Visco-resistive magnetic reconnection due to steady inertialess flows. Part 1. Exact analytical solutions

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Magnetic reconnection at an X-type neutral point of a two-dimensional magnetic field is studied in an incompressible viscous resistive fluid whose flow is assumed to be slow enough that its inertia is negligible. In both the ideal and resistive magnetohydrodynamic approximations current singularities appear at the X-point and along the separatrices. It is shown here analytically that the combined effect of viscosity and resistivity can resolve these singularities with the flow crossing the separatrices. A wide class of exact solutions describing the structure of the flow and current density distribution is found. The results suggest that reconnection may occur with localized distributions of strong current density restricted to finite regions.

1. Introduction

Magnetic reconnection is a key fundamental process in magnetohydrodynamics (MHD) and thereby of importance for basic fluid mechanics. The analogy between the behaviour of a magnetic field in MHD and the vorticity of a non-magnetic fluid (Elsasser 1946) gives it an added interest, although of course the analogy is not exact since the velocity in the nonlinear vector diffusion equation relates differently to the magnetic field and the vorticity (Moffatt 1978).

In a magnetofluid of large magnetic Reynolds number, which is the norm in solar and astrophysical plasmas as well as in the Earth's magnetosphere and laboratory fusion machines, the magnetic field is frozen to the fluid and moves everywhere with it. The exception in two dimensions is at X-type neutral points where the magnetic field vanishes and extremely large magnetic gradients and asociated electric currents are formed, so that the magnetic field lines can slip through the fluid and reconnect. This leads to a change of topology of the magnetic field and a conversion of magnetic energy into other forms such as heat and kinetic energy. As a result, magnetic reconnection is thought to be at the core of many dynamic processes in the universe, including solar flares, mechanisms for heating the Sun's corona, geomagnetic substorms and tokamac disruptions. Understanding it is therefore of major significance for these physical processes. However, it is inherently a highly nonlinear problem which present a tough challenge for applied mathematicians. Indeed, so far, no analytical solutions have yet been discovered with the expected physical feature of an isolated diffusion region and current spike localized around the magnetic null point where the reconnection is taking place. It is the purpose of this article to present such solutions with the physically reasonable property of possessing a continuous pressure and to show how more general solutions with continuity up to any derivative in the pressure may be constructed.

1.1. *Previous history of the problem*

The problem considered in this article has an interesting history. It started when Syrovatskii (1979) analysed slow (inertialess) steady flows of plasma in the vicinity of a magnetic X-point, using the so-called 'strong magnetic field approximation' for ideal magnetohydrodynamics (MHD). In particular, he found that such flows are necessarily singular at the separatrices, i.e. for any small velocity of the flow on the boundary, the velocity at the separatrices must be infinitely large to support the constancy of pressure along field lines. He considered this result as evidence that, even for very slow plasma movement far from an X-point, the resulting rarefaction of plasma near this point cannot be compensated by a corresponding inflow of material.

Recently, Craig & Rickard (1994) have shown in a similar but resistive and incompressible approximation that one can find a steady regular solution for the fluid motion near the X-point configuration provided one supposes an absence of flow across the separatrices, which is not the usual property of classical reconnection solutions where magnetic flux is transported across the separatrices. These solutions have the undesirable feature that current concentrations extend out along the separatrices to infinity. Then it was proved by Priest *et al.* (1994) as an anti-reconnection theorem that the absence of trans-separatrix flow is an inherent property of inertialess resistive inviscid flows. A necessary condition under which this theorem breaks down for nonlinear flows has been found by Neukirch & Priest (1996).

Can, however, solutions for magnetic reconnection with non-vanishing slow flow across the separatrices be obtained when both resistivity and viscosity are included? The first attempt in this direction (Priest *et al.* 1994) was only partially successful, since it was realized at the time that the resulting analytical expression for the stream function has a discontinuity in the third derivative across the separatrices, which is not physically acceptable, since it implies a corresponding discontinuity in pressure distribution. We demonstrate here that such a mathematically weak discontinuity can be resolved by the combined effect of resistivity and viscosity. A short summary has been given in Titov & Priest (1997a), but here we present a full account of the necessary details and several possible generalizations.

1.2. *Basic equations*

We shall study slow MHD flows of a resistive and viscous incompressible fluid in an X-type magnetic configuration. Suppose that these flows are slow enough that their influence on the magnetic field is a small inertialess perturbation of some unperturbed magnetostatic configuration. As the latter unperturbed state we adopt a potential magnetic field in the neighbourhood of a two-dimensional null point characterized at a distance L_e by magnetic field B_e, so that the unperturbed dimensional field is

$$\tilde{\boldsymbol{B}}_0 \equiv (\tilde{B}_{0x},\, \tilde{B}_{0y}) = B_e/L_e\,(-\tilde{x},\, \tilde{y})$$

in a system of coordinates for which the axes coincide with the separatrix field lines. It transpires that these $(\tilde{x}, \tilde{y})$-coordinates are more suitable for studying the problem than the conventional $(\tilde{x}', \tilde{y}')$-coordinates (figure 1), and so only the final results will be presented in $(\tilde{x}', \tilde{y}')$-coordinates.

