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Formation of current layers in three-dimensional, inhomogeneous coronal magnetic fields by photospheric motions

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Abstract

We present a method for calculating analytical expressions for the velocity, magnetic field, electric field, and current density arising from the quasi-static stressing of an initially potential, inhomogeneous coronal magnetic field by photospheric flows. Four exact solutions that are produced by a wide variety of driving flows are examined. We have found that large gradients in the vorticity can lead to the generation of strong current layers, across which the magnetic field is continuous. In two of the exact solutions these appear as logarithmic singularities in the current density and first spatial derivatives of the vorticity. Two other solutions, on the other hand, driven by continuous and differentiable photospheric flows lacking large gradients in the vorticity, exhibit little intensification of the current density. When current layers appear, they are located at the quasi-separatrix layers (QSL) of the magnetic configuration, where small displacements of the magnetic field footpoints at one part of the photosphere lead to large displacements of the footpoints at the other end of the field line. This extends previous QSL work by providing an additional constraint on the driving flow in order that strong currents are generated at such layers in the course of a quasi-static evolution of the magnetic field.

1. Introduction

For some time the investigation of the heating of coronal active regions has followed an approach proposed by *Parker* [1972, 1983], who considered a coronal loop that emerges from one part of the photosphere and submerges in another area, having passed through the corona in between. Any motions of the photospheric plasma stress the magnetic field and drive currents into the corona, usually aligned with the magnetic field. Parker straightened out his loop so that the photospheric regions of emergence and submergence became parallel planes, with the magnetic field being predominantly perpendicular to these planes. He considered an initial state consisting of a uniform, unidirectional field that passes from one plane to the other and considered the effects of motions at each plane, concluding that equilibrium is possible only if the magnetic field is invariant in the direction perpendicular to the planes. This being unlikely, general photospheric motions would then lead to the generation of tangential discontinuities in the magnetic field, with associated current sheets of zero thickness and infinite current density. Later, *van Ballegooijen* [1985] challenged these results, proposing instead that the expansions used by *Parker* [1972, 1983] in deriving his results should be replaced by another expansion scheme, and concluded also that invariance of the magnetic field in the perpendicular direction is not a necessary condition for equilibrium. Instead, van Ballegooijen proposed that discontinuities in the photospheric velocity lead to the formation of tangential discontinuities in the initially continuous magnetic field. More recently, *Longcope and Strauss* [1994] proposed that, although it may not be possible for current sheets to form from such a magnetic field, one can still form intense current layers of small but finite thickness.

Another direction of investigation has been started by *Priest and Forbes* [1992], who consider the mapping of the footpoints of the initial magnetic field from one photospheric plane to another. Their starting magnetic field is geometrically more complex than the uniform fields of *Parker* [1972, 1983] and *van Ballegooijen* [1985]. *Priest and Forbes* [1992] propose the existence of magnetic flipping layers, in which the small displacement of a footpoint at one plane leads to a large displacement of the footpoint at the other end of the field line, and suggest that such layers are likely sites for reconnection in magnetic fields that initially have no neutral points. These layers have since received further consideration under the label of quasi-separatrix layers (QSLs) [*Priest and Démoulin,* 1995; *Démoulin et al.,* 1996a, b]. It is shown herein that the presence of a QSL itself does not guarantee the generation of strong currents in a quasi-static evolution of the magnetic configuration; in the approximations of this paper such large currents actually require large vorticity gradients in the photospheric velocity distribution when starting from a continuous and differentiable magnetic field.

2. Determination of Linearized, Force-Free Perturbations to a Potential Magnetic Configuration

We consider an initially potential magnetic configuration $\mathbf{B}_0$ (where $\nabla\cdot\mathbf{B}_0 = 0, \nabla\times\mathbf{B}_0 = \mathbf{0}$), whose field lines are anchored in the photosphere with no velocity ($\mathbf{v}_0 = \mathbf{0}$). Flows at the photosphere carry the loop footpoints and drive currents into the corona, which are force-free because of the assumed slowness of the flows. The loop curvature is neglected, so that we have the ends of the loop intersecting two parallel planes representing the photosphere. The direction perpendicular to the planes is the z direction in a Cartesian system and is termed the vertical direction, the x and y directions lying parallel to the plane being the horizontal direction. The equations required to model this situation are then the force-free magnetic field equation, the induction equation, the definitions of the current density $\mathbf{J}$ and electric field $\mathbf{E}$ in terms of the magnetic and velocity fields $\mathbf{B}$ and $\mathbf{v}$, and the condition that the magnetic field be solenoidal:

$$\begin{aligned}\mathbf{J}\times\mathbf{B} &= \mathbf{0}, &(1)\\ \frac{\partial\mathbf{B}}{\partial t} &= -\nabla\times\mathbf{E}, &(2)\\ \mathbf{J} &= \frac{1}{\mu}\nabla\times\mathbf{B}, &(3)\\ \mathbf{E} &= -\mathbf{v}\times\mathbf{B}, &(4)\\ \nabla\cdot\mathbf{B} &= 0, &(5)\end{aligned}$$

where t and μ are the time and magnetic permeability of the medium, respectively. The variables can be nondimensionalized with respect to the loop length L, the z component of the potential magnetic field B_{0z}, and some characteristic speed v_c by $\mathbf{B}' = \mathbf{B}/B_{0z}$, $\mathbf{v}' = \mathbf{v}/v_c$, $\mathbf{J}' = \mathbf{J}\mu L/B_{0z}$, $\mathbf{E}' = \mathbf{E}/(v_c B_{0z})$, $t' = tv_c/L$, and $\nabla' = L\nabla$. Now we linearize about the potential equilibrium by writing $\mathbf{B}' = \mathbf{B}'_0 + \mathbf{B}'_1$, $\mathbf{v}' = \mathbf{v}'_0 + \mathbf{v}'_1$, $\mathbf{J}' = \mathbf{J}'_0 + \mathbf{J}'_1$, and $\mathbf{E}' = \mathbf{E}'_0 + \mathbf{E}'_1$, with

$\mathbf{v}_0' = \mathbf{J}_0' = \mathbf{E}_0' = \mathbf{0}$, and drop the primes so that

$$\begin{aligned} \mathbf{J}_1 \times \mathbf{B}_0 &= \mathbf{0}, & (6)\\ \frac{\partial \mathbf{B}_1}{\partial t} &= -\nabla \times \mathbf{E}_1, & (7)\\ \mathbf{J}_1 &= \nabla \times \mathbf{B}_1, & (8)\\ \mathbf{E}_1 &= -\mathbf{v}_1 \times \mathbf{B}_0, & (9)\\ \nabla \cdot \mathbf{B}_1 &= 0, & (10) \end{aligned}$$

after neglecting terms quadratic in the perturbed quantities $\mathbf{v}_1$, $\mathbf{B}_1$, $\mathbf{J}_1$, and $\mathbf{E}_1$.

