_1O_2 (see Fig. 6.13). We want to prove that K is the centre of similitude of the circles S_1 and S_2. We will construct the required similitude in a rather indirect manner.

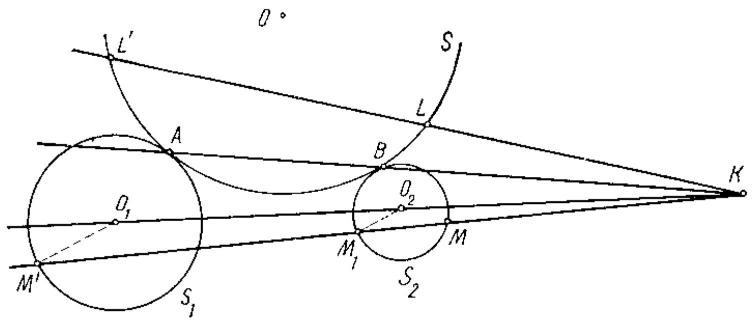


Figure 6.13: Three tangent circles

Let f be the transformation of the plane which, with every point M, associates the point M' belonging to the half-line KM at a distance from K that satisfies $KM \cdot KM' = KA \cdot KB = \text{const}$. Obviously, $f(A) = B$ and $f(B) = A$.

We claim that the circle S goes into itself under f. This follows from a well-known theorem of elementary geometry (if you don't know it, try to prove it yourself): *for a fixed circle S, a fixed point K and an arbitrary line l that passes through K and meets S in the two points L and L', the product of lengths of the two segments KL and KL' does not depend on the choice of the line l.*

Now take a point $M \in S_2$. Its image M' lies on the half-line KM and satisfies

$$KM' = \frac{KA \cdot KB}{KM}.$$

Let M_1 be the second intersection point of KM with S_2. By the theorem that we quoted, $KM \cdot KM_1 = C = \text{const}$. Therefore,

$$KM' = \frac{KA \cdot KB}{C} KM_1,$$

which means that M' is obtained from M_1 by a homothety centred at K! Therefore, the image of S_2 under f is a circle, say S_2'. Since S_2 passes through B and is tangent to the circle S, the circle S_2' passes through the point A and is tangent in that point to S, thus, $S_2' = S_1$.

We have proved that the two circles S_1 and S_2 can be transformed one into another by a homothety with centre K.

The transformation f that we used in the previous problem, is an example of the inversion.

Definition 40 *The inversion with respect to the circle T with centre O and radius r is the transformation that maps every point M into the point M' that belongs to the half-line OM and satisfies the equality $OM \cdot OM' = r^2$.*

The inversion maps the inner part of the circle outside and the outer part of the circle inside. It preserves the circle itself. A well-known joke of H. Pétard ("A contribution to the mathematical theory of big game hunting") suggests the following method to catch a lion. The hunter gets into a cage and waits. When the lion appears, he makes inversion. Now the lion is inside the cage.

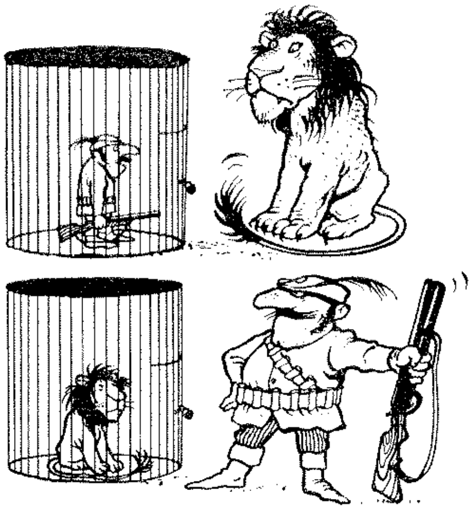


Figure 6.14: Use of inversion to catch a lion (after H. Pétard)

The inversion is an *almost* one-to-one transformation of the plane: it is defined and one-to-one everywhere except at the centre of the circle O. When the point M moves towards O, its image M' moves infinitely far from O. This is why it is natural to add the point ∞ ("infinity") to the plane, similarly to what we did for projective transformations of the line on page 131, and consider the inversion as a one-to-one transformation of the extended plane.

Suppose that our plane is the plane of complex numbers We know that every complex number z and its conjugate $\bar{z}$ satisfy the relation $z\bar{z} = |z|^2$. Therefore, the algebraic formula for an inversion of radius r with centre 0 is $z \mapsto r^2/\bar{z}$.

Exercise 150. What transformation group is generated by the set of all inversions with a fixed centre 0?