Slow steady MHD flow of a uniform incompressible fluid with density $\tilde{\rho}$, magnetic permeability μ, resistivity η and kinematic viscosity ν can be described by linearized

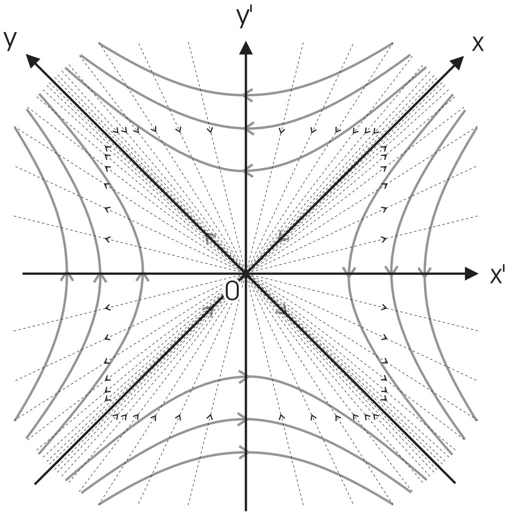

FIGURE 1. The convenient (x, y) and conventional (x', y') systems of dimensionless coordinates used for studying MHD flow near an X-point. The field lines of the unperturbed magnetic configuration are shown by thick grey curves, while thin dashed lines represent the stream lines of an ideal MHD flow perturbing such a configuration.

equations of motion, incompressibility and Ohm's law for the pressure $\tilde{p}_0 + \tilde{p}$, velocity $\tilde{\boldsymbol{v}}$, magnetic field $\tilde{\boldsymbol{B}}_0 + \tilde{\boldsymbol{B}}$ and current density $\tilde{\boldsymbol{j}} = \mu^{-1}\, \tilde{\boldsymbol{\nabla}} \times \tilde{\boldsymbol{B}}$ as follows

$$0 = -\tilde{\boldsymbol{\nabla}}\tilde{p} + \tilde{\boldsymbol{j}} \times \tilde{\boldsymbol{B}}_0 + \tilde{\rho}\nu\tilde{\nabla}^2\tilde{\boldsymbol{v}}, \tag{1.1}$$

$$\tilde{\boldsymbol{\nabla}} \cdot \tilde{\boldsymbol{v}} = 0, \tag{1.2}$$

$$\tilde{\boldsymbol{E}} + \tilde{\boldsymbol{v}} \times \tilde{\boldsymbol{B}}_0 = \eta\tilde{\boldsymbol{j}}. \tag{1.3}$$

Here the subscript 1 denoting perturbed values is omitted for brevity. The electric field $\tilde{\boldsymbol{E}}$ has a uniform value, which can be expressed in terms of the speed v_e and magnetic field B_e at a sufficiently large distance L_e from the X-point where the frozen-in flux condition is fulfilled, so that $\tilde{\boldsymbol{E}} = v_e B_e \hat{\boldsymbol{z}}$. Owing to the assumed slowness of the flow (the Alfvén Mach number $M_A \equiv v_e/v_{Ae} \ll 1$) the inertial term in the equation of motion (1.1) has been neglected. Also the Lorentz force ($\simeq \tilde{j}\hat{\boldsymbol{z}} \times \tilde{\boldsymbol{B}}_0$) is due to the interaction of the perturbed current density $\tilde{j}$ with the potential field $\tilde{\boldsymbol{B}}_0$; the induction electric field ($\tilde{\boldsymbol{v}} \times \tilde{\boldsymbol{B}}_0$) in Ohm's law (1.3) is caused by the interaction of the flow with the same field.

Owing to the incompressibility condition (1.2), the velocity $\tilde{\boldsymbol{v}}$ may be expressed in terms of a stream function $\tilde{\psi}$ as

$$\tilde{\boldsymbol{v}} = \tilde{\boldsymbol{\nabla}} \times (\tilde{\psi}\hat{\boldsymbol{z}}). \tag{1.4}$$

So, after substituting (1.3) and (1.4) into (1.1) and then taking the curl of the obtained equation, one can derive the following linear dimensionless equation for the stream function ψ (Priest *et al.* 1994):

$$(\boldsymbol{B}_0 \cdot \boldsymbol{\nabla})^2\psi - \epsilon\,\nabla^2\nabla^2\psi = 0. \tag{1.5}$$

Here $\boldsymbol{B}_0 = \tilde{\boldsymbol{B}}_0/B_e$, $\boldsymbol{\nabla} = L_e\,\tilde{\boldsymbol{\nabla}}$, $\psi = \tilde{\psi}/(v_e L_e)$ are dimensionless variables (as are $j = \tilde{j}\,\mu L_e/(M_A B_e)$, $p = \tilde{p}\,2\mu/(M_A\, B_e^2)$, $E = \tilde{E}/(v_e B_e)$) and the dimensionless parameter

$$\epsilon = \frac{1}{Re\, R_m} \equiv \frac{\nu\eta}{\mu(L_e\, v_{Ae})^2}$$

is defined in terms of the ordinary ($Re = L_e v_{Ae}/\nu$) and magnetic ($R_m = \mu L_e v_{Ae}/\eta$)

Reynolds numbers, both based on the Alfvén speed $v_{Ae} = B_e/(\mu\tilde{\rho})^{1/2}$. The current density $\boldsymbol{j} = j\hat{\boldsymbol{z}}$ determined by Ohm's law (1.3) acquires as a result of these transformations the following dimensionless form:

$$j/R_m = 1 + \boldsymbol{B}_0 \cdot \boldsymbol{\nabla}\psi. \tag{1.6}$$

If the viscosity is negligible so that $\epsilon \ll 1$, then (1.5) becomes simply

$$(\boldsymbol{B}_0 \cdot \boldsymbol{\nabla})^2\, \psi = 0.$$

After one integration along $\boldsymbol{B}_0$ this leads to (1.6) in which, however, the current density j is now not arbitrary but is constant along field lines, i.e. $j \equiv j(A_0)$, where

$$A_0 = -xy$$

is the unperturbed dimensionless flux function corresponding to $\boldsymbol{B}_0$.

A subsequent integration of (1.6) along $\boldsymbol{B}_0$ with the condition of flow symmetry (requiring that $\psi = 0$ at $x = y$) yields the resistive solution

$$\psi = \tfrac{1}{2}\left(1 - {R_m}^{-1} j(A_0)\right) \log|x/y|\,. \tag{1.7}$$

In the limit of small resistivity, i.e. $R_m \to \infty$, this reduces to the ideal solution

$$\psi = \tfrac{1}{2}\log|x/y| \tag{1.8}$$

such that the velocity components

$$v_x = -1/(2y) \quad \text{and} \quad v_y = -1/(2x)$$

become singular at the separatrices $y = 0$ and $x = 0$, respectively, and therefore at the X-point itself.

