Chapter 7

Symmetries of Differential Equations

In this chapter we will apply the machinery of transformation groups to the solution of differential equations. We assume that the reader is acquainted with the notions of derivative, definite and indefinite integral.

7.1 Ordinary differential equations

In this book we will only study the simplest class of differential equations: *ordinary differential equations of first order resolved with respect to the derivative.*

Definition 41 *A differential equation is an equation of the form*

$$y' = f(x, y), \tag{7.1}$$

where y is a variable depending on x, the prime means the derivative over x and $f(x, y)$ is a given function of two variables, x and y, which is supposed to be "good enough" (continuous and differentiable).

Definition 42 *A solution to equation 7.1 is a function $y = \phi(x)$ which, upon substitution into the equation, makes it a true identity, so that*

$$\phi'(x) = f(x, \phi(x))$$

holds for any value of x.

Since we are interested in the way y depends on x, we call x *independent* and y *dependent* variable. Equation (7.1) can also be written as $dy/dx = f(x, y)$, where dy and dx are *differentials*, i.e. infinitesimal ("infinitely small") increments of y and x whose ratio is by definition equal to the derivative y'. [1]

Here is an example of a differential equation:

$$y' = y - x. \tag{7.2}$$

[1] From the modern viewpoint, the notion of differential is formalized using *differential forms*, but, as we cannot touch upon that in this book, we are going to treat the differentials in the above intuitive sense, following the mathematicians of 17th and 18th centuries.

As you can check by a direct substitution, either of the functions $y = x+1$ and $y = e^x+x+1$ is a solution of this equation.

The main theorem of the theory of ordinary differential equations implies that every differential equation has a one-parameter family of solutions that can be described by a formula $y = \psi(x, c)$ containing a constant c whose value may be arbitrary. Such a function $\psi(x, c)$ is referred to as the *general solution* of the given equation. For example, Equation 7.2 has the general solution $y = ce^x + x + 1$ which gives the two particular solutions quoted above, when $c = 0$ and $c = 1$.

Note that the family of solutions of a differential equation $\psi(x, c)$ cannot be an arbitrary one-parameter family of functions.

Exercise 159. Is there a (first order) differential equation that has the following pair of particular solutions: (a) $y = 0$ and $y = 1$ (constant functions)? (b) $y = 1$ and $y = x$?

The function $f(x, y)$ in the right hand side of the differential equation 7.1 can be free of x, of y, or of both. For example, we can consider the following equations:

$$y' = 2, \tag{7.3}$$

$$y' = \cos x, \tag{7.4}$$

$$y' = y^2. \tag{7.5}$$

Exercise 160. Find the general solution of Equations 7.3 and 7.4. Try to guess a particular solution of Equation 7.5.

Equations 7.3 and 7.4 belong to the class of equations, whose right hand side depends only on x:

$$y' = f(x). \tag{7.6}$$

The reader knows that the general solution to such an equation is obtained by indefinite integration:

$$y = \int f(x)dx, \tag{7.7}$$

where the right hand side is defined "up to an additive constant". More exactly, if $F(x)$ is a certain *primitive* of $f(x)$, i.e. a function such that $F'(x) = f(x)$, then the general solution to Equation 7.6 is written as

$$y = F(x) + C. \tag{7.8}$$

This formula, for arbitrary values of the constant C, gives all the solutions of Equation 7.6. The graphs of all functions 7.8 do not intersect and fill all the plane (x, y). E.g., for the equation 7.4 we obtain the picture shown in Fig. 7.1a.

Not only the set of solutions, but the differential equation itself can be represented as a geometric object. The equality $y' = f(x, y)$ means that the slope (more exactly, the tangent of the slope angle) of the graph of the unknown solution at the point (x, y) should be equal to the known number $f(x, y)$. Therefore, at every point of the plane (x, y) we know the direction in which the integral curve (the graph of a solution) should pass. We arrive at the following conclusion: *the geometric object associated with the differential equation 7.1*

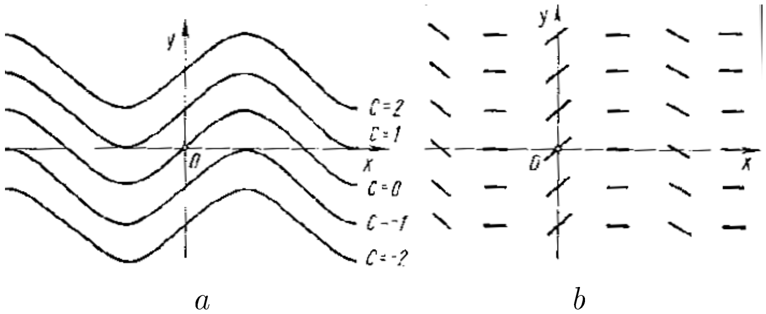

Figure 7.1: Graphs of solutions and field of directions of a differential equation

is a field of directions in the plane. A field of directions is fixed whenever, for every point of the plane, one defines a line passing through that point.

Geometrically, the problem of integration of a differential equation is formulated as follows: *given a field of directions in the plane, find all the curves that are everywhere tangent to the given field.* Such curves are referred to as *integral curves* of the field of directions. Figures 7.1 and 7.2 show the direction fields and the families of integral curves corresponding to the equations 7.4 and 7.2, respectively.

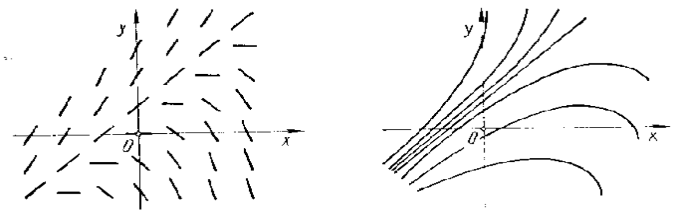

Figure 7.2: Field of directions and solutions of another differential equation

Exercise 161. Draw the direction fields and the families of integral curves for the equations 7.3 and 7.5.

Exercise 162. What are the integral curves of the field of directions shown in Fig. 7.3? Does this field correspond to any differential equation?