The equation for the time evolution of the current density in terms of the driving flow can be found by taking the curl of (2), applying (3) and (4), and using the vorticity, $\boldsymbol{\omega} = \nabla \times \mathbf{v}$:

$$\begin{aligned} \frac{\partial \mathbf{J}}{\partial t} = & (\mathbf{B} \cdot \nabla)\boldsymbol{\omega} - (\mathbf{v} \cdot \nabla)\mathbf{J} - (\nabla \cdot \mathbf{v})\mathbf{J} \\ & + \mathbf{B} \times \nabla(\nabla \cdot \mathbf{v}) + (\nabla \mathbf{B} \cdot \nabla) \times \mathbf{v} \\ & - (\nabla \mathbf{v} \cdot \nabla) \times \mathbf{B}. \qquad (11) \end{aligned}$$

In linearized form with an initially potential magnetic field this becomes

$$\begin{aligned} \frac{\partial \mathbf{J}_1}{\partial t} = & (\mathbf{B}_0 \cdot \nabla)\boldsymbol{\omega}_1 + \mathbf{B}_0 \times \nabla(\nabla \cdot \mathbf{v}_1) \\ & + (\nabla \mathbf{B}_0 \cdot \nabla) \times \mathbf{v}_1 - (\nabla \mathbf{v}_1 \cdot \nabla) \times \mathbf{B}_0, \quad (12) \end{aligned}$$

which implies, in particular, that large local gradients of the vorticity give rise to regions of intense current density, even when the vorticity itself is small.

We now examine the solutions for the initial equilibrium

$$\mathbf{B}_0 = hx\hat{\mathbf{x}} - hy\hat{\mathbf{y}} + \hat{\mathbf{z}}, \qquad (13)$$

where h is a dimensionless parameter measuring the ratio of horizontal to vertical magnetic field strength and $-1 \leq x, y \leq 1$. Taking the cross product of $\mathbf{B}_0$ with the curl of (7) and invoking (6) and (8) gives

$$\mathbf{B}_0 \times [\nabla \times (\nabla \times \mathbf{E}_1)] = \mathbf{0}. \qquad (14)$$

From this it can be seen that $\nabla \times (\nabla \times \mathbf{E}_1)$ is parallel to $\mathbf{B}_0$ so that we may write

$$\nabla \times (\nabla \times \mathbf{E}_1) = \chi(x, y, z, t)\mathbf{B}_0, \qquad (15)$$

where x, y, and z are the dimensionless Cartesian coordinates of the loop. Considerable progress can be made by writing $\mathbf{B}_0 = \nabla\xi \times \nabla\eta$, where ξ and η are the Euler potentials of the magnetic field satisfying $\mathbf{B}_0 \cdot \nabla\xi = \mathbf{B}_0 \cdot \nabla\eta = 0$. For the field of (13) these potentials are

$$\begin{aligned} \xi &= xe^{-hz}, & (16)\\ \eta &= ye^{hz}. & (17) \end{aligned}$$

We proceed by working in terms of the coordinates ξ, η, and z. Taking the divergence of (15) then shows that $\mathbf{B}_0 \cdot \nabla\chi = 0$, implying that $\chi = \chi(\xi, \eta, t)$. From (15) we have $\partial \mathbf{j}_1/\partial t = -\chi(\xi, \eta, t)\mathbf{B}_0$, so

$$\mathbf{j}_1 = -\mathbf{B}_0 \int_0^t \chi(\xi, \eta, \tau)\mathrm{d}\tau. \qquad (18)$$

Since χ is an arbitrary function, we can write it in terms of another arbitrary function $\kappa(\xi, \eta, t)$ so that

$$\chi = \frac{\partial \kappa}{\partial \xi} - \frac{\partial \kappa}{\partial \eta}. \qquad (19)$$

This leads to the identity $\nabla \times (\kappa\nabla\xi + \kappa\nabla\eta) = \chi\mathbf{B}_0$, which can be used to uncurl (15) to obtain

$$\nabla \times \mathbf{E}_1 = \kappa(\nabla\xi + \nabla\eta) + \nabla u, \qquad (20)$$

where u is again an arbitrary function of ξ, η, z, and t. Taking the divergence of (20) gives an equation for u in terms of κ:

$$\begin{aligned} \nabla^2 u = & -\kappa(\nabla^2\xi + \nabla^2\eta) - \left(\frac{\partial \kappa}{\partial \xi} + \frac{\partial \kappa}{\partial \eta}\right) \nabla\xi \cdot \nabla\eta \\ & - \frac{\partial \kappa}{\partial \xi}(\nabla\xi)^2 - \frac{\partial \kappa}{\partial \eta}(\nabla\eta)^2, \qquad (21) \end{aligned}$$

where $\nabla^2\xi = h^2\xi$, $\nabla^2\eta = h^2\eta$, $(\nabla\xi)^2 = h^2\xi^2 + \exp(-2hz)$, $(\nabla\eta)^2 = h^2\eta^2 + \exp(2hz)$, and $\nabla\xi \cdot \nabla\eta = -h^2\xi\eta$. The reason for writing χ in the particular form used in (19) is to obtain a simple solution to (15), which possesses symmetry or antisymmetry when ξ and η are interchanged and z changes sign.