At first sight, it may be thought that a regular solution can be obtained from the inviscid resistive expression (1.7) by a suitable choice of the current distribution $j(A_0)$. This possibility has been explored by Craig & Rickard (1994), who considered electric currents that are highly concentrated near the separatrices. However, in this case there is only a simple magnetic diffusion (with no advection) of magnetic flux across the separatrices – a result that has been derived in a general form by Priest *et al.* (1994) as the so-called anti-reconnection theorem.

So let us turn now to the combined effect of viscosity and resistivity described by (1.5). We shall for simplicity assume that the asymptotic form in the non-ideal case is the ideal solution (1.8) except for the narrow regions around the separatrices. This is a natural asymptotic form for physically interesting solutions, but there may well be other solutions with different (non-ideal) asymptotics. Numerical solutions (as reviewed in Priest 1996) suggest that this is a reasonable assumption at slow and intermediate reconnection rates, but that for fast reconnection the central current sheet becomes a sheet which bifurcates to give two standing shock waves in a manner that is analogous to boundary layer separation. Such a highly complex behaviour is not amenable to the exact analytical treatment that we are attempting here. It should be noted first that the parameter ϵ in (1.5) may be absorbed by a simple change of variable such that

$$x = \epsilon^{1/4}\,\bar{x}, \qquad y = \epsilon^{1/4}\,\bar{y}. \tag{1.9}$$

Applying this transformation when $\epsilon \neq 0$ (as assumed in this paper) to (1.5) and then omitting the bars, we obtain

$$\left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right)^2 \psi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2 \psi \tag{1.10}$$

as our basic equation whose solutions give those of (1.5) by a simple rescaling of variables in accord with (1.9). Thus the solutions will be universal in the sense that they hold for all values of ϵ. It may be noted that at large values of $\bar{x}$ and $\bar{y}$ (corresponding to small ϵ) the right-hand side of (1.10) becomes small and we recover the ideal MHD equation.

Note that (1.10) is elliptic, since its highest (fourth-order) derivatives form a biharmonic operator, while its lower-order derivatives do not influence the ellipticity of the equation. So according to the general theory of elliptical equations (see e.g. Bers, John & Schechter 1964) it is possible to formulate for (1.10) in some domain $\mathcal{D}$ an internal Dirichlet problem with given values of the stream function ψ and its normal derivative $\partial\psi/\partial n$ on the boundary $\partial\mathcal{D}$. We shall prove in Part 2 of this article (Titov & Priest 1977b) that such a boundary-value problem can indeed be formulated and develop an approximate method for solving this problem. In this Part, however, it is worth noting that, although (1.10) is a fourth-order partial differential equation, the fact that it is elliptic means that the general global solution involves only two arbitrary functions rather than four. Similarly, the solution of Laplace's equation (of second order) depends on only one free function (which may be imposed on a boundary).

2. A universal family of exact solutions resolving the separatrix singularity

2.1. *Exact solution partially resolving the separatrix singularity*

Let us first try to find a particular analytical solution to equation (1.10) which tends asymptotically to the ideal solution (1.8) far from the separatrices. It is instructive to note that the ideal solution can be written in an additively separable form

$$\psi = \tfrac{1}{2}\log|x| - \tfrac{1}{2}\log|y|. \tag{2.1}$$

The variables x and y are also separated in equation (1.10), since the coefficients of the derivatives in the operator $\boldsymbol{B}_0\cdot\boldsymbol{\nabla} = y\,\partial/\partial y - x\,\partial/\partial x$ are just the appropriate variables of differentiation. This important intrinsic property of equation (1.5) is manifested only in the present coordinate system with axes parallel to the separatrices and is masked in other systems.

Both these observations prompt us to search for a particular solution of (1.10) in the additively separable form:

$$\psi = f(x) - g(y). \tag{2.2}$$

Substituting this expression into (1.10), we find that the function $f(x)$ must satisfy

$$x^2\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + x\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\mathrm{d}^4 f}{\mathrm{d}x^4}. \tag{2.3}$$

Although this equation may generally contain an arbitrary constant as an additional term, for simplicity we set it here equal to zero, which does not affect our final conclusions (see Appendix C). The same equation is obtained for the function $g(y)$, so the assumed form of solution (2.2) is compatible with our basic equation (1.10). In other words, any pair of functions $f(x)$ and $g(y)$, satisfying (2.3) and being superimposed in accord with (2.2), determines an exact solution of (1.10). Such solutions describe a linear superposition of two unidirectional fluid flows parallel to the separatrices. Generally such flows are different, since the functions $f(x)$ and $g(y)$ are not necessarily identical. However, we shall here assume them to be the same so as to give the simplest symmetrical flows.

Equation (2.3) has appeared already in Priest *et al.* (1994) as an equation describ-