Using the mere indefinite integration, one can solve not only equations of class 7.6 (independent of y), but also equations of the form

$$y' = f(x)g(y), \tag{7.9}$$

This can be done by the following classical trick. Write y' as the ratio of two differentials dy/dx and then rewrite Equation 7.9 as

$$\frac{dy}{g(y)} = f(x)\,dx. \tag{7.10}$$

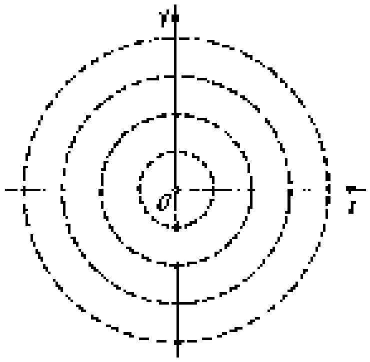

Figure 7.3: A field of directions in the plane

As you see, the variables are separated: on the left, we have only y, on the right, only x. This is why equations of type 7.9 are referred to as *equations with separating variables.* Integrating both sides of Equation 7.10, we obtain

$$\int \frac{dy}{g(y)} = \int f(x)\, dx \tag{7.11}$$

(it is understood that one of the two parts of this equality contains an arbitrary additive constant C). This is an implicit formula for the general solution of Eq. 41. If y is expressed through x, we will get an explicit general solution.

Problem 58 *Find the general solution of the differential equation*

$$y' = (2x+1)/(3y^2). \tag{7.12}$$

> **Solution.** Rewrite the equation in terms of differentials: $3y^2\, dy = (2x+1)\, dx$. Finding the indefinite integral gives $y^3 = x^2 + x + C$, whence $y = \sqrt[3]{x^2+x+C}$. This is the general solution of Equation 7.12.

Special cases of equations with separating variables consist of equations whose right hand side depends on only one variables x or y.

Exercise 163. Find the general solution of Equation 7.5.

7.2 Change of variables

We now pass to the discussion of the relation between the differential equations and the main theme of the book — transformations of the plane.

It turns out that in various methods of finding solutions of differential equations a crucial role is played by *changes of variables.* If we pass from variables x, y to new variables u, v according to some formulas

$$\left\{ \begin{array}{lcl} u & = & \varphi(x,y), \\ v & = & \psi(x,y), \end{array} \right. \tag{7.13}$$

then the equation $y' = f(x, y)$ transforms to another equation

$$v' = g(u, v) \tag{7.14}$$

where the prime means the derivative over u, not x. If it turns out that in this equation variables separate, then we can solve it and then, using formulas 7.13, return to the initial variables x, y and obtain the solution of the initial equation.

What we have described is the simplest, but very effective method of integration: make a change of variables that lead to an equation with separating variables.

Problem 59 *Solve the equation $y' = y - x$ (7.2) by reducing it to an equation with separating variables.*

Solution. Let us make the following change of variables:

$$\begin{cases} u = x, \\ v = y - x - 1. \end{cases}$$

Since $u = x$, the derivative over u is the same thing as the derivative over x, therefore no confusion arises, if we denote both of them by a prime. Next we have $y' = v' + 1$ and, substituting this into the given equation, we obtain the equation $v' = v$. This is an equation with separating variables, it has the general solution $v = Ce^u$. Coming back to the variables x, y, we get the general solution of the initial equation in the form $y = Ce^x + x + 1$.

Exercise 164. Find the change of variables that transforms the equation $y' = y^2 + 2xy + x^2 - 1$ to an equation with separating variables.

The formulas of the change of variables 7.13 have a double geometrical meaning.

First, regarding x, y as Cartesian coordinates of a point in the plane, we can view u, v as the coordinates of the same point in another curvilinear coordinate system. For example, formulas

$$\begin{cases} u = \sqrt{x^2 + y^2}, \\ v = \arctan \dfrac{y}{x} \end{cases}$$

or equivalent formulas

$$\begin{cases} x = u \cos v, \\ y = u \sin v \end{cases}$$

introduce the system of polar coordinates (u, v).

Second, we can think that we deal with a transformation of the plane, which takes a point with coordinates (x, y) into the point with coordinates (u, v) where $u = \varphi(x, y)$, $v = \psi(u, v)$. All the coordinates are in this case calculated in one and the same coordinate system. To visualize the transformation in this case, it is useful to draw the images of the coordinate lines $x = \text{const}$ and $y = \text{const}$.

7.3 The Bernoulli equation

Historically, the first person who successfully applied transformations of variables to differential equations was probably Johann Bernoulli who solved the equation (now bearing his name)

$$y' = Ay + By^n, \tag{7.15}$$

where A and B are given functions of x. He managed to reduce this equation to a simpler (*linear*) equation

$$y' = Py + Q, \tag{7.16}$$

where P and Q are again some functions of x.

Let us first explain how the linear equation is solved. Write the unknown function y as a product $y = uv$, where u and v are unknown functions of x. Substituting this into Equation 7.16, we get:

$$u'v + uv' = Puv + Q.$$

This equation is satisfied, if the following two relations hold: $u' = Pu$, $v' = Q/u$. The first one of them is an equation with separating variables, from which we can find the function $u(x)$. Feeding it into the second one, we can find $v(x)$ by a simple integration. We thus get the solution of the initial equation $y = u(x)v(x)$.

Exercise 165. Find the general solution of the equation

$$y' = 2\frac{y}{x} - x^3 + x.$$

Problem 60 *Find a transformation that reduces Bernoulli's equation 7.15 to the linear equation 7.16.*

Solution. Equations 7.15 and 7.16 differ only in the exponents of the dependent variable y. Therefore, it is natural to try the transformation of the form $y = v^k$ (preserving the independent variable x). Let us substitute this expression into the equation and see what happens:

$$kv^{k-1}v' = Av^k + Bv^{kn},$$

or

$$v' = \frac{A}{k}v + \frac{B}{k}v^{kn-k+1}.$$

If $k = 1/(1-n)$, the second exponent $kn - k + 1$ becomes 0 and we arrive at a linear equation! Therefore, the required transformation is $y = v^{1/(1-n)}$.

Exercise 166. Find the general solution of the equation

$$y' = \frac{xy^2 + 1}{2y}.$$

A reader who has solved this exercise (or looked at the answer in the end of the book) might be perplexed by the fact that the solution is not given by a conventional formula, as a closed expression in elementary functions. We must therefore say a few words about *integration in closed form.* The function

$$\int e^{-x^2/2}dx,$$

although it is not an elementary function (it cannot be written as a combination of polynomials, trigonometric functions, logarithms and exponents), is in fact almost as good as any elementary function. Its numeric values can be found by computer to any degree of precision and its properties are well known, because this function is widely used in probability theory and statistics. The same refers also to the integral of any elementary function. This makes ground for the following important notion.