Before proceeding to solve this equation, we should consider the necessary boundary conditions. First of all, the function κ is constant along each magnetic field line since it is a function of the Euler potentials and time only. Hence specifying κ at one photospheric plane will completely specify κ throughout the volume under consideration. Now (21) is just a Poisson equation for u, which requires the specification of either u or its normal derivative at the surfaces $z = 0$ and $z = 1$ for a Parker-type problem. So specifying three free functions of two spatial variables and time completely determines the evolution of the magnetic field. Note that this paragraph as well as our basic equations, (20) and (21), are equally applicable to any other potential configuration $\mathbf{B}_0$ since they are not connected with the particular form of the Euler potentials, (16) and (17).

The ansatz is now introduced that $u = u(\xi, \eta, t)$; that is, the explicit z dependence is dropped. The coefficients of $\exp(2hz)$ and $\exp(-2hz)$ in (21) must then both be zero, leading to the overdetermined system of equations for u and κ:

$$0 = \frac{\partial^2 u}{\partial \xi^2} + \frac{\partial \kappa}{\partial \xi}, \quad (22)$$

$$0 = \frac{\partial^2 u}{\partial \eta^2} + \frac{\partial \kappa}{\partial \eta}, \quad (23)$$

$$0 = \xi^2 \frac{\partial^2 u}{\partial \xi^2} + \eta^2 \frac{\partial^2 u}{\partial \eta^2} - 2\xi\eta \frac{\partial^2 u}{\partial \xi \partial \eta} + \xi \frac{\partial u}{\partial \xi} + \eta \frac{\partial u}{\partial \eta} + \xi(\xi-\eta)\frac{\partial \kappa}{\partial \xi} - \eta(\xi-\eta)\frac{\partial \kappa}{\partial \eta} + (\xi+\eta)\kappa. \quad (24)$$

It turns out that the general solution of (22)–(24) is spanned by the linearly independent solutions (see appendix for details)

$$\kappa_1 = 0, \quad (25)$$

$$\kappa_2 = \frac{1}{2}(\xi - \eta), \quad (26)$$

$$\kappa_3 = \xi \log|\xi| - \eta \log|\eta| - \xi + \eta, \quad (27)$$

$$u_1 = 1, \quad (28)$$

$$u_2 = \frac{1}{4}(\eta^2 - \xi^2), \quad (29)$$

$$u_3 = \frac{1}{2}(\eta^2 \log|\eta| - \xi^2 \log|\xi|) + \frac{3}{4}(\xi^2 - \eta^2) \quad (30)$$

where κ_1 corresponds to u_1, etc. These have all been expressed in forms that are symmetric or antisymmetric in ξ and η. The resulting values of χ are

$$\chi_1 = 0, \quad (31)$$

$$\chi_2 = 1, \quad (32)$$

$$\chi_3 = \log|\xi\eta|. \quad (33)$$

Any solution to the problem with the ansatz that $u = u(\xi, \eta, t)$ can now be obtained by taking linear combinations of the above functions multiplied by functions of t, noting that for u_1, κ_1, and χ_1 the current density vanishes.

Another ansatz is to take $u = u_4(\xi,\eta,t)\exp(2hz) + u_5(\xi,\eta,t)\exp(-2hz) + u_6(\xi,\eta,t)$. The solution space of (21) is then spanned by 15 basis solutions, including those already presented, the 12 new solutions each possessing a term in $\exp(2hz)$ and $\exp(-2hz)$ to maintain symmetry or antisymmetry as $z \to -z$. Two of these that are of interest will have their MHD variables presented in Cartesian form but without the details of the analysis leading to them. It should be noted, however, that the number of solutions obtained is a result of the ansatz used for u and not of the boundary conditions.

The electric field can also be written in terms of Euler potentials by noting that $\mathbf{E}_1 \cdot \mathbf{B}_0 = 0$ from (9). Since $\nabla\xi$ and $\nabla\eta$ are not parallel and each are perpendicular to $\mathbf{B}_0$, this enables the electric field to be written as

$$\mathbf{E}_1 = E_\xi \nabla\xi + E_\eta \nabla\eta. \quad (34)$$

The components of (20) in the $\nabla\eta$, $\nabla\xi$, and $\hat{\mathbf{z}}$ directions are then, respectively,

$$\frac{\partial E_\xi}{\partial z} = -h^2\left(\kappa + \frac{\partial u}{\partial \xi}\right)\xi\eta + h\eta\frac{\partial u}{\partial z} + \left(\kappa + \frac{\partial u}{\partial \eta}\right)\left(h^2\eta^2 + e^{2hz}\right), \quad (35)$$

$$\frac{\partial E_\eta}{\partial z} = -\left(\kappa + \frac{\partial u}{\partial \xi}\right)\left(h^2\xi^2 + e^{-2hz}\right) + h^2\left(\kappa + \frac{\partial u}{\partial \eta}\right)\xi\eta + h\xi\frac{\partial u}{\partial z}, \quad (36)$$

$$\frac{\partial E_\eta}{\partial \xi} - \frac{\partial E_\xi}{\partial \eta} = -h\xi\left(\kappa + \frac{\partial u}{\partial \xi}\right) + h\eta\left(\kappa + \frac{\partial u}{\partial \eta}\right) + \frac{\partial u}{\partial z}. \quad (37)$$

The electric fields arising from the above five solutions, except that of zero current density, are now examined separately.

3. First Logarithmic Solution

Selecting the third basis solution multiplied by $4h$ to ensure that the current density tends to zero as $h \to 0$, (35)–(37) are integrated to give the antisymmetric solutions

$$E_\xi = 2\xi(\log|\xi| - 1)e^{2hz} + 4h^3\xi\eta^2 z(\log|\xi\eta| - 2) - \xi(\log\xi^2 + h^2\eta^2\log\eta^2 - 3h^2\eta^2 - 2), \quad (38)$$

$$E_\eta = 2\eta(1 - \log|\eta|)e^{-2hz} + 4h^3\xi^2\eta z(\log|\xi\eta| - 2) + \eta(h^2\xi^2\log\xi^2 + \log\eta^2 - 3h^2\xi^2 - 2). \quad (39)$$

The derivative $\partial E_\xi/\partial\xi$ possesses a logarithmic singularity at $\xi = 0$ that corresponds to a singularity in $\partial E_x/\partial x$ at $x = 0$, except at the photospheric plane $z = 0$. Similarly, $\partial E_\eta/\partial\eta$ and $\partial E_y/\partial y$ possess singularities at $\eta = 0$ and $y = 0$, respectively. These singularities do not affect the value of the perturbed magnetic field since they are absent from the curl of the perturbed electric field. However, they do appear in the second-order inertial term $(\mathbf{v}_1 \cdot \nabla)\mathbf{v}_1$, with the result that this supposedly negligible quadratic term is, in fact, infinite on the separatrices $x = 0$ and $y = 0$, whenever $z \neq 0$. At the photosphere the inertial term remains finite.