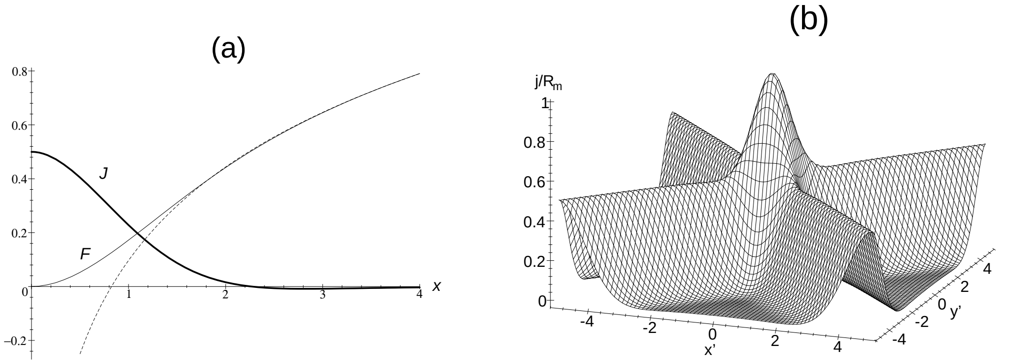

FIGURE 2. (a) The profiles of the stream function $F(x)$ (thin solid curve) and corresponding current density $J(x)$ (thick solid curve) describing a visco-resistive unidirectional shear flow, which has a partially resolved logarithmic singularity at $x = 0$ of the ideal MHD flow. The asymptotic behaviour ($\frac{1}{2}\log x + 0.097$) of $F(x)$ is shown by the dashed curve. (b) The current density distribution in (x', y')-coordinates for the simplest visco-resistive flow as a superposition of the two one-dimensional distributions $J\left((x'-y')/\sqrt{2}\right)$ and $J\left((x'+y')/\sqrt{2}\right)$ parallel to the separatrices $x' = y'$ and $x' = -y'$, respectively.

ing an approximate self-similar flow at the separatrix layer with x being a self-similar variable rather than the x-coordinate. One can see now, however, that its significance is surprisingly much wider – in fact it describes a family of exact solutions (2.2) of equation (1.10). In spite of the differences in meaning and in independent variables, we can use all the results obtained by Priest *et al.* (1994) about the properties of the solutions of equation (2.3).

First of all, there is a single solution $F(x)$ of (2.3) which has the desired logarithmic asymptotics, i.e. $F(x) \to \frac{1}{2}\log|x|$ as $x \to \infty$. It may be written explicitly as

$$F(x) = \frac{1}{4}\int_0^x \xi\, I_{1/4}(\tfrac{1}{4}\xi^2)\, K_{1/4}(\tfrac{1}{4}\xi^2)\, \mathrm{d}\xi, \tag{2.4}$$

where $I_{\frac{1}{4}}$ and $K_{\frac{1}{4}}$ are modified Bessel functions of order one quarter. Thus the solution of the form (2.2) with asymptotics (2.1) is simply

$$\psi = F(x) - F(y). \tag{2.5}$$

When substituted into Ohm's law (1.6) it yields a current density

$$j/R_m = J(x) + J(y), \tag{2.6}$$

where

$$J(x) = \tfrac{1}{2} - \tfrac{1}{4}x^2\, I_{1/4}\left(\tfrac{1}{4}x^2\right)\, K_{1/4}\left(\tfrac{1}{4}x^2\right),$$

and $J(y)$ is the same function of y.

Since the above solution is a linear superposition of two unidirectional flows, it may be well understood with the help of the stream function $F(x)$ of one of the flows represented in figure 2(a). This demonstrates that the logarithmic singularity of the ideal MHD flow (2.1) at the separatrix $x = 0$ (or $y = 0$) is indeed resolved by the combined effect of resistivity and viscosity which is manifested in the appearance of separatrix current spikes. When superimposed, these spikes give the total current distribution shown in figure 2(b). The results are qualitatively the same as in Priest *et al.* (1994), whose approximate solution behaves in a similar way. Moreover, near the separatrices and far from the origin both solutions become asymptotically identical, so the above exact solution (2.5) is approximated by the previous solution.

Even though the solution (2.5) is exact, it has the same disadvantage as the approximate solution of possessing a discontinuous third derivative across the separatrices. This is because both of them are constructed on the basis of the function (2.4) which has a jump in F''' at the origin. Such a jump implies a discontinuity in the tangential component of the viscous force $\tilde{\rho}\nu\nabla^2\boldsymbol{v}$ at the separatrices, which in turn means a corresponding discontinuity of pressure there. This fact is explicitly demonstrated by the dimensionless expression for the pressure perturbation

$$p = (\nu/\eta)^{1/2}\left[-xy + y\left(x^2\,F'(x) - F'''(x)\right) + x\left(y^2\,F'(y) - F'''(y)\right)\right],$$

which can be obtained from (1.1) with the help of (2.4)–(2.6) and the assumption that $M_A\,B_e^2/\mu$ is a scale unit for the perturbed pressure. Thus the above visco-resistive solution (2.5) resolves the singularity in the ideal MHD flow (2.1) only partially, i.e. to within this weak discontinuity. So we need to seek other more physical solutions with better smoothness properties.

2.2. *Physically acceptable family of piecewise analytical solutions*

Formally the above-mentioned discontinuity means that $F(x)$ satisfies

$$x^2\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + x\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\mathrm{d}^4 f}{\mathrm{d}x^4} + \frac{2}{\pi}\,\Gamma\left(\tfrac{3}{4}\right)^2\,\delta(x) \tag{2.7}$$

rather than (2.3), where the additional inhomogeneous term is the Dirac delta-function $\delta(x)$ with a coefficient depending on the value of the gamma function $\Gamma(x)$ at 3/4. This term provides the weak discontinuity which may be interpreted physically as a forced vortex sheet at the separatrix $x = 0$. One may expect to obtain the desired smooth solution if this infinitesimally thin sheet spreads somehow into a corresponding layer of a finite thickness. This implies formally that the equation determining the flow must be inhomogeneous like (2.7) but having instead of $\delta(x)$ some smooth and localized function. So the question is whether such an equation may be derived from (1.10) or not.

The above consideration gives a hint of how to try to answer this question. Indeed, it suggests that the assumption about the unidirectional form of the flow leads inevitably to the presence of a forced vortex layer, which, however, can be described by terms that are additional to the unidirectional form (2.5). So one needs to give more freedom to the fluid motion by generalizing this form.

