Lecture 8

Subrings and ideals

Theorem. The ring of residues $\mathbb{Z}_m$ is a field if and only if m is a prime number.

The construction of the ring of residues is a particular case of the general construction of a quotient ring. To define this consruction, we need the notions of subring and ideal.

In the theory of rings, the notions of subring and ideal play the role of subgroups and normal subgroups, respectively:

Groups	Rings
Subgroup	Subring
Normal subgroup	Ideal

A subring of a given ring A is a smaller ring B which is contained in A. An ideal the construction of the quotient ring A/I makes sense.

Examples.

1. We know that any subgroup of the additive group of integers $\mathbb{Z}$ has the form $n\mathbb{Z}$, i.e. consists of all whole numbers divisible by a fixed number $n \in \mathbb{Z}$. Note that if a number divisible by n gets multiplied by any integer m, the result is also divisible by n. Therefore, for every number n the subgroup $n\mathbb{Z}$ is an ideal (in particular, a subring) in $\mathbb{Z}$.

2a. Let $A = \mathcal{F}(\mathbb{R}, \mathbb{R})$ be the ring of real-valued functions on the real line. Then the set of all even functions $B = \{f \in A \mid f(-x) = -f(x)\}$ is a subring which is not an ideal.

2b. In the same notations let $I = \{f \in A \mid f(0) = 0\}$ be the set of all functions that take value 0 at the point 0. Then I is an ideal in the ring A.