Note that the gradient of a function ϕ can always be subtracted from the electric field without affecting (20). The only restriction on ϕ is that it must satisfy $\mathbf{B}_0 \cdot \nabla\phi = 0$ since $\mathbf{E}_1 \cdot \mathbf{B}_0 = 0$. Thus ϕ is a function of ξ, η, and t only.

Reverting to Cartesian coordinates gives the electric field as

$$\begin{aligned}\mathbf{E}_1 = -x &\left[\left(e^{-2hz} - 2h^3y^2z - 1\right)\log x^2 - h^2y^2\times\right.\\ &(2hz-1)\log y^2 - 2e^{-2hz} - 2hz\left(e^{-2hz}\right.\\ &\left.\left.-5h^2y^2-1\right) - 3h^2y^2 + 2\right]\hat{\mathbf{x}} + y\left[h^2x^2\times\right.\\ &(2hz+1)\log x^2 + \left(e^{2hz} + 2h^3x^2z - 1\right)\log y^2\\ &+2hz\left(e^{2hz} - 5h^2x^2 - 1\right) - 2e^{2hz} - 3h^2x^2\\ &\left.+2\right]\hat{\mathbf{y}} + h\left[x^2\left(e^{-2hz} + h^2y^2 - 1\right)\log x^2\right.\\ &+y^2\left(e^{2hz} + h^2x^2 - 1\right)\log y^2 - 2hz\left(x^2e^{-2hz}\right.\\ &\left.-y^2e^{2hz} - x^2 + y^2\right) - 2\left(x^2e^{-2hz} + y^2e^{2hz}\right.\\ &\left.\left.-x^2 - y^2 + 3h^2x^2y^2\right)\right]\hat{\mathbf{z}}.\end{aligned} \tag{40}$$

The velocity arising from the above electric field and the potential magnetic field can be computed from

$$\mathbf{v}_1 = \frac{\mathbf{E}_1 \times \mathbf{B}_0}{B_0^2} + a(x,y,z,t)\mathbf{B}_0, \tag{41}$$

where $a(x,y,z,t)$ is an arbitrary function that is related to the parallel velocity component by $v_\parallel = a|\mathbf{B}_0|$. Imposing the condition that the z component of the velocity is zero at the photosphere leads to the boundary condition

$$\begin{aligned}&(1+h^2x^2+h^2y^2)a(x,y,0,t) =\\ &\quad h^3xy(x^2\log x^2 - y^2\log y^2 - 3x^2 + 3y^2).\end{aligned} \tag{42}$$

Applying this value to (41) gives the photospheric velocity as

$$\begin{aligned}&\mathbf{v}_1(x,y,0,t) =\\ &\quad h^2xy\left[x(\log x^2 - 3)\hat{\mathbf{x}} + y(\log y^2 - 3)\hat{\mathbf{y}}\right],\end{aligned} \tag{43}$$

and the inertial term as

$$\begin{aligned}(\mathbf{v}_1\cdot\nabla)\mathbf{v}_1(x,y,0,t) = h^4x^2y^2&\left[x(\log x^2 - 3)\times\right.\\ (\log x^4y^2 - 7)\hat{\mathbf{x}} + y(\log y^2 - 3)&\left.(\log x^2y^4 - 7)\hat{\mathbf{y}}\right].\end{aligned} \tag{44}$$

If, in addition, it is desired that the velocity be incompressible at the photosphere, this can be achieved by imposing the condition

$$\left.\frac{\partial a}{\partial z}\right|_{z=0} = -\frac{2h^4xy(x^2+y^2)}{1+h^2x^2+h^2y^2}. \tag{45}$$

The perturbed electric field is not unique, as it can have the gradient of an arbitrary function of ξ, η, and t added to it. The resulting velocity may then also have an arbitrary component parallel to $\mathbf{B}_0$ added and still satisfy (6)–(10), giving considerable freedom in the choice of driving flow that can maintain the magnetic field evolution.

Finally, the perturbed magnetic field and current density are

$$\begin{aligned}\mathbf{B}_1 &= 4ht\left[y(\log|y| + hz - 1)\hat{\mathbf{x}} - x(\log|x| - hz\right.\\ &\quad\left.-1)\hat{\mathbf{y}} - hxy(\log|xy| - 2)\hat{\mathbf{z}}\right], &(46)\\ \mathbf{J}_1 &= -4ht\log|xy|(hx\hat{\mathbf{x}} - hy\hat{\mathbf{y}} + \hat{\mathbf{z}}). &(47)\end{aligned}$$

Since $\zeta\log|\zeta| \to 0$ as $\zeta \to 0$ (where ζ is x or y) the perturbed magnetic field $\mathbf{B}_1$ is continuous at $x = 0$ and $y = 0$. On the other hand, the corresponding current density $\mathbf{J}_1$ diverges logarithmically at $x = 0$ and $y = 0$, representing thereby singular current layers.