A differential equation is said to be *integrable in closed form*, if its general solution can be written by a formula involving elementary functions, indefinite integrals and inverse functions.

The simplest example of an equation which is not integrable in closed form is the equation

$$y' = y^2 + x$$

(this fact was proved by J. Liouville in 1841). This equation is a particular case of the so called *Riccati equation*

$$y' = a(y^2 + x^n), \tag{7.17}$$

Exercise 167. Find the general solution of the Riccati equation 7.17 for $n = 0$.

In the year 1742, D. Bernoulli and J. Riccati discovered a discrete series of values of the parameter n for which the equation 7.17 can be integrated in closed form. This was done by a very elegant trick, actually by means of a cyclic group of transformations. The idea was to find a change of variables which takes an equation 7.17 into an equation of the same form, but with a different value of the exponent n, and then try to reduce the equation to the case $n = 0$ (which is integrable, as you know from exercise 167).

Let us first make the change of the dependent variable according to the formula

$$y = \frac{1}{x^2 v} - \frac{1}{ax}.$$

Feeding this into the equation 7.17, after some simplifications we obtain an equation for v:

$$v' = a\left(-\frac{1}{x^2} - x^{n+2}v^2\right), \tag{7.18}$$

where the prime, as before, means the derivative with respect to x.

This is not yet a Riccati equation, but we will get one, if we change the independent variable according to the rule[2] $u = x^{n+3}$ (or $x = u^{1/(n+3)}$). Indeed, by the chain rule we have:

$$v' = \frac{dv}{dx} = \frac{dv}{du}\frac{du}{dx} = \frac{dv}{du}(n+3)x^{n+2} = (n+3)u^{\frac{n+2}{n+3}}\frac{dv}{du}.$$

[2]We only quote the transformations invented by Bernoulli and Riccati. Nobody knows *how* they found them!

A simple calculation shows that after this change equation 7.18 becomes

$$\frac{dv}{du} = -\frac{a}{n+3}\left(v^2 + u^{-\frac{n+4}{n+3}}\right).$$

This is again a Riccati equation, but with the exponent n changed to $-(n+4)/(n+3)$.

If, for example, we had an equation with $n = -4$, after this transformation we obtain the equation with $n = 0$ — which is integrable. Therefore, Riccati's equation with $n = -4$ is integrable, too.

Exercise 168. Find the general solution of the equation $y' = y^2 + x^{-4}$.

Exercise 169. Find one more value of n for which the Riccati equation is integrable in closed form.

Now let us make our observations into a general theory. We know that, if the Riccati equation is integrable for a certain exponent m, then it is also integrable for the exponent n such that $-(n+4)/(n+3) = m$, i.e. $n = -(3m+4)/(m+1)$.

Consider the fractional linear function

$$q(m) = -\frac{3m+4}{m+1}.$$

By the previous argument, if the Riccati equation 7.17 is integrable for some exponent m, then it is also integrable for the value $q(m)$. Repeating the transformation, we deduce that it is also integrable for the exponents $q(q(m))$, $q(q(q(m)))$ and in general for any $q^k(m)$, where q^k means the k-th power of the transformation q.

Exercise 170. Find an explicit formula for $q^k(m)$.

Note that the inverse transformation q^{-1} has the same property: it takes an "integrable" exponent into an "integrable" exponent. We obtain an infinite cyclic group generated by the fractional linear transformation q. This group acts on the set of all real numbers (exponents of the Riccati equation). The property of the equation to be integrable in closed form is an invariant of this action. Therefore, each orbit either consists entirely of the exponents, for which the equation is integrable, or contains only such exponents, for which the equation is not integrable.

In particular, the orbit of the number 0 furnishes an infinite series of Riccati equations that are integrable in closed form. Their exponents are

$$q^k(0) = \frac{4k}{1-2k},$$

where $k \in \mathbb{Z}$ is an arbitrary integer. Note that $q^k(0)$ tends to the value -2, when k goes to infinity.

Exercise 171. Prove that the Riccati equation 7.17 is also integrable for the exponent $n = -2$.

We must, however, warn the reader that, starting from the "integrable" value $n = -2$ and using transformation the q, it is impossible to find any new integrable cases, because the number -2 is a fixed point of q and its orbit consists of only one point.

We have thus found two integrable orbits in the set of Riccati exponents. J. Liouville proved that for all the remaining values Riccati's equation cannot be solved in closed form.

7.4 Point transformations

So far, we have only encountered changes of variables of the form

$$\begin{cases} x = \varphi(u), \\ y = \psi(u,v), \end{cases} \tag{7.19}$$

i.e. where the independent variable x is expressed through the new independent variable only. In this case it is easy to express the derivative dy/dx through u, v and dv/du using the chain rule $dy/dx = dy/du \cdot du/dx$.

One can, however, use arbitrary transformations of the independent and dependent variables $x = \varphi(u,v)$, $y = \psi(u,v)$ (*point transformations*). To derive the transformation formula for the derivative dy/dx in this case, we will need the notion of *partial derivatives.*

Definition 43 *Let $z = h(x,y)$ be a function of two variables. If the value of y is fixed, $y = y_0$, we obtain a function of one variables $z = h(x,y_0)$. The derivative of this function at the point x_0 is referred to as the partial derivative of the function $h(x,y)$ over the variable x at the point (x_0,y_0). The partial derivative is denoted by $\frac{\partial h}{\partial x}(x_0,y_0)$. Symbolically:*

$$\frac{\partial h}{\partial x}(x_0,y_0) = \frac{dh(x,y_0)}{dx}\Big|_{x=x_0} = \lim_{\varepsilon\to 0}\frac{h(x_0+\varepsilon,y_0)-h(x_0,y_0)}{\varepsilon}.$$

Problem 61 *Compute the partial derivative over x of the function $z = \sqrt{9-x^2-y^2}$ at the point $(2,1)$.*

> **Solution.** Assigning $y = 1$, we get a function of one variable $z = \sqrt{8-x^2}$. Its derivative is $-x/\sqrt{8-x^2}$. For $x = 2$ we obtain:
>
> $$\frac{\partial z}{\partial x}(2,1) = -1.$$