During the discussion of Problem 54, we have found that an arbitrary circle that does not pass through the centre of the inversion f, maps under f into a certain circle.

Exercise 151. What is the image under an inversion, of a circle that passes through the centre of inversion?

Exercise 152. What is the image under an inversion, of a straight line?

All these facts, put together, mean that *the inversion preserves the set of all lines and circles.* Viewing a straight line as a circle passing through infinity, we can say that inversions are *circular transformations*, i.e. transformations that preserve the class of all (generalized) circles. A little later we will see that the set of circular transformations is not exhausted by inversions.

Now we give some more applications of the inversions in elementary geometry.

Problem 55 *Each of the four circles touches two of its neighbours (Fig. 6.15a). Prove that the points of contact lie on one circle.*

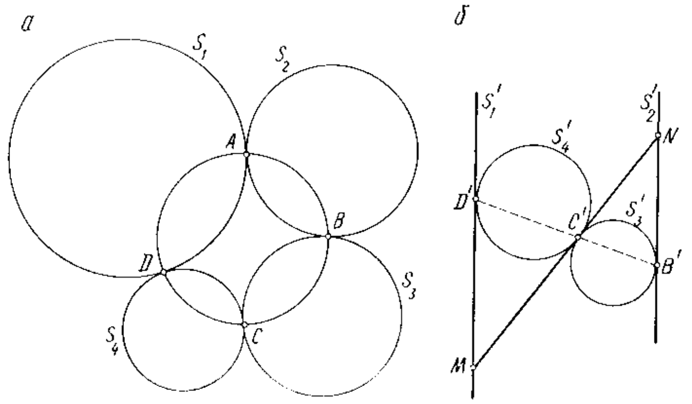


Figure 6.15: Four circles touching each other

Solution. Let us apply an inversion with centre A (one of the contact points) and an arbitrary radius. We will see that the problem becomes simpler after this transformation.

Denote by S_i' the image of S_i. It follows from our previous considerations that S_1' and S_2' are straight lines, and S_3' and S_4' are circles. The relations of tangency between S_i' are the same as between S_i, i.e. S_1' is tangent to S_2', S_2' to S_3', S_3' to S_4', and S_4' to S_1'. Note also that the lines S_1' and S_2' must be parallel, because S_1 and S_2 have only one common point A, which goes to infinity under the inversion. We arrive at the configuration shown in Fig. 6.15b. The problem is to prove that the three points of contact B', C', D' belong to one straight line — this will imply that the inverse images of these points B, C, D belong to a circle that passes through the centre of the inversion A.

To prove that $B'C'D'$ is a straight line, let us draw the common tangent of the circles S_3' and S_4' until the intersection with the lines S_1' and S_2' in the points M and N. Consider the two triangles $MC'D'$ and $NC'B'$. They are equilateral and have equal angles $\angle M = \angle N$. Therefore their angles at the vertex C' are also equal to each other. Hence $B'C'D'$ is a straight line. The proof is complete.

In problems 54 and 55 we used the evident property that if two lines are tangent to each other, then their images under the inversion are also tangent. The next exercise is a generalization of this fact.

Exercise 153. Call the angle between the two circles at a point of intersection, the angle made by their tangent lines drawn through that point. Prove that the inversion with respect to any circle preserves the angles between circles.

6.5 Circular transformations

The set of all inversions in the plane is not a transformation group. In this section, we shall study the group *generated* by all inversions. This group is called *the group of circular transformations*. It consists of two halfs: the subgroup of orientation preserving transformations, which coincides with the complex projective group $\mathrm{PGL}(1, \mathbb{C})$ (see page 131), and a coset consisting of orientation reversing transformations.

We start with an illustrative problem.

Problem 56 *Fix a circle C with centre O. Let A be the midpoint of its radius OB. Suppose that we are allowed to perform two transformations: inversion with respect to the circle C and half turn around point A. What is the maximal number of different points that can be obtained from a given point by successive applications of these transformations?*

Solution. Let us write both transformations as functions of a complex variable, assuming that the point O has complex coordinate 0 and point B complex coordinate 1.