We now consider the origin of the singular current density. First of all, the vorticity at $z = 0$ is given by

$$\begin{aligned}\boldsymbol{\omega}_1 = -2hx(h^2y^2 - \log x^2 + 2)\hat{\mathbf{x}} - 2hy(h^2x^2\\ -\log y^2 + 2)\hat{\mathbf{y}} - h^2(x^2\log x^2 - y^2\log y^2 - 3x^2\\ +3y^2)\hat{\mathbf{z}},\end{aligned} \tag{48}$$

which can be seen to be finite everywhere. Next we examine (12). Just as we imposed $\nabla\cdot\mathbf{v}_1 = 0$ at the photosphere, we can ensure that $\nabla(\nabla\cdot\mathbf{v}_1) = 0$ at $z = 0$ by choosing

$$\left.\frac{\partial^2 a}{\partial z^2}\right|_{z=0} = \frac{8h^3xy\log|y/x|}{1+h^2x^2+h^2y^2}. \tag{49}$$

The components of the vorticity gradient matrix are then given by

$$\begin{aligned}\frac{\partial\omega_{1x}}{\partial x} &= 2h(\log x^2 - h^2y^2), &(50)\\ \frac{\partial\omega_{1x}}{\partial y} &= -4h^3xy, &(51)\\ \frac{\partial\omega_{1x}}{\partial z} &= -4h^2x(\log|x^3y| - 1), &(52)\\ \frac{\partial\omega_{1y}}{\partial x} &= -4h^3xy, &(53)\\ \frac{\partial\omega_{1y}}{\partial y} &= 2h(\log y^2 - h^2x^2), &(54)\\ \frac{\partial\omega_{1y}}{\partial z} &= 4h^2y(\log|xy^3| - 1), &(55)\\ \frac{\partial\omega_{1z}}{\partial x} &= -4h^2x(\log|x| - 1), &(56)\end{aligned}$$

$$\frac{\partial\omega_{1z}}{\partial y} = 4h^2 y(\log|y|-1), \tag{57}$$
$$\frac{\partial\omega_{1z}}{\partial z} = 2h[h^2(x^2+y^2)-\log x^2y^2]. \tag{58}$$

This is the only part of (12) that contributes to the singularities in the current density.

The above forms have been chosen as the simplest ones to demonstrate the essential physical features of this class of solution. The photospheric flow and the initial magnetic field are shown in Figure 1.

4. First Nonlogarithmic Solution

We now examine the second basis solution multiplied by $4h$. Following the methods of section 3 leads to the antisymmetric solutions

$$E_\xi = \xi\left(e^{2hz}+4h^3\eta^2 z-h^2\eta^2-1\right), \tag{59}$$
$$E_\eta = -\eta\left(e^{-2hz}-4h^3\xi^2 z-h^2\xi^2-1\right). \tag{60}$$

In Cartesian coordinates this is

$$\begin{aligned}\mathbf{E}_1 = &-x\left(e^{-2hz}-4h^3y^2z+h^2y^2-1\right)\hat{\mathbf{x}} \\ &+y\left(e^{2hz}+4h^3x^2z+h^2x^2-1\right)\hat{\mathbf{y}} \\ &+h\left(x^2e^{-2hz}+y^2e^{2hz}-x^2-y^2+2h^2x^2y^2\right)\hat{\mathbf{z}}.\end{aligned} \tag{61}$$

The velocity is again computed from (41). Requiring the z component of the velocity to be zero at the photosphere then forces

$$a(x,y,0,t)=\frac{h^3xy(x^2-y^2)}{1+h^2x^2+h^2y^2}, \tag{62}$$

giving rise to

$$\mathbf{v}_1(x,y,0,t) = h^2xy(x\hat{\mathbf{x}}+y\hat{\mathbf{y}}), \tag{63}$$
$$(\mathbf{v}_1\cdot\nabla)\mathbf{v}_1(x,y,0,t) = 3h^4x^2y^2(x\hat{\mathbf{x}}+y\hat{\mathbf{y}}). \tag{64}$$

Additionally, enforcing incompressibility at the photosphere requires the choice of a such that

$$\left.\frac{\partial a}{\partial z}\right|_{z=0}=0. \tag{65}$$

Similar restrictions can be imposed at the $z=1$ plane to give a solution akin to that of *Parker* [1972]:

$$\begin{aligned}&(1+h^2x^2+h^2y^2)a(x,y,1,t)= \\ &\quad hxy\left[e^{2h}-e^{-2h}+4h^3(x^2+y^2)+h^2(x^2-y^2)\right], \\ &(1+h^2x^2+h^2y^2)\left.\frac{\partial a}{\partial z}\right|_{z=1}=2h^2xy\left(e^{2h}+e^{-2h}-2\right),\end{aligned} \tag{66}$$

$$\begin{aligned}\mathbf{v}_1(x,y,1,t) = &\, y\left[h^2x^2(1+4h)+e^{2h}-1\right]\hat{\mathbf{x}} \\ &+x\left[h^2y^2(1-4h)+e^{-2h}-1\right]\hat{\mathbf{y}},\end{aligned} \tag{67}$$

$$\begin{aligned}(\mathbf{v}_1\cdot\nabla)\mathbf{v}_1(x,y,1,t) = &\, x\left[h^2x^2(1+4h)+e^{2h}-1\right] \\ &\times\left[h^2y^2(3+4h)+e^{-2h}-1\right]\hat{\mathbf{x}} \\ &+y\left[h^2x^2(3-4h)+e^{-2h}-1\right] \\ &\times\left[h^2y^2(1-4h)+e^{-2h}-1\right]\hat{\mathbf{y}}.\end{aligned} \tag{68}$$

The perturbed magnetic field and current density are

$$\mathbf{B}_1 = 2ht(y\hat{\mathbf{x}}-x\hat{\mathbf{y}}-2hxy\hat{\mathbf{z}}), \tag{69}$$
$$\mathbf{J}_1 = -4ht(hx\hat{\mathbf{x}}-hy\hat{\mathbf{y}}+\hat{\mathbf{z}}). \tag{70}$$