When the point (x_0,y_0) varies, the value $\frac{\partial z}{\partial x}(x_0,y_0)$ becomes a function of the variables x_0 and y_0. Using the normal notations (x,y) instead of (x_0,y_0), one gets a function in the variables x and y denoted by $\partial z/\partial x$ or simply z_x. Thus, for the function $z = \sqrt{9-x^2-y^2}$ we obtain:

$$\frac{\partial z}{\partial x} = -\frac{x}{\sqrt{9-x^2-y^2}}.$$

Once again: to compute the partial derivative z_x, one has to differentiate $z(x,y)$ over x, treating the variable y as an arbitrary constant. The partial derivative with respect to y is defined in the similar way, treating x as a parameter. For the function $z = \sqrt{9-x^2-y^2}$ we have:

$$\frac{\partial z}{\partial y} = -\frac{y}{\sqrt{9-x^2-y^2}}.$$

We shall also explain the geometrical meaning of partial derivatives. Consider the surface in 3-space consisting of all points $(x,y,z(x,y))$ — the *graph* of the given function of two variables. Figure 7.4 depicts the graph of our favourite function $\sqrt{9-x^2-y^2}$.

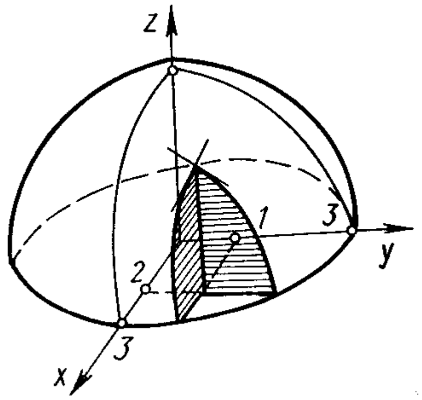

Figure 7.4: Partial derivatives

Given a point (x_0, y_0), in which the function $z(x, y)$ is defined, draw the plane $y = y_0$. It cuts the surface along a certain plane curve. The slope of the tangent line to this curve at the point (x_0, y_0) gives the value of the partial derivative $z_x(x_0, y_0)$.

Another partial derivative $z_y(x_0, y_0)$ is the slope of the tangent line to the section of the surface by the plane $x = x_0$. Figure 7.4 shows both sections and their tangent lines for the function $\sqrt{9 - x^2 - y^2}$ at the point $(2, 1)$.

The plane passing through the two tangent lines is the tangent plane to the graph of the function at the given point. Its equation is

$$z - z_0 = p(x - x_0) + q(y - y_0), \tag{7.20}$$

where $z_0 = h(x_0, y_0)$, $p = \frac{\partial h}{\partial x}(x_0, y_0)$ and $q = \frac{\partial h}{\partial y}(x_0, y_0)$. Indeed, substituting $x = x_0$ into 7.20, we get the equation of the tangent line to the graph of $h(x_0, y)$ viewed as a function in y, and substituting $y = y_0$, the similar equation for $h(x, y_0)$.

The point of the surface and the point of the tangent plane, both corresponding to one and the same point (x, y) of the horizontal plane, are very close to each other, if the point (x, y) is close enough to (x_0, y_0). Therefore, the difference $z - z_0$, computed by the formula 7.20, can be viewed as increment of the function $h(x, y)$, when its argument moves from (x_0, y_0) to (x, y), provided that this shift is "infinitely small". Denoting the infinitesimal increments of the three variables by dx, dy and dz, we can write:

$$dz = z_x dx + z_y dy \tag{7.21}$$

(the formula of the *differential of a function of two variables*).

With the help of this formula, we will now obtain the transformation rule for the derivative dy/dx, when the variables x and y undergo an arbitrary point transformation

$$\left\{ \begin{array}{lcl} x & = & \varphi(u, v), \\ y & = & \psi(u, v). \end{array} \right. \tag{7.22}$$

Using the notation $y' = dy/dx$, $v' = dv/du$ (note that the prime has different meaning in either case), by formula 7.21 we can write:

$$y' = \frac{dy}{dx} = \frac{y_u du + y_v dv}{x_u du + x_v dv} = \frac{y_u + y_v \frac{dv}{du}}{x_u + x_v \frac{dv}{du}} = \frac{y_u + y_v v'}{x_u + x_v v'}. \tag{7.23}$$

We see that the derivative y' is expressed as a fractional linear function of v' with coefficients depending on u and v, i.e. as a projective transformation of v' (see 6.2). This remarkable fact, by the way, leads to a deep connection between projective geometry and ordinary differential equations, but we will not discuss that in this book.

As we noticed before, formulas 7.22 can be viewed either as the passage from one coordinate systems in the plane to another, or as a mapping of the plane into itself according to the rule $(u, v) \mapsto (x, y)$. In the latter case formula 7.23 describes how the slope of the plane curves changes under this mapping (see Fig. 7.5): v' is the tangent of the angle to the horizontal line for a given curve, while y' is the tangent of the similar angle for the image of this curve.

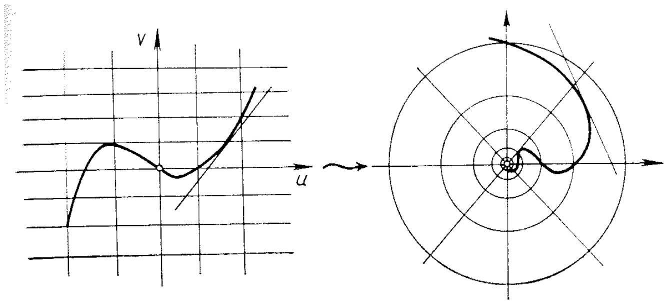

Figure 7.5: Point transformation

Let us call a *contact element* a point of the plane together with a direction (a straight line) attached to this point. A contact element is described by three numbers (x, y, p), where (x, y) are the coordinates of the given point and p is the slope of the line (the tangent of the angle it makes with the horizontal axis). The set of all contact elements thus forms a three-dimensional space — the *space of contact elements*. Some of its elements are shown in Fig. 7.6.

Formula 7.23, taken together with 7.22, defines a certain transformation of the space of contact elements, which corresponds to the plane transformation given by formula 7.22 alone.