We achieve success by seeking a solution in the quasi-quadratic form

$$\psi = f_0(x) + f_1(x)\tfrac{1}{2}y^2 - f_0(y) - f_1(y)\tfrac{1}{2}x^2, \tag{2.8}$$

where the unknown functions f_0 and f_1 represent, respectively, the unidirectional and non-unidirectional parts of the flow connected with one another by the following system of equations:

$$x^2\frac{\mathrm{d}^2 f_0}{\mathrm{d}x^2} + x\frac{\mathrm{d}f_0}{\mathrm{d}x} = \frac{\mathrm{d}^4 f_0}{\mathrm{d}x^4} + 2\frac{\mathrm{d}^2 f_1}{\mathrm{d}x^2}, \tag{2.9}$$

$$x^2\frac{\mathrm{d}^2 f_1}{\mathrm{d}x^2} - 3x\frac{\mathrm{d}f_1}{\mathrm{d}x} + 4f_1 = \frac{\mathrm{d}^4 f_1}{\mathrm{d}x^4}, \tag{2.10}$$

which is obtained by substituting (2.8) into (1.10). One can see from (2.9) that the equation for the unidirectional component f_0 really has the desired form if the second derivative $f_1''(x)$ defining the forced vorticity distribution at the separatrix layer is a localized function. The details of how the solutions $f_0(x)$ and $f_1(x)$ may be obtained

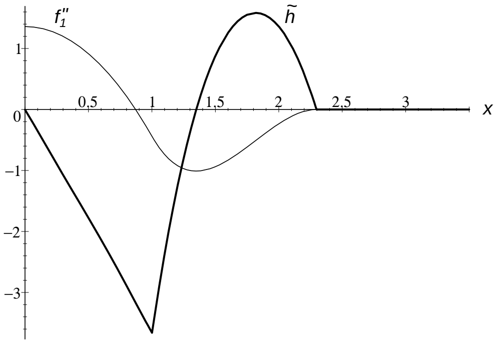

FIGURE 3. The profiles of the piecewise analytical odd function $\tilde{h}(x) \equiv f_1'''(x)$ (thick curve) and even function $f_1''(x)$ (thin curve) which determine a physically acceptable solution; the smoothness of the functions breaks down at the points $x_0 = 0$, $|x_1| = 1.0$ and $|x_2| = 2.2933$.

are given in Appendices A and B, while here we just present a sketch of this non-trivial procedure.

Let us formulate first the minimal requirements on f_0 and f_1 to determine physically acceptable solutions of the form (2.8):

(i) the function $f_0(x)$ must have logarithmic asymptotics, i.e. $f_0(x) \sim \frac{1}{2}\log|x|$ as $|x| \to \infty$;

(ii) the function $f_1(x)$ must be asymptotically vanishing, i.e. $f_1(x) \to 0$ as $|x| \to \infty$;

(iii) both functions $f_0(x)$ and $f_1(x)$ must be continuous up to and including their third derivatives, i.e. $f_0(x),\ f_1(x) \in C^3$.

Thus the requirements (i) and (ii) guarantee the same asymptotic behaviour far from the separatrices as the ideal solution, while the requirement (iii) provides the physically acceptable degree of smoothness for the solution.

Since equation (2.10) contains only f_1, it is natural to solve it first. Note that differentiating this equation twice leads again to equation (2.3) for the new unknown $f(x) = f_1''(x)$. So we can use again our previous knowledge about the general solution of (2.3) (Priest *et al.* 1994) to find this unknown:

$$\frac{\mathrm{d}^2 f_1}{\mathrm{d}x^2} = \int_0^x h(\xi)\,\mathrm{d}\xi + c_4, \tag{2.11}$$

where

$$h = c_1 u_1^2 + c_2 u_2^2 + c_3 u_1 u_2, \tag{2.12}$$

$$u_1 = \xi^{1/2} I_{1/4}(\tfrac{1}{4}\xi^2), \qquad u_2 = \xi^{1/2} K_{1/4}(\tfrac{1}{4}\xi^2) \tag{2.13}$$

at positive ξ and the values c_i, $i = 1, \ldots, 4$, are arbitrary constants. We can try to determine them to satisfy the requirements (i)–(iii). The corresponding analysis shows that it is impossible to do so in the whole range of ξ. However, one can construct a piecewise smooth function

$$\tilde{h} = c_1^i u_1^2 + c_2^i u_2^2 + c_3^i u_1 u_2, \tag{2.14}$$

where i = 1, 2 or 3 when, respectively, ξ belongs to the interval $[0,\ x_1]$, $(x_1,\ x_2]$ or

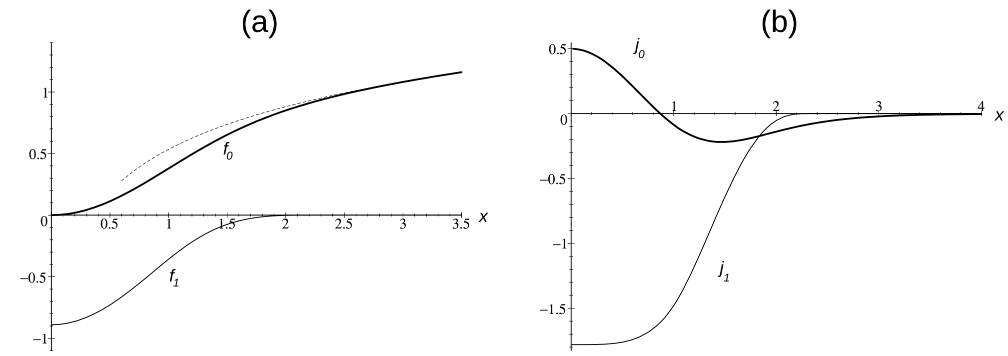

FIGURE 4. (*a*) The stream function components $f_0(x)$ (thick solid curve) and $f_1(x)$ (thin solid curve), respectively, for unidirectional and non-unidirectional parts of the flow in the particular case of $\tilde{h}(x)$ shown in figure 3. The asymptotic behaviour ($\frac{1}{2}\log x{+}0.5343$) of $f_0(x)$ is represented by a dashed curve. (*b*) The components of the current density $j_0(x)$ (thick curve) and $j_1(x)$, respectively, for unidirectional and non-unidirectional parts of the flow in the particular case of $\tilde{h}$ shown in figure 3.