This time we choose

$$\left.\frac{\partial^2 a}{\partial z^2}\right|_{z=0} = 0, \tag{71}$$
$$\left.\frac{\partial^2 a}{\partial z^2}\right|_{z=1} = \frac{4h^3xy\left(e^{2h}-e^{-2h}\right)}{1+h^2x^2+h^2y^2}, \tag{72}$$

and find

$$\boldsymbol{\omega}_1 = 2hx\hat{\mathbf{x}}+2hy\hat{\mathbf{y}}-h^2(x^2-y^2)\hat{\mathbf{z}}, \tag{73}$$

with

$$\nabla\boldsymbol{\omega}_1 = 2h\begin{pmatrix}1 & 0 & -4hx\\ 0 & 1 & 4hy\\ -hx & hy & -2\end{pmatrix}, \tag{74}$$

at $z=0$ and

$$\begin{aligned}\boldsymbol{\omega}_1 = &\, 2hxe^{-2h}\hat{\mathbf{x}}+2hye^{2h}\hat{\mathbf{y}}-\left[e^{2h}-e^{-2h}\right. \\ &\left.+4h^3(x^2+y^2)+h^2(x^2-y^2)\right]\hat{\mathbf{z}},\end{aligned} \tag{75}$$

with $\nabla\boldsymbol{\omega}_1 = 2h\times$

$$\begin{pmatrix}e^{-2h} & 0 & -2hx\left(e^{-2h}+1\right)\\ 0 & e^{2h} & 2hy\left(e^{2h}+1\right)\\ -hx(4h+1) & -hy(4h-1) & -\left(e^{2h}+e^{-2h}\right)\end{pmatrix}, \tag{76}$$

at $z=1$.

The streamlines of the perturbed velocity and the potential magnetic field lines are plotted in Figure 2.

5. Second Logarithmic Solution

The MHD variables are

$$\begin{aligned}\mathbf{E}_1 = x&\left[\left(1-e^{-2hz}+2h^3y^2z\right)\log x^{12}-h^2y^2\times\right. \\ &(2hz-1)\log y^{12}-48h^4y^2z^2+12\left(e^{-2hz}-1\right)\end{aligned}$$

$$+12hz\left(e^{-2hz}+2h^2y^2-1\right)+h^2x^2\left(1-e^{-4hz}\right)\big]\,\hat{\mathbf{x}}$$
$$+y\left[h^2x^2(2hz+1)\log x^{12}-\left(e^{2hz}+2h^3x^2z-1\right)\right.$$
$$\times\log y^{12}-48h^4x^2z^2-12hz\left(e^{2hz}+2h^2x^2-1\right)$$
$$\left.+12\left(e^{2hz}-1\right)+h^2y^2\left(1-e^{4hz}\right)\right]\hat{\mathbf{y}}+h\left[x^2\left(e^{-2hz}\right.\right.$$
$$\left.+h^2y^2-1\right)\log x^{12}-y^2\left(e^{2hz}+h^2x^2-1\right)\log y^{12}$$
$$-12hz\left(x^2e^{-2hz}+y^2e^{2hz}+4h^2x^2y^2-x^2-y^2\right)$$
$$+h^2\left(x^4e^{-4hz}-y^4e^{4hz}\right)-12\left(x^2e^{-2hz}-y^2e^{2hz}\right)$$
$$\left.+(x^2-y^2)[12-h^2(x^2+y^2)]\right]\hat{\mathbf{z}}, \qquad (77)$$

$$\mathbf{v}_1(x,y,0,t)=12h^2xy(x\log|x|\hat{\mathbf{x}}-y\log|y|\hat{\mathbf{y}}), \qquad (78)$$

$$\mathbf{B}_1=4ht\left\{-y[\log y^6+h^2(3x^2+y^2)+6(hz-1)]\hat{\mathbf{x}}\right.$$
$$-x[\log x^6+h^2(x^2+3y^2)-6(hz+1)]\hat{\mathbf{y}}$$
$$\left.+6hxy\left(\log\left|\frac{y}{x}\right|+2hz\right)\hat{\mathbf{z}}\right\}, \qquad (79)$$

$$\mathbf{J}_1=24ht\left(\log\left|\frac{y}{x}\right|+2hz\right)(hx\hat{\mathbf{x}}-hy\hat{\mathbf{y}}+\hat{\mathbf{z}}), \qquad (80)$$

the properties of which are similar to those of the first logarithmic solution. The streamlines of the perturbed velocity and the potential magnetic field lines are plotted in Figure 3.

6. Second Nonlogarithmic Solution

Here

$$\mathbf{E}_1=x\left\{4h^2x^2e^{-4hz}+e^{2hz}-[4h^2(x^2-y^2)-4hz\right.$$
$$\left.+1]e^{-2hz}-4h^2y^2\right\}\hat{\mathbf{x}}+y\left\{4h^2y^2e^{4hz}+e^{-2hz}\right.$$
$$\left.+[4h^2(x^2-y^2)-4hz-1]e^{2hz}-4h^2x^2\right\}\hat{\mathbf{y}}$$
$$+h\left\{4h^2\left(y^4e^{4hz}-x^4e^{-4hz}\right)+[4h^2y^2(x^2-y^2)\right.$$
$$-x^2-y^2-4hy^2z]e^{2hz}+[4h^2x^2(x^2-y^2)$$
$$\left.+x^2+y^2-4hx^2z]e^{-2hz}\right\}\hat{\mathbf{z}}, \qquad (81)$$

$$\mathbf{v}_1(x,y,0,t)=\mathbf{0}, \qquad (82)$$

$$\mathbf{v}_1(x,y,1,t)=y\left\{4h^2y^2e^{4h}+[4h^2(x^2-y^2)-4h\right.$$
$$\left.-1]e^{2h}-4h^2x^2+e^{-2h}\right\}\hat{\mathbf{x}}$$
$$+x\left\{-4h^2x^2e^{-4h}+[4h^2(x^2-y^2)-4h+1]e^{-2h}\right.$$
$$\left.+4h^2y^2-e^{2h}\right\}\hat{\mathbf{y}}, \qquad (83)$$

$$\mathbf{B}_1=4ht\left\{y\left[(2h^2y^2-1)e^{2hz}+(2h^2x^2-1)e^{-2hz}\right]\hat{\mathbf{x}}\right.$$
$$+x\left[(2h^2y^2-1)e^{2hz}+(2h^2x^2-1)e^{-2hz}\right]\hat{\mathbf{y}}$$
$$\left.-2hxy\left(e^{2hz}-e^{-2hz}\right)\hat{\mathbf{z}}\right\}, \qquad (84)$$

$$\mathbf{J}_1=16h^4\left(x^2e^{-2hz}-y^2e^{2hz}\right)(hx\hat{\mathbf{x}}-hy\hat{\mathbf{y}}+\hat{\mathbf{z}}), \qquad (85)$$

the properties of which are similar to those of the first nonlogarithmic solution. The main difference between these two solutions is that in this one the velocity at the $z=0$ plane is everywhere zero, this being the only solution for which we could obtain this feature. The streamlines of the perturbed velocity and the potential magnetic field lines are plotted in Figure 4.