The point A corresponds to the number $1/2$, and the symmetry in this point is described by the function
$$f_1(z) = 1 - z.$$

To find the formula for the second allowed transformation, note that the points z and w that correspond to each other under the inversion, satisfy the two relations: $|z||w| = 1$ and $\arg z = \arg w$. It follows that $w = 1/\bar{z}$ and thus

$$f_2(z) = 1/\bar{z}.$$

Each of the two allowed transformations is involutive, this is why the only way to obtain different compositions of the two functions is to apply them by turns. Starting from z and applying first f_1, then f_2, then again f_1 etc., we obtain the following list: z, $1 - z$, $1/(1 - \bar{z})$, $\bar{z}/(\bar{z} - 1)$, $1 - 1/z$, $1/z$, $\bar{z}$, $1 - \bar{z}$, $1/(1 - z)$, $z/(z - 1)$, $1 - 1/\bar{z}$, $1/\bar{z}$. After which we obtain z once again, and the sequence begins looping. Therefore, the inversion and the central symmetry generate a group G of 12 elements, and its orbit cannot contain more than 12 points. An example where the orbit contains exactly 12 points, is given below.

Exercise 154. Find all the possibilities for the number of points in the orbits of the group G. Draw pictures of different types of orbits.

Now let us find the *fundamental domain* of the group G, i.e. the part of the plane whose images under the group action cover the plane without overlapping. The image of the circle C under the transformation f_1 is the circle C' (see Fig. 6.16).

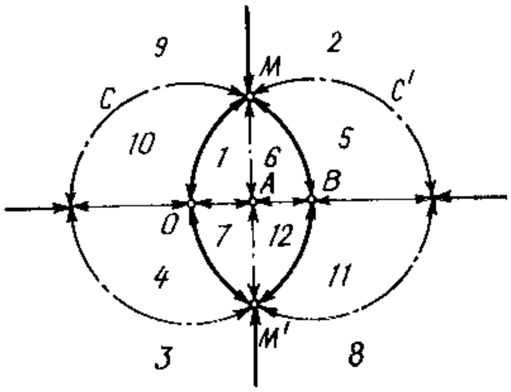


Figure 6.16: Group of complex functions generated by $z \mapsto 1 - z$ and $z \mapsto 1/\bar{z}$.

The image of C' under f_2 is the line MM'. One more important line is the straight line OB which separates the two regions corresponding to each other under the complex conjugation $z \mapsto \bar{z}$ (a mapping belonging to our group). These lines divide the plane into 12 domains that go one into another under the group action. Each of these domains has the property that they do not contain inner points equivalent under G.

Any of the 12 domains, e.g. domain 1, can be taken as the fundamental domain of the group action under study. The reader is invited to check what are the images of domain 1 under different transformations of the group.

The union of the four lines drawn in Fig. 6.16 is the set of all points on the plane left invariant by some nontrivial element of the group. The orbit of any inner point consists of exactly 12 points. The orbits of the points of the lines different from their intersection points have cardinality 6. The intersection points split into three orbits: two of length 3 ($1/2, 2, -1$ and $0, 1, \infty$) and one of length 2 (points M and M', corresponding to the numbers $1/2 \pm i\sqrt{3}/2$).

The four lines shown in Fig. 6.16, are divided into 18 segments by the intersection points. These segments go one into another by the group action and split into 3 orbits of length 6, shown in the figure as normal, bold and dotted lines.

Now we shall study the geometrical meaning of plane transformations described by fractional linear functions with arbitrary complex coefficients, i.e. elements of the group $\mathrm{PGL}(1, \mathbb{C})$, as well as similar functions with z replaced by the conjugate variable $\bar{z}$.

Theorem 17 *Let a, b, c, d be any complex numbers such that $ad - bc \neq 0$. Then:*

1. *The transformation defined by the function*

$$w = \frac{a\bar{z} + b}{c\bar{z} + d}$$

(an improper fractional linear transformation) is a composition of an inversion and a spiral similarity.

2. *The transformation defined by the function*

$$w = \frac{az + b}{cz + d}$$

(a proper fractional linear transformation) is a composition of an inversion, a spiral similarity and a reflection.

Proof If $c = 0$, then the second formula gives a linear function $a_1 z + b_1$ which, as we know, corresponds to a similitude transformation. The first formula gives $a_1\bar{z} + b_1$, a linear function in $\bar{z}$, which is the composition of the reflection $z \mapsto \bar{z}$ and a similitude.

Suppose now that $c \neq 0$. Then the fraction $\dfrac{a\bar{z} + b}{c\bar{z} + d}$ can be written as

$$\frac{a\bar{z} + b}{c\bar{z} + d} = p\left(\frac{1}{\bar{z} - \bar{z}_0} + z_0\right) + r,$$

where $z_0 = -\overline{d}/\overline{c}$, $p = (bc - ad)/c^2$ and $r = a/c - pz_0$. The expression inside the parentheses is the conjugation of the standard inversion (with centre 0 and radius 1) by the translation $z \mapsto z + z_0$, therefore, it represents the inversion of radius 1 centred at the point z_0. To the result of the inversion, the similitude transformation $z \mapsto pz + r$ is applied, and we get the required composition.