Problem 62 *Find the transformation of the space of contact elements that corresponds to the inversion with respect to the circle $x^2 + y^2 = 1$.*

Solution. Under the inversion, the coordinates of a point transform as follows:

$$\begin{aligned} u &= \frac{x}{x^2 + y^2}, \\ v &= \frac{y}{x^2 + y^2}. \end{aligned}$$

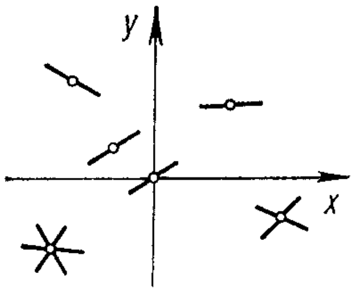

Figure 7.6: Contact elements

The partial derivatives of these functions are:

$$\begin{aligned} u_x &= \frac{y^2 - x^2}{(x^2+y^2)^2}, \quad u_y = \frac{-2xy}{(x^2+y^2)^2}, \\ v_x &= \frac{-2xy}{(x^2+y^2)^2}, \quad v_y = \frac{x^2 - y^2}{(x^2+y^2)^2}. \end{aligned}$$

By formula 7.23 we obtain:

$$v' = \frac{v_x + v_y y'}{u_x + u_y y'} = \frac{(x^2 - y^2)y' - 2xy}{-2xyy' + y^2 - x^2}. \tag{7.24}$$

As a corollary of this result, we can prove the following theorem which generalizes the assertion of Exercise 153 (see page 141).

Theorem 18 *A plane inversion preserves the angles between curves.*

Proof The angle between the two curves is, by definition, the angle between their tangent lines. If we have two curves in the (x, y)–plane such that the tangents of the slope angles are p_1 and p_2, then the angle α between the curves satisfies

$$\tan \alpha = \frac{p_1 - p_2}{1 + p_1 p_2}.$$

Let q_1 and q_2 be the corresponding tangents for the images of the two curves after inversion and β be the angle between them. Then

$$\tan \beta = \frac{q_1 - q_2}{1 + q_1 q_2}. \tag{7.25}$$

According to formula 7.24, we have

$$\begin{aligned} q_1 &= \frac{ap_1 + b}{bp_1 - a}, \\ q_2 &= \frac{ap_2 + b}{bp_2 - a}, \end{aligned}$$

where a and b are certain constants depending on the point (x, y). Feeding this into equation 7.25, we get

$$\tan\beta = \frac{\dfrac{ap_1+b}{bp_1-a} - \dfrac{ap_2+b}{bp_2-a}}{1+\dfrac{ap_1+b}{bp_1-a}\dfrac{ap_2+b}{bp_2-a}} = -\frac{p_1-p_2}{1+p_1p_2} = -\tan\alpha,$$

which means that the inversion preserves the angles, but changes the orientation of the plane.

We now give an example where a generic point transformation is used to solve a differential equation.

Problem 63 *Solve the differential equation*

$$\frac{(y^2-x^2)\left(x^2+(x^2+y^2)^2\right)+2\,xy\left(y^2+(x^2+y^2)^2\right)}{2\,xy\left(x^2+(x^2+y^2)^2\right)+(x^2-y^2)\left(y^2+(x^2+y^2)^2\right)}$$

Solution. Let us make the change of variables

$$\begin{aligned} x &= \frac{u}{u^2+v^2}, \\ y &= \frac{v}{u^2+v^2}, \\ y' &= \frac{(u^2-v^2)v'-2uv}{-2uvv'+v^2-u^2}. \end{aligned}$$

Feeding these expressions into the given equation, we obtain after simplifications:

$$v' = \frac{u^2+1}{v^2-1}.$$

Here the variables separate: $(v^2-1)dv = (u^2+1)du$, and the general solution is given by the following implicit function:

$$\frac{v^3}{3} = \frac{u^3}{3} + C.$$

Going back to the variables (x, y), we obtain the answer:

$$y^3 - x^3 = 3(x+y)(x^2+y^2)^2 + C(x^2+y^2)^3,$$

C being an arbitrary constant.

Exercise 172. (a) Find the expression of $dr/d\varphi$ through dy/dx, if (x, y) are Cartesian coordinates and (r, φ) are polar coordinates in the plane. (b) Using the formulas obtained, solve the differential equation $yy'+x = (x^2+y^2)(xy'-y)$ by passing to polar coordinates.

7.5 One-parameter groups

Definition 44 *A one-parameter group of plane transformations is an action of the additive group $\mathbb{R}$ on the plane.*

This means that for every real number t a transformation g_t is defined in such a way that the equality $g_t \circ g_s = g_{s+t}$ holds for every pair $s, t \in \mathbb{R}$. In other words, we deal with a homomorphism from the group $\mathbb{R}$ into the group of plane transformations. In this case we say that $\{g_t\}$ is a *one-parameter group of transformations* of the plane. Let us stress that a one-parameter group is not simply the set $\{g_t\}$, but this set together with the parametrization $t \mapsto g_t$.

The simplest example of a non-trivial one-parameter group is the group of parallel translations, say, in the direction of the axis Ox: g_t is the translation by vector $t\mathbf{e}_1$, where $\mathbf{e}_1$ is the horizontal unit vector.

Exercise 173. Prove that any two transformations belonging to a one-parameter group commute.

A one-parameter group can be written in coordinates as a pair of functions of three variables:

$$\left\{ \begin{array}{rcl} x_t &=& \varphi(x, y, t), \\ y_t &=& \psi(x, y, t). \end{array} \right. \tag{7.26}$$

Here (x, y) are the coordinates of an arbitrary point in the plane, and (x_t, y_t) are the coordinates of its image under the transformation g_t. For any fixed value of t we obtain a pair of functions of two variables that define a concrete transformation.

The group law, i.e. the relation $g_t \circ g_s = g_{s+t}$, can be written in terms of functions φ, ψ as follows:

$$\left\{ \begin{array}{l} \varphi(\varphi(x, y, s), \psi(x, y, s), t) = \varphi(x, y, s + t), \\ \psi(\varphi(x, y, s), \psi(x, y, s), t) = \psi(x, y, s + t). \end{array} \right. \tag{7.27}$$

This is the definition of a one-parameter group written in coordinates.

For example, the group of horizontal translations is represented by the functions

$$\left\{ \begin{array}{rcl} x_t &=& x + t, \\ y_t &=& y, \end{array} \right.$$

for which the relations 7.27 are obviously fulfilled.

Exercise 174. Consider the set of all homotheties with a common centre and positive coefficients. Is this a one-parameter group?