7. Quasi-Separatrix Layers

All four solutions presented here start from the same initial potential magnetic field that has quasi-separatrix layers at $x=0$ and $y=0$ [*Priest and Démoulin,* 1995]. It is found that in two of the examples that have logarithmic singularities in the vorticity gradient matrix, singular current layers can be generated at the QSLs with logarithmically divergent current density distribution. The other two solutions, lacking these logarithmic singularities, have finite current density and, indeed, have zero current density on the QSLs. There is freedom in the addition of the gradient of an arbitrary function of the Euler potentials to the electric field without affecting the three-dimensional incompressibility of the boundary flows or the need to have the flows lying in the plane of the photosphere. We have thus found an additional condition for the appearance of current layers at QSLs.

Démoulin et al. [1996a] based their argument that smooth motions lead to concentrated currents at the QSLs on the expression for the displacement of the field-line footpoints at one plane in terms of the new equilibrium force-free magnetic field generated. Writing

$$\mathbf{B}_1=U_a(\xi,\eta,t)\nabla\xi+V_a(\xi,\eta,t)\nabla\eta+\nabla P(\xi,\eta,S,t), \qquad (86)$$

where ∇S is parallel to $\mathbf{B}_0$, the displacement is found to be

$$\mathbf{d}_1 = \left(U_a+\frac{\partial P}{\partial\xi}\right)\int_{fl}\frac{\nabla\xi}{|\mathbf{B}_0|}dl + \left(V_a+\frac{\partial P}{\partial\eta}\right)\int_{fl}\frac{\nabla\eta}{|\mathbf{B}_0|}dl, \qquad (87)$$

where dl is a line element along the field line and $\int_{fl}$ denotes a line integral taken along the field line. Using the magnetic field of (13), we compute the integrals to be

$$\int_{fl}\frac{\nabla\xi}{|\mathbf{B}_0|}dl = \frac{1}{h}\left(1-e^{-h}\right)\hat{\mathbf{x}}-h\xi\hat{\mathbf{z}}, \qquad (88)$$

$$\int_{fl}\frac{\nabla\eta}{|\mathbf{B}_0|}dl = \frac{1}{h}\left(e^{h}-1\right)\hat{\mathbf{y}}+h\eta\hat{\mathbf{z}}. \qquad (89)$$

The QSLs are located at $\xi=0$ and $\eta=0$. For small h the first term of each integral is of order unity, while the second is of order h, so that the first terms dominate the expression for the displacement. For large h, on the other hand, the first term of the second integral dominates, being of order $\exp(h)/h$, while in the first integral it is the second term that is dominant. Hence for both large and small h the dominant term of the second integral depends only on h and not on how near the field line under consideration is to the QSL. For the first integral this is true for small h; for large h, however, there is a dependence on location but with the result that the magnitude of the integral is at its minimum near the QSL. *Démoulin et al.* [1996a, b] went on to argue that having large changes in the integrals near the QSLs forced the functions U_a and V_a to behave in such a way that the current density is large. In fact, this implies large changes in the current density and not necessarily that the current density itself is large.

8. Conclusions

We have found exact solutions to the equations of linearized, ideal MHD describing the quasi-static evolution of force-free magnetic field perturbations to an initially potential magnetic field driven by photospheric motions. Four solutions have been presented explicitly, two of which possess strong current layers, while the other two do not. We call them singular current layers since the current density logarithmically grows to infinity at the separatrix surfaces of the transverse magnetic field, although the magnetic field itself remains continuous everywhere. This arises because the magnetic field behaves as $\zeta \log|\zeta|$, where ζ is a transverse coordinate in the layers, which in turn is due to a similar behavior of the vorticity distribution at the photosphere. Thus, to get strong current concentrations in the corona, there is no need to have strong vorticity layers in the corresponding places at the photosphere; it is probably sufficient to have only a rapid spatial variation in vorticity. The latter is in contrast with the mechanism of current sheet formation process considered by *Parker* [1972] and *van Ballegooijen* [1985], who imposed tangential discontinuities in the photospheric velocity field to obtain current sheets in the corona. Thus our solutions possess much weaker singularities than those of Parker and van Ballegooijen. In reality, these singularities are resolved owing to finite scales of variation of the vorticity distribution, while their presence in our simple solutions indicates only that a rapid spatial variation of the photospheric vorticity is sufficient for the formation of strong, force-free current layers.

These solutions are also relevant to recent studies of quasi-separatrix layers, where small footpoint displacements at one plane lead to large displacements at the other. It has been conjectured that QSLs are likely sites for reconnection [*Démoulin et al.,* 1996a]. All four of our solutions arise from the situation considered by *Priest and Forbes* [1992] with QSLs at $x = 0$ and $y = 0$. We have shown, however, that the presence of a QSL is not in itself sufficient to generate a strong current layer, with reconnection and heating in likely accompaniment. We find that the generation of our current layers at the QSLs is associated with a rapid spatial variation in the vorticity of the photospheric flow.

Finally, it seems that this type of singular current layer, across which the magnetic field is continuous, has so far been neglected from considerations of coronal current concentrations. We can thus generalize the conclusion of *van Ballegooijen* [1985] to say that it is possible to generate singular current layers without nulls being present and with continuous driving flow when logarithmic singularities are present in the vorticity gradients. This in turn implies that, even for small values of the photospheric vorticity, large gradients in the vorticity may generate strong current layers in the corona during its quasi-static evolution.