The proper fraction $(az + b)/(cz + d)$ is reduced to the improper one by the change $z \mapsto \bar{z}$, and we obtain the second part of the theorem. It is funny that in this case improper transformations, those that change orientation, are easier to handle than the proper transformations. This observation is accounted for by the importance of inversions in this context — and inversions are improper transformations.

Exercise 155. Check that all (proper and improper) fractional linear transformations form a group, and the set of proper transformations is its normal subgroup. Find the quotient group.

Theorem 17 implies that both classes of fractional linear transformations are *circular*, i.e. preserve the set of generalized circles (circles and straight lines) in the plane. It is also true that any circular transformation is described by either proper or improper fractional linear function. This is why the group of all such transformations is called *the circular group.* Another noteworthy property of these transformations is that they are *conformal*, i.e. preserve angles between curves. However, the class of all conformal mappings is much wider than that of circular transformations — for example, it includes complex functions $P(z)/Q(z)$, where P and Q are arbitrary polynomials.

Exercise 156. Prove that all transformations of the complex plane given by formulas

$$w = \frac{az+b}{cz+d}, \quad a,b,c,d \in \mathbb{R}, \quad ad-bc>0, \tag{6.9}$$

and

$$w = \frac{a\bar{z}+b}{c\bar{z}+d}, \quad a,b,c,d \in \mathbb{R}, \quad ad-bc<0, \tag{6.10}$$

form a group.

6.6 Hyperbolic geometry

Let us check that transformations 6.9 and 6.10 map the upper half-plane $y>0$ into itself. If $z = x+iy$, $w = u+iv$, then a simple calculation shows that the complex formula 6.9 is equivalent to the pair of real formulas

$$\begin{aligned} u &= \frac{(ax+b)(cx+d)+acy^2}{(cx+d)^2+y^2}, \\ v &= \frac{(ad-bc)y}{(cx+d)^2+y^2}, \end{aligned}$$

and we see that v has the same sign as y. The formula 6.10 is considered in a similar way.

Let L be the group of all transformations 6.9 and 6.10 acting on the upper half-plane $H = \{(x,y)|y>0\}$. The half-plane H is called *hyperbolic plane*, or *Lobachevsky plane*, and the group L, the group of *hyperbolic movements* of H. This terminology has the following meaning.

As we know, the transformation belonging to the group L take any circle into a circle (or a line, which we view as a particular case of the circle). In the plane H, there is a distinguished set of circles which is preserved by the group L. These are the (half)-circles and (half)-lines perpendicular to the line Ox (see Fig. 6.17). We will call these circles the *L-lines* for the reason that through any two points of H there passes one and only one L-line — the property owned also by the set of all usual lines in the usual plane.

The group L of hyperbolic movements, or *L-movements* has properties, similar to those of the group of plane movements acting on the usual plane. In particular, any point can be

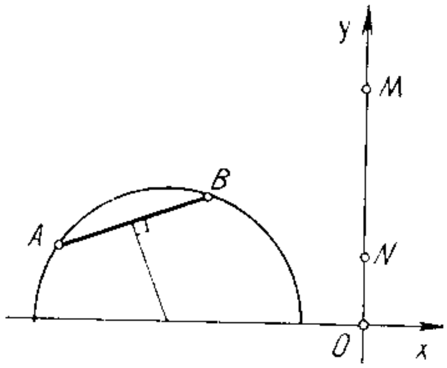


Figure 6.17: L-lines in hyperbolic plane

taken into any other by a hyperbolic movement, but the action of L on the set of L-segments (arcs of L-lines) is not transitive. The main geometric difference between the hyperbolic and the usual planes appears, when we think about parallel lines.

In the ordinary Euclidean geometry, two lines are called parallel, if they do not have common points, and the main property of parallel lines is that for any line a and any point A outside of a there is exactly one line passing through A and parallel to a. Now look at Fig. 6.18 which shows an L-line l and an L-point A. Among the four lines drawn through A, there is one (l) that intersects the line a, and there are three (k, n, m) that have no common points with a (we remind that the points of the boundary horizontal line do not belong to H). We thus see that in Lobachevsky geometry one can draw many lines passing through the given point and not intersecting the given line.