A watchful reader can notice that the question of this exercise is not correctly posed, because a one-parameter group presupposes a fixed parametrization of the given set of transformations. If we assign, to every number $t \in \mathbb{R}$, the homothety with stretching coefficient t, we won't obtain a one-parameter group, because the composition of homotheties with coefficients s and t is the homothety with coefficient st, not $s+t$. Fortunately, we know the trick that turns addition into multiplication: this is the exponential function. Assigning the homothety with coefficient e^t to the number t, we get a genuine one-parameter group. Placing the centre of homotheties into the origin, we can describe the group by the functions

$$\begin{cases} x_t &= e^t x, \\ y_t &= e^t y. \end{cases}$$

Relations 7.27 obviously hold.

A one-parameter group can be visualized through the set of its orbits. Figure 7.7 shows the orbits of the two groups that we have mentioned: translations and homotheties.

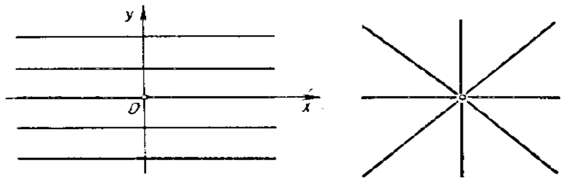

Figure 7.7: Orbits of one-parameter groups of translations and homotheties

We must note that the set of orbits does not uniquely define the one-parameter group. A simple example of this is provided by the group of translations with a double speed

$$\begin{cases} x_t &= x+2t, \\ y_t &= y, \end{cases}$$

which has the same orbits as the group of translations previously discussed (shown in Fig. 7.7a).

Exercise 175. Define a one-parameter group of rotations with a common centre, write its coordinate representation and draw its orbits.

Exercise 176. Check that the relations

$$\begin{cases} x_t &= e^{at}(x\cos bt - y\sin bt), \\ y_t &= e^{at}(x\sin bt + y\cos bt) \end{cases}$$

define a one-parameter group. Explain its geometrical meaning and draw its orbits.

Exercise 177. Let x_t and y_t be the roots of the quadratic equation for the unknown w

$$(w-x)(w-y)+t=0$$

chosen in such a way that x_t, y_t continuously depend on t and $x_0 = x$, $y_0 = y$.

The numbers x_t and y_t are functions of three variables x, y and t. Prove that these functions define a one-parameter group of transformations and draw its orbits.

7.6 Symmetries of differential equations

A differential equation, viewed as a field of directions in the plane, may possess some symmetry. One glance at Figure 7.1b is sufficient to understand that this field of directions is preserved by any translation along Oy as well as translations along Ox by whole multiples of 2π. Transformations of the first kind form a one-parameter group $x_t = x$, $y_t = y + t$. Transformations of the second kind form a infinite cyclic group (see 1).

It turns out that *the knowledge of a one-parameter group of symmetries enables one to find the general solution of the equation under study in closed form.*

Passing to exact definitions, suppose that we are given a differential equation

$$y' = f(x, y) \tag{7.28}$$

and a transformation of the plane

$$\begin{cases} x &= \varphi(u, v), \\ y &= \psi(u, v). \end{cases} \tag{7.29}$$

By formula 7.23, we can find the corresponding expression of $y' = dy/dx$ through u, v and $v' = dv/du$.

Definition 45 *Transformation 7.29 is called a symmetry of differential equation 7.28, if the equation for $v(u)$ obtained after the expressions for u, v and v' are substituted into 7.28, has the same function f in its right hand side:*

$$v' = f(u, v).$$

In geometric language this means that the transformation of the space of contact elements given by formulas 7.22–7.23, preserves the surface in this space that consists of all contact elements belonging to the given field of directions.

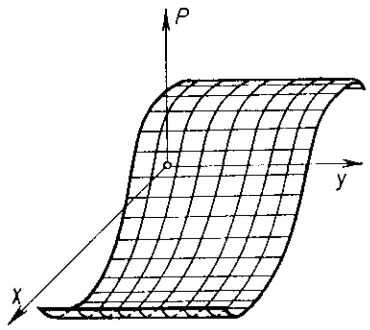

Figure 7.8: Differential equation as a surface in 3-space

Figure 7.8 shows such surface for the differential equation $y' = \cos x$, and you can see that the transformations

$$\begin{aligned} x_t &= x, \\ y_t &= y + t, \\ p_t &= p \end{aligned}$$

and

$$\begin{aligned} x_t &= x + 2\pi k, \\ y_t &= y, \\ p_t &= p \end{aligned}$$

map this surface into itself.

There are two main problems about the interrelation of differential equations and one-parameter groups:

1. Given a differential equation, find all (or some) groups of its symmetries.
2. Given a one-parameter group of plane transformations, find all (or some) differential equations preserved by this group, i.e. such that the group consists of their symmetries.

For practical needs (solving differential equations) the first question is more important. But it is also more difficult. Therefore, let us first discuss the second question.

We start with two simple examples, where the answer is obvious:

- The general equation preserved by the group of x-translations is

$$y' = f(y). \tag{7.30}$$

  We have earlier considered a particular case of this: equation 7.5.

- The general equation preserved by the group of y-translations is

$$y' = f(x).$$

  We have encountered equation 7.4 belonging to this class.

Let us now consider more interesting groups.

Problem 64 *Find the general form of a differential equation preserved by the group of rotations of the* (x, y) *plane centred at* $(0, 0)$.

> **Solution.** Under the rotation through an angle α every contact element moves together with the point of attachment and turns by the same angle α (Fig. 7.9).
>
> Hence, the angle it makes with the radius-vector of the point (x, y) does not change. Therefore, a field of directions (= a differential equation) is invariant under the group of rotations, if and only if the angle between the direction of the field and the radius-vector depends only on the distance from the origin. We can take the tangent of the angle instead of the angle itself, and the square of the distance instead of the distance. Using the formula for the tangent of the difference of two angles, we obtain the general form of the equation admitting the group of rotations around the origin:
>
> $$\frac{xy' - y}{yy' + x} = f(x^2 + y^2),$$

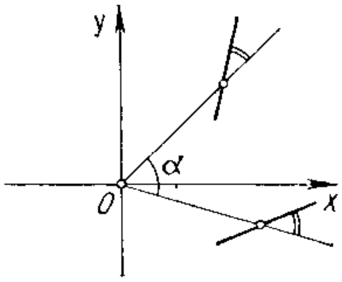

Figure 7.9: Action of rotations on contact elements

f being an arbitrary function of its argument. Resolving this with respect to y', we can write the answer as follows:

$$y' = \frac{xf(x^2+y^2)+y}{x-yf(x^2+y^2)}. \tag{7.31}$$

Exercise 178. Find all differential equations that admit the one-parameter group of homotheties.