(x_2, ∞); for negative ξ the function $\tilde{h}(\xi)$ is assumed to be continued as an odd function. The sets of constants c^i_j are here chosen so that, first, $\tilde{h}$ has a discontinuity in $\tilde{h}'$ and therefore in f_1^{iv} at any two different points x_1 and x_2 and, second, the requirements (ii) and (iii) are satisfied. An example of such a function together with the corresponding f_1'' is shown in figure 3. In this example we have put $x_1 = 1.0$ and $x_2 = 2.2933$; the value of x_2 is chosen here for simplicity so that $c^3_2 = 0$ and, since $c^3_1 = c^3_3 = 0$ (see Appendix A), $\tilde{h}(\xi) \equiv 0$ at $\xi \geq x_2$.

Integrating (2.11) twice yields

$$f_1(x) = \frac{1}{2}\int_0^x (x-\xi)^2\,\tilde{h}(\xi)\,\mathrm{d}\xi + c_4\tfrac{1}{2}x^2 + f_1'(0)\,x + f_1(0), \tag{2.15}$$

where we have used Cauchy's formula for repeated integration to reduce the three-fold integral to a single integral (e.g. Oldham & Spanier 1974, p. 38). The values $f_1(0)$ and $f_1'(0)$ here are

$$f_1(0) = \tfrac{1}{4}\tilde{h}'(0), \tag{2.16}$$

$$f_1'(0) = \tilde{h}''(0), \tag{2.17}$$

which may be verified by substituting (2.15) into (2.10), Taylor expanding about $x = 0$ and comparing the coefficients of the corresponding powers of x.

On the basis of (2.14)–(2.17) and the requirements (i)–(iii) one can derive a linear system of equations for the constants appearing in (2.14). It is not difficult then to calculate these constants and to transform the expression (2.15) to the form (see Appendices A and B)

$$f_1(x) = -\frac{1}{2}\int_x^\infty (x-\xi)^2\,\tilde{h}(\xi)\,\mathrm{d}\xi.$$

This expression makes it clear that the function $f_1(x)$ is indeed localized if one assumes an exponential decay of $\tilde{h}$ or a complete vanishing at $x \geq x_2$. The resulting solution $f_1(x)$ in the above particular case (figure 3) with $f_1(x) = 0$ at $x \geq x_2$ is shown in figure 4(*a*).

The appropriate solution $f_0(x)$ is also shown in the same figure; it is obtained from

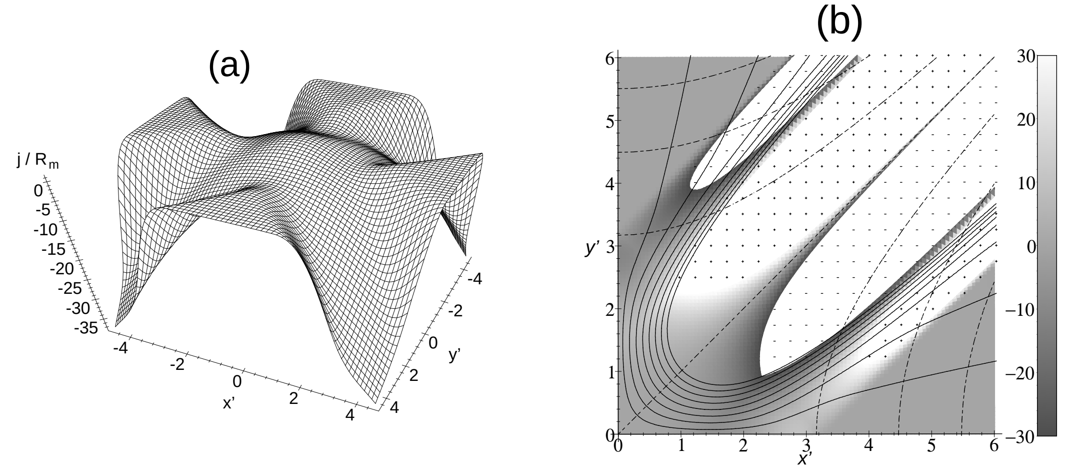

FIGURE 5. (*a*) The current density distribution in perspective for the flows described by the quasi-quadratic stream function. (*b*) The corresponding pressure distribution whose variation is depicted by the distribution with light and dark grey half-tones representing, respectively, the higher and lower values of pressure; a sharp dark border of this distribution outlines the regions of lowest pressure, which are filled by minuses, while the regions of highest pressure are filled by pluses.

(2.9) with the help of the corresponding Green's function $G(\xi, x)$ in the usual way:

$$f_0(x) = 2 \int_{-\infty}^{+\infty} f_1''(\xi)\, G(\xi, x)\, \mathrm{d}\xi. \tag{2.18}$$

This Green's function is determined from an inhomogeneous equation similar to (2.7) but with $\delta(x - \xi)$ on the right-hand side. All the necessary details of its determination and properties are discussed in Appendix B. Here we only note that $G(\xi, x)$ has a logarithmic asymptotic behaviour at large x. Owing to this property any solution $f_0(x)$ based on a localized solution $f_1(x)$ has the required logarithmic asymptotics at infinity.