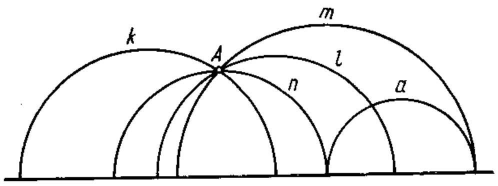


Figure 6.18: Mutual position of two L-lines

Let us do a computational exercise in Lobachevsky geometry. The angle between two L-lines is by definition measured as the usual Euclidean angle between the tangent lines (note that with this definition of the angle we have the property that L-movements preserve L-angles).

Problem 57 *Find the sum of angles of the Lobachevsky triangle with vertices* $A(0,7)$, $B(4,3)$, $K(0,5)$.

Solution. The given triangle ABK is the hatched region in Fig. 6.19.

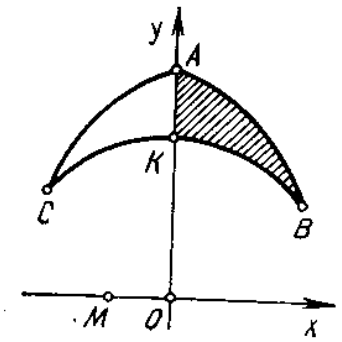


Figure 6.19: A triangle in hyperbolic plane

The side AK lies on the axis Oy. The side KB is an arc of a circle with the centre at O. The side AB is an arc of the circle with the centre at the point M.

Exercise 157. Find the coordinates of the point M.

The angle K of our triangle is a right angle, because it is formed by a circle and its radius:

$$\angle K = 90^\circ.$$

The angle B between the two circles is equal to the angle between the tangents and hence to the angle between the radii:

$$\angle B = \angle OBM.$$

Similarly,

$$\angle A = \angle OMA.$$

If you have found the coordinates of M, then you can find:

$$\begin{aligned}
\tan \angle OMA &= \frac{7}{3}, \\
\tan \angle OBM &= \tan(\angle BOB_1 - \angle BMB_1) = \frac{9}{37}, \\
\tan(\angle A + \angle B) &= \frac{\frac{7}{3} + \frac{9}{37}}{1 - \frac{7}{3} \cdot \frac{9}{37}} = \frac{143}{24}.
\end{aligned}$$

Since the tangent is positive, we infer that the angle $\angle A + \angle B$ is less than 90°. Therefore, the sum of the three angles of the triangle ABK is less than 180°.

It is interesting to note that the bigger is the Lobachevsky triangle (in a certain sense), the smaller is its sum of angles. For example, you can check that the isosceles triangle ABC, which is twice the triangle ABK, has a smaller sum of angles.

Finally, we will give one example of a crystallographic group in the hyperbolic plane — the so called *modular group* U that consists of all proper fractional linear transformations with integer coefficients:

$$U = \left\{ \frac{az+b}{cz+d} \mid a, b, c, d \in \mathbb{Z} \right\}.$$

This group is generated by two elements

$$S : z \mapsto -1/z \quad \text{and} \quad T : z \mapsto 1 + z.$$

Exercise 158. Check the relations $S^2 = (ST)^3 = \mathrm{id}$.

Figure 6.20 shows the fundamental domain of the group U

$$\Phi = \{z = x + yi \mid |z| \geq 1, \ |x| \leq \frac{1}{2}\}$$

and its images under T, S, T^{-1}, TS, ST etc.

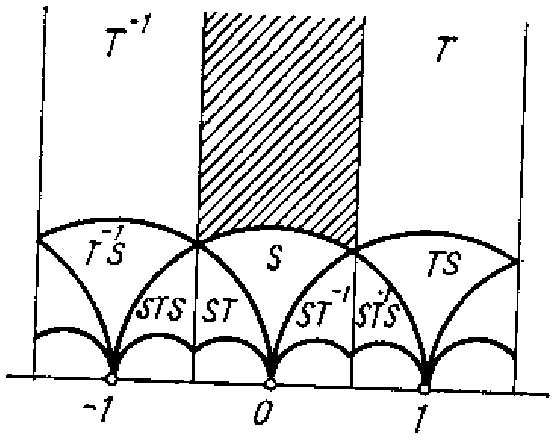


Figure 6.20: Fundamental domain of the modular group

The domain Φ is in fact an L-triangle, and it has a finite area (we have not defined the area in the Lobachevsky geometry, so you cannot check that!). Copies of Φ cover all the upper half-plane without overlapping. Thus U is really a crystallographic group. The reader is invited to draw a motive and repeat it throughout the Lobachevsky plane using the action of the group U and thus obtain a *hyperbolic ornament*.