Exercise 179. Find all differential equations that admit the one-parameter group of spiral homotheties described in Exercise 176.

7.7 Solving equations by symmetries

In this section, we will prove the following fact: *if a one-parameter group of symmetries of a differential equation is known, then it can be reduced, by a change of variables, to an equation with separating variables — and hence its general solution can be found in closed form.*

To find the new coordinate system in which the variables separate, we will use *invariants* of one-parameter groups, so let us first mention some of their properties and consider some examples.

We recall (see Sec. 5.6) that an invariant of a group action is a function which is constant on the orbits. In other words, a function is an invariant, if it has equal values at any two points that map one into another by a transformation of the group. To give a simple example, any function of y is an invariant of the one-parameter group of translations along Ox. The function y itself is the universal (complete) invariant of this group action, because its values on all orbits are different.

In the same way, the function x is the universal invariant of the group of translations in the direction of the axis Oy.

What is the universal invariant of the group of homotheties with centre 0 acting on the plane without the origin? One is tempted to think that it is the polar angle φ. Indeed, the polar angle φ takes equal values at all points of every ray (half-line) that issues from the origin, and different rays correspond to different values of the function. However, the polar angle φ is not a normal single-valued function on the plane, for example, to the point $(-1, 0)$ one can, with an equal success, assign the values $180°$ and $-180°$ (and an infinite number of others). Of course, one can make φ a single-valued function using, for example, the convention that φ must always take values between $0°$ (including) and $360°$ (excluding) — but then it will become a discontinuous function. In fact, the group of homotheties does not have any continuous universal invariant with values in $\mathbb{R}$. Of course, it does have continuous invariants which are not universal, for example, the function $\sin\varphi$.

Exercise 180. Find an invariant of the group of rotations around the origin. Does this group have a universal continuous real-valued invariant?

There are two approaches to the problem of finding the invariants of a given one-parameter group:

1. The formal approach. Assuming that the group is given in coordinates by a pair of functions 7.26, the problem is to make up a combination of the expressions $\varphi(x, y, t)$ and $\psi(x, y, t)$ that does not contain the variable t, i.e. find a function $h(\varphi, \psi)$ that does not depend on t.

2. The geometric approach. In the plane, we draw a curve K that meets every orbit of the group exactly in one point (see Fig. 7.10). We choose an arbitrary function on this curve that takes different values at different points, and then prolong it to all the plane following the rule: the value of the function at any point A is set to be equal to its value at the point B where the orbit meets the chosen curve K.

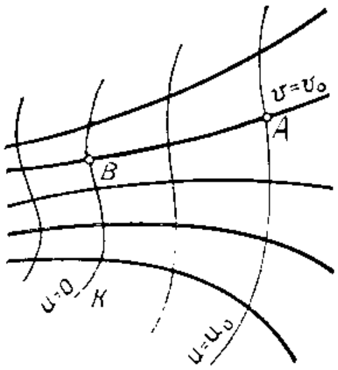

Figure 7.10: Invariant of a one-parameter group

Problem 65 *Find an invariant of the one-parameter group*

$$\begin{cases} x_t &= x + t, \\ y_t &= e^t y. \end{cases}$$

Solution.

Formal method. Looking at the above formulas for a while, one can guess that the combination $e^t y \cdot e^{x+t} = ye^{-x}$ does not depend on t and thus provides an invariant of the group.

Geometric method. The orbits of the given group are shown in Fig. 7.11.

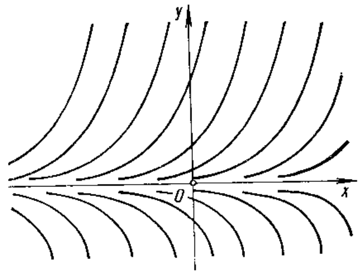

Figure 7.11: Orbits of the group studied in Problem 65

The coordinate axis Oy meets every orbit in one point, so we can choose it as the curve K. The function $v(0, y) = y$ takes different values at different points of this line, so let us find its prolongation to all the plane along the orbits of the group. Let $A(x, y)$ be an arbitrary point of the plane and let $B(0, v)$ be the intersection point of the corresponding orbit with the y-axis. From the equations of the group we have that there is a number t such that $x = t$, $y = e^t v$. From these equations we find: $v = ye^{-x}$. Thus, the invariant is ye^{-x}.

Exercise 181. Find a non-trivial invariant of the group of spiral homotheties (exercise 176) by the above two methods.

We finally get to the question of integrating the differential equations with a known symmetry group. The first principle is that it is useful to pass to the new coordinates, such that one of the new coordinate functions is an invariant of the group. After such transformation it often (but not always) happens that the variables separate, and the equation can be solved in closed form. Let us consider the example that comes up quite often.

Problem 66 *Find the general solution of the differential equation*

$$y' = f(y/x). \tag{7.32}$$

Solution. Such equations are called *homogeneous*. As we know from exercise 178, homogeneous equations are invariant with respect to the one-parameter group of homotheties with the centre $(0, 0)$. Since the function y/x is an invariant of this group, let us take it for the new dependent variable, leaving the independent variable x unchanged. We thus set $v = y/x$, or $y = xv$, hence $y' = v + xv'$. Feeding this into equation 7.32, we get: $v + xv' = f(v)$, or

$$v' = \frac{f(v) - v}{x}.$$

This is indeed an equation with separating variables.

Exercise 182. Continuing the previous argument, find an explicit answer in the particular case of the equation $y' = 1 + 2y/x$.

Exercise 183. Do the variables in equation 7.32 separate in polar coordinates?

Exercise 184. Adapt the argument of problem 66 to the equations of the form

$$y' = f\left(\frac{ax + by + c}{a_1x + b_1y + c_1}\right).$$

We must stress that taking the new dependent variables to be an invariant of the group does not guarantee the separation of variables. The choice of the *independent* variable is also very important. For example, if, in the equation 7.31 that admits the group of rotations, one goes over to polar coordinates, then the new equation has the form $u\,dv/du = f(v)$ where the variables separate. However, the transformation $u = x$, $v = x^2 + y^2$ (the second variable is a group invariant) does not lead to an equation with separating variables (please check).