The current density distribution is found by substituting (2.8) into (1.6) and may be transformed to the form

$$j/R_m = j_0(x) + \tfrac{1}{2} j_1(x)\, y^2 + j_0(y) + \tfrac{1}{2} j_1(y)\, x^2, \tag{2.19}$$

where

$$j_0(x) = \tfrac{1}{2} - x\, f_0'(x), \tag{2.20}$$

$$j_1(x) = 2\, f_1(x) - x\, f_1'(x). \tag{2.21}$$

The resulting profiles of $j_0(x)$ and $j_1(x)$ for the above example (figure 3) are shown in figure 4(*b*), which demonstrates that both components of the current density vanish, as required, far from the origin. The value $j_1(x)$ is a monotonic negative function, while $j_0(x)$ is non-monotonic with a much more noticeable region of reverse current than a similar value $J(x)$ (see figure 2*a*) in the previous solution (2.6). This provides a rather non-trivial distribution of the current density (2.19) at the separatrix layer (figure 5*a*). In the central region there is a nearly axisymmetrical current spike which turns gradually into a region of reverse current growing in value quadratically with increasing the distance from the centre along the separatrices. Physically, the structure in the middle of the separatrix layer allows the viscous stress of the moving fluid tangential to the separatrices to be balanced by a pressure gradient, since the tangential component of the Lorentz force vanishes at the separatrices. However, this also produces a normal pressure gradient which is in turn counterbalanced by the resulting Lorentz force whose direction is consistent with the presence of reverse current at the separatrices. These qualitative arguments are in agreement with the real distribution of the (dimensionless) perturbed pressure which can be determined from (1.1), using (2.8) and (2.19)–(2.21), to be

$$p = \left(\frac{\mu\nu}{\eta}\right)^{1/2} \left[-xy + y\left(x^2 f_0'(x) - f_0'''(x) - f_1'(x)\right) + x\left(y^2 f_0'(y) - f_0'''(y) - f_1'(y)\right)\right.$$
$$\left. + \tfrac{1}{6} y^3 \left(x^2 f_1'(x) - f_1'''(x) - 2x\, f_1(x)\right) + \tfrac{1}{6} x^3 \left(y^2 f_1'(y) - f_1'''(y) - 2y\, f_1(y)\right)\right],$$

where $M_A\, B_e^2/\mu$ is used again as a scale unit. The corresponding distribution of pressure (see figure 5*b*) confirms the above arguments that the reverse currents in the middle of the separatrix layers are necessary elements of the structure for the present flow geometry.

We can estimate *a posteriori* the region of applicability of our solution by comparying the neglected inertia term with the Lorentz force term (the left-hand side of (1.10)). The corresponding procedure includes dimensionalizing, rescaling (1.9), and substituting (2.8) into these terms; then our results can be evaluated at the separatrix layer, say, along the x-axis far from the origin. Such an analysis shows that the dimensional inertia term $-(\boldsymbol{v}\cdot\boldsymbol{\nabla})\nabla^2\psi \equiv \hat{\boldsymbol{z}}\cdot\boldsymbol{\nabla}\psi\times\boldsymbol{\nabla}(\nabla^2\psi)$ has a leading term of the order of $M_A\, Re\, x^3\,(f_1(y)\, f_1'''(y) + f_1'^2(y))/2$. The same procedure for the left-hand side of (1.10) yields $x^2\,(3y\, f_1'(y) - y^2\, f_1''(y) - 4f_1(y))/2$. In both expressions the factors depending on y are of the order of unity in the separatrix layer, so the inertial effect is negligible for $M_A\, Re\, x \ll 1$. The value $x = 1/\epsilon^{1/4}$ corresponds to the dimensional size L_e of the X-point neighbourhood, and so we obtain the inequality $M_A\, Re/\epsilon^{1/4} \ll 1$ or

$$Re \ll M_A^{-4/5}\, R_m^{1/5}$$

which determines the range of dimensionless parameters for which our solution is applicable in a region of size L_e.

2.3. *Other possible generalizations of piecewise analytical solutions*

The solutions of § 2.2 have the disadvantage that the fourth-order derivative of the stream function possesses discontinuities and therefore so does the pressure gradient. However, as indicated in the last paragraph of Appendix A, our method enables us to construct smoother solutions by an appropriate increase of the number of 'patching' points. This is because introducing an extra such point introduces three additional arbitrary constants in the expression for the function $\tilde{h}(x)$, which smoothly interpolates the third derivative of some coefficient in the solution between adjacent 'patching' points. So one of these constants is fixed by the requirement of continuity (or 'patching') of $\tilde{h}(x)$, while the other two can be used for making the first and higher derivatives of $\tilde{h}(x)$ continuous at the 'patching' points. Thus, the degree of smoothness of the solution constructed in this way depends on the number of such points – the larger the number, the better the smoothness of the resulting solution. For example, introducing two extra 'patching' points, say, x_3 and x_4, would make our quasi-quadratic solution smooth up to its fourth derivatives, which means that the pressure gradient would be continuous.

In this approach it is important to note two points. First, there are no evident restrictions on the positions of the 'patching' points; however, physically the visco-resistive effect is likely to be effective only on a length scale of order unity, so the coordinates of the 'patching' points must also lie in the same region. Second, the corresponding continuity conditions form a linear inhomogeneous system of equations for the free constants. So one can expect that in the generic case a unique solution is obtained for fixed positions of the 'patching' points. Both of these have been confirmed by other examples that we have computed.

A large freedom in the positions of the 'patching' points is also understandable as follows. A solution of the basic equation (1.10) is uniquely determined in a finite region, such as a circle, by prescribing values of the stream function and its normal derivative on the boundary of the region. On the other hand, our family of solutions is determined from the asymptotic requirement that they describe ideal MHD flows outside the separatrix layers. This requirement fixes the boundary conditions on most of the boundary but leaves them free close to the sepatrices. Owing to this freedom the positions of the 'patching' points within the separatrix layers are relatively arbitrary.

The finite smoothness of the resulting solutions is due to the assumed Ansatz – its relatively simple form turns out to be too restrictive for representing infinitely differentiable solutions with the proper asymptotic behaviour. However, their value lies in their simplicity and that fact that the addition of extra 'patching' points increases their degree of smoothness.

It is unlikely that the 'patching' points have any particular physical meaning, except that they are the locations where the smoothness of the vorticity distribution is finite. They are similar to the nodes in cubic spline interpolation, where the cubic parabola corresponds in our case to the interpolating function $\tilde{h}(x)$. Such nodes have no particular physical meaning, but this does not reduce the value of the spline method.