Appendix

Equations (22) and (23) may be integrated to give

$$\frac{\partial u}{\partial \xi} + \kappa + f_1(\eta, t) = 0, \tag{A1}$$

$$\frac{\partial u}{\partial \eta} + \kappa + f_2(\xi, t) = 0. \tag{A2}$$

Differentiating (A1) with respect to η and (A2) with respect to ξ and subtracting then gives the integrability condition

$$\frac{\partial \kappa}{\partial \xi} - \frac{\partial \kappa}{\partial \eta} = \frac{\partial f_1}{\partial \eta} - \frac{\partial f_2}{\partial \xi}. \tag{A3}$$

Two more equations can be generated by, in turn, solving (A1) and (A2) for κ and substituting the results into (24):

$$\xi\eta\frac{\partial^2 u}{\partial \xi^2} + \eta^2\frac{\partial^2 u}{\partial \eta^2} - \eta(\xi+\eta)\frac{\partial^2 u}{\partial \xi \partial \eta} - \eta\frac{\partial u}{\partial \xi} + \eta\frac{\partial u}{\partial \eta}$$
$$+\eta(\xi-\eta)\frac{\partial f_1}{\partial \eta} - (\xi+\eta) f_1 = 0, \tag{A4}$$

$$\xi^2\frac{\partial^2 u}{\partial \xi^2} + \xi\eta\frac{\partial^2 u}{\partial \eta^2} - \xi(\xi+\eta)\frac{\partial^2 u}{\partial \xi \partial \eta} + \xi\frac{\partial u}{\partial \xi} - \xi\frac{\partial u}{\partial \eta}$$

$$-\xi(\xi-\eta)\frac{\partial f_2}{\partial \xi}-(\xi+\eta)f_2=0. \tag{A5}$$

Now $\xi \times (\text{A4}) - \eta \times (\text{A5}) + 2\xi\eta \times (\text{A1}) - 2\xi\eta \times (\text{A2})$ gives an equation relating f_1 and f_2:

$$\xi\eta\left(\frac{\partial f_1}{\partial \eta}+\frac{\partial f_2}{\partial \xi}\right)-\xi f_1-\eta f_2=0. \tag{A6}$$

Differentiating this with respect to ξ gives

$$\xi\eta\frac{\partial^2 f_2}{\partial \xi^2}+\eta\frac{\partial f_1}{\partial \eta}-f_1=0, \tag{A7}$$

which can be separated in terms of a function $C_1(t)$ as

$$\begin{aligned} \xi\frac{\partial^2 f_2}{\partial \xi^2} &= C_1, & \text{(A8)}\\ \eta\frac{\partial f_1}{\partial \eta}-f_1 &= -C_1\eta, & \text{(A9)} \end{aligned}$$

with solutions

$$\begin{aligned} f_1 &= \eta(C_4-C_1\log|\eta|), & \text{(A10)}\\ f_2 &= C_1\xi\log|\xi|+(C_3-C_1)\xi+C_2, & \text{(A11)} \end{aligned}$$

C_1 to C_4 being functions of t. The integrability condition, (A3), can now be solved by the method of characteristics for κ:

$$\begin{aligned} \kappa &= -C_1(\xi\log|\xi|-\eta\log|\eta|)+(C_4-C_3)\xi-C_1\eta\\ &\quad +f_3(\xi+\eta,t), \end{aligned} \tag{A12}$$

where f_3 is an arbitrary function. Equations (A1) and (A2) together yield u as

$$\begin{aligned} u &= \frac{1}{2}C_1(\xi^2\log|\xi|-\eta^2\log|\eta|) \\ &\quad +\frac{1}{4}(2C_3-2C_4-C_1)\xi^2+\frac{3}{4}C_1\eta^2\\ &\quad +(C_1-C_4)\xi\eta-C_2\eta-C_5-\int_0^{\xi+\eta}f_3(\zeta,t)\mathrm{d}\zeta, \end{aligned} \tag{A13}$$

where $C_5=C_5(t)$. Equation (24) next requires $C_2=0$. The function χ can now be obtained from (19) as

$$\chi=-C_1(\log|\xi\eta|+1)+C_4-C_3, \tag{A14}$$

which may be spanned by the linearly independent solutions given in (31)–(33). The corresponding values for κ and u, written in symmetric form, appear in (25)–(30). Any solution to the problem with the ansatz that $u=u(\xi,\eta,t)$ can now be obtained by taking linear combinations of these functions multiplied by functions of t.

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Figure 1. Streamlines at $z = 0$ for the first logarithmic χ solution (dotted curves), stagnation lines $x = 0$ and $y = 0$ (solid lines), and unperturbed potential magnetic field (solid curves emerging from the photospheric plane). The streamlines are independent of h, but for the magnetic field lines we have taken $h = 0.1$. In addition to the stagnation lines, there are two source stagnation points at $\pm(e^{3/2}, -e^{3/2})$ and two sink stagnation points at $\pm(e^{3/2}, e^{3/2})$.

Figure 2. The streamlines at $z = 0$ and $z = 1$ for the constant χ solution (dotted curves) for (top) $h = 0.1$ and (bottom) $h = 10$. Other features are represented as in Figure 1. In addition, the stagnation point at $(0, 0, 1)$ is denoted by an asterisk. The driving flow for $z = 1$ and $h = 10$ is actually directed in the y direction at $y = 0$, but its speed is much less than the x-directed flow that appears for $y \neq 0$.

Figure 3. The streamlines at $z = 0$ for the second logarithmic χ solution (dotted curves). As in Figure 1, the streamlines are independent of h, but here $h = 0.1$ when representing the initial magnetic field. The stagnation lines are emphasized as in Figures 1 and 2. In addition, there are X-type stagnation points at $(0, 1)$, $(1, 1)$, $(1, 0)$, $(1, -1)$, $(0, -1)$, $(-1, -1)$, $(-1, 0)$, $(-1, 1)$ and an O-type stagnation point at $(0, 0)$, but this time there are no sources or sinks present.

Figure 4. The streamlines for (top) $h = 0.1$ and (bottom) $h = 10$. All features are represented as in Figures 1–3. In particular, the five stagnation points in the $z = 1$ plane for $h = 0.1$ are indicated by asterisks, while the velocity is everywhere zero in the $z = 0$ plane.