To get a universal rule for the integration of equations with a known one-parameter group of symmetries, we will try to find such a coordinate system (u, v) that in this coordinates the group looks as simple as possible, for example, consists of parallel translations along the axis u. The general form of an equation that admit this group is, as we know, $dv/du = \varphi(v)$. Here the variables separate, and the equation is solvable in closed form. It remains to understand how we can reduce the initial one-parameter group to this simple form by a change of coordinates.

We want that in the coordinates (u, v) the transformations of the group have the form $u_t = u + t$, $v_t = v$. These transformations map the coordinate line $u = 0$ into the parallel lines $u = t$. Choose a real-valued function $v(x, y)$ which is an invariant of the group. The orbits are given by the equation $v = \text{const}$. Choose a curve K which meets every orbit in one point and assume that K is the v-axis of the new coordinate system, i.e. is described by the equation $u = 0$. Then, to fulfill our plane, we must assume that the image of K under the group transformation g_t should be described by the equation $u = t$.

If, instead of this function u, we take for the independent variable another function w that has the same level lines $w = \text{const}$ (the line $w = t$ should be the image of $w = 0$ under some transformation of the group, but not necessarily g_t), then we will also have an equation with separating variables in the new coordinates. Indeed, in this case w is a function of u: $w = h(u)$, which, upon the substitution into the equation $dv/du = \varphi(v)$ gives an equation $dv/dw = \varphi(v)\psi(w)$.

We can state the obtained result as the following *variable separation theorem.*

Theorem 19 *Suppose that we know a one-parameter group $G = \{g_t\}$ of symmetries of a differential equation E. Then the equation E becomes an equation with separating variables in any coordinate system (u, v), such that the coordinate lines $v =$ const are the orbits of the group G, and the lines $u =$ const go one into another under the transformations g_t.*

Problem 67 *Solve the differential equation*

$$y' = \frac{2}{5}(y^2 + x^{-2}).$$

Solution. If x is multiplied by a constant k and y by its inverse k^{-1}, then y' gets multiplied by k^{-2} (you can also check that using formula 7.23). All the terms of the given equation increase by the same factor, so that the equation actually does not change. This means that the group

$$\begin{cases} x_t &= e^{-t}x, \\ y_t &= e^{t}y \end{cases}$$

is a group of symmetries of the given equation.

This group is called the group of *hyperbolic rotations.* Its orbits are branches (connected components) of the hyperbolas $xy =$ const (see Figure 7.12). The product xy is an invariant of this group. We take it as the new dependent variable: $v = xy$.

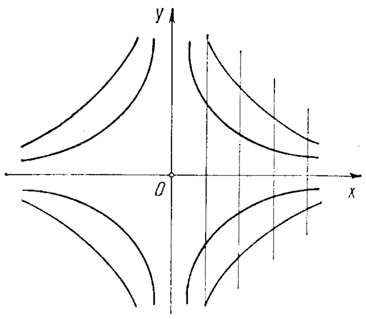

Figure 7.12: Orbits of hyperbolic rotations

Now note that the vertical lines $x =$ const are mapped one into another by hyperbolic rotations: the image of the line $x = a$ is the line $x = e^{-t}a$. Therefore, we can set x to be the new coordinate u. The required change of variables is thus

$$\begin{aligned} x &= u, \\ y &= \frac{v}{u}. \end{aligned}$$

After this change of variables the given equation becomes

$$v' = \frac{2v^2 + 5v + 2}{5u}.$$

The variables separate, and the general integral is

$$\frac{v+\frac{1}{2}}{v+2} = Cu^{3/5}.$$

Expressing u and v through x and y, we obtain the answer:

$$y = \frac{3}{2(x - Cx^{8/5})} - \frac{2}{x}.$$

Remark. The same procedure solves the equation $y' = a(y^2 + x^{-2})$ for an arbitrary value of a. The reason why we only considered a special case is that the general answer is rather cumbersome.

Exercise 185. Find the general solution of the equation $y' = (x + y^2)/(xy)$.

As the last example, we consider an equation where it is impossible to separate the variables by the change of the form $x = \varphi(u)$, $y = \psi(x, y)$.

Problem 68 *Solve the differential equation*

$$y' = \frac{(y^2 - x^2)^2 - 5(y - x) + 4}{(y^2 - x^2)^2 + 5(y - x) + 4}$$

using the one-parameter group

$$\begin{cases} x_t &= \frac{e^t+e^{-t}}{2}x + \frac{e^t-e^{-t}}{2}y, \\ y_t &= \frac{e^t-e^{-t}}{2}x + \frac{e^t+e^{-t}}{2}y. \end{cases}$$

Solution. These transformations are hyperbolic rotations (see 7.7), but considered in a coordinate system rotated through 45° with respect to the initial one. The function $v = y^2 - x^2$ is a group invariant. Note that the families of vertical or horizontal lines are not preserved by the transformations of the group, hence it is no good to take $u = x$ or $u = y$. However, the lines $y - x = \text{const}$ do have this property. Set $u = y - x$. Then after the transformation the equation becomes

$$\frac{dv}{du} = \frac{5v - v^2 - 4}{u}.$$

The variables separate!

Solving this equation and making the inverse change of variables, we obtain the implicit solution to the initial equation:

$$\frac{y^2 - x^2 - 1}{y^2 - x^2 - 4} = C(y - x)^3.$$

Exercise 186. Solve the differential equation $y' = e^{-x}y^2 - y + e^x$ with the help of the symmetry group $x_t = x + t$, $y_t = e^t y$.

We have thus learned how to solve a differential equation, if a one-parameter group of its symmetries is known. "This, however, by no means implies that any differential equation $X dy - Y dx = 0$ can be solved in closed form. The difficulty consists in *finding* the one-parameter group that leaves it invariant". These words belong to Sophus Lie, the Norwegian mathematician who created the theory of continuous groups and found its applications to the differential equations, the simplest case of which was described in the previous pages. As a last remark, let us mention that during the last 30 years or so some algorithmic methods for finding the symmetries of differential equations have been elaborated and implemented in numerous software systems of computer algebra.