The previous consideration suggests that we may seek a more general class of exact solutions in the quasi-polynomial form

$$\psi = \sum_{k=0}^{n} \left(f_k(x)\, y^{2k} - f_k(y)\, x^{2k} \right) / (2k)!. \tag{2.22}$$

Substituting (2.22) into (1.10), one can derive in a similar way as before a system of equations for the functions $f_k(x)$, namely

$$x^2 \frac{\mathrm{d}^2 f_k}{\mathrm{d}x^2} + (1-4k)x\frac{\mathrm{d}f_k}{\mathrm{d}x} + 4k^2 f_k = \frac{\mathrm{d}^4 f_k}{\mathrm{d}x^4} + 2\frac{\mathrm{d}^2 f_{k+1}}{\mathrm{d}x^2} + f_{k+2}, \tag{2.23}$$

$$x^2 \frac{\mathrm{d}^2 f_{n-1}}{\mathrm{d}x^2} + [1-4(n-1)]x\frac{\mathrm{d}f_{n-1}}{\mathrm{d}x} + 4(n-1)^2 f_{n-1} = \frac{\mathrm{d}^4 f_{n-1}}{\mathrm{d}x^4} + 2\frac{\mathrm{d}^2 f_n}{\mathrm{d}x^2}, \tag{2.24}$$

$$x^2 \frac{\mathrm{d}^2 f_n}{\mathrm{d}x^2} + (1-4n)x\frac{\mathrm{d}f_n}{\mathrm{d}x} + 4n^2 f_n = \frac{\mathrm{d}^4 f_n}{\mathrm{d}x^4}, \tag{2.25}$$

where $k = 0, \ldots, n-2$. Each of these equations generally contains an additional arbitrary constant term which for simplicity is put equal to zero here. Its presence probably would yield as before (see Appendix C) only some addition to $f_k(x)$ that grows at infinity.

Physically acceptable solutions of the above system must satisfy two major requirements: first, the functions $f_k(x)$, $k = 1, \ldots, n$, must be localized, i.e. $f_k(x) \to 0$ as $|x| \to \infty$, and, second, they must be sufficiently smooth that at least $f_k(x) \in C^3$. These requirements provide the necessary smoothness and logarithmic asymptotics for the function $f_0(x)$.

The structure of (2.23)–(2.25) implies that we determine the unknowns $f_k(x)$ in descending order of k, starting from $k = n$: one needs first to solve (2.25), then substitute $f_n(x)$ into (2.24) and solve it for $f_{n-1}(x)$, then substitute again $f_n(x)$ and $f_{n-1}(x)$ into (2.23) at $k = n-2$ and solve it for $f_{n-2}(x)$ and so forth, until $k = 0$. It is interesting that at each step of this procedure the homogeneous part of the kth equation ($k = 1, \ldots, n$) can be reduced by $2k$-fold differentiation to the same form as (2.3) but with $f = \mathrm{d}^{2k} f_k / \mathrm{d}x^{2k} \equiv f_k^{(2k)}$, so that (2.25), for instance, becomes

$$x^2 \frac{\mathrm{d}^2 f_n^{(2n)}}{\mathrm{d}x^2} + x \frac{\mathrm{d} f_n^{(2n)}}{\mathrm{d}x} = \frac{\mathrm{d}^4 f_n^{(2n)}}{\mathrm{d}x^4}. \tag{2.26}$$

Then one can apply again the method described in § 2.2, which implies that the general solution of (2.26) may be expressed in terms of the function $\tilde{h}(\xi)$ similar to (2.14). Since the number of points x_i where the smoothness of $\tilde{h}(\xi)$ breaks down may be as large as we like, we may always take enough of them to determine the fitting constants in $\tilde{h}(\xi)$ to satisfy the above requirements of smoothness and asymptotic behaviour. In particular, the solution $f_n(x)$ can be localized and written in a form similar to (2.18):

$$f_n(x) = -\frac{1}{(2n)!} \int_x^{\infty} (x - \xi)^{2n} \tilde{h}(\xi)\, \mathrm{d}\xi. \tag{2.27}$$

In the case of $k = n - 1$, we then obtain

$$x^2 \frac{\mathrm{d}^2 f_{n-1}^{(2n-2)}}{\mathrm{d}x^2} + x \frac{\mathrm{d} f_{n-1}^{(2n-2)}}{\mathrm{d}x} = \frac{\mathrm{d}^4 f_{n-1}^{(2n-2)}}{\mathrm{d}x^4} + 2 f_n^{(2n)},$$

which can be solved for f_{n-1}, using the Green's function (B 14) and Cauchy's formula for repeated integration. This may be performed in a similar way for any other k, and so, in principle, the solutions of (2.23)–(2.25) may be expressed in terms of the corresponding integrals containing $\tilde{h}(\xi)$ and $G(\xi, x)$.

One can see now that the above general method has already been applied in § 2.2 to construct the physical solution in the particular case of $n = 1$. It is not so hard to apply it for $n = 2$, but unfortunately for larger n the method becomes rather complicated because of the increasing number of repeated integrations and asymptotic conditions to be satisfied.

However, some important conclusions can be drawn here without computing particular examples, just by appealing to the quasi-polynomial form of these solutions: as the nth term in (2.27) dominates the rest sufficiently far from the origin along the separatrices, the corresponding absolute value of current density monotonically grows there or, in other words, it is unbounded in the whole plane. On the other hand, in a given finite region it is quite possible to obtain localized current distributions with values completely or almost vanishing near the boundary. This is certainly true in the simplest case of quasi-quadratic solutions (see § 2.2) and seems easier for the quasi-polynomial solutions of higher orders since they have more freedom for variations at small and intermediate distances from the centre of the configuration. Moreover, taking into account the wideness of this class of solutions, one can generally expect that the latter is a characteristic feature of physically acceptable solutions.

3. Summary and discussion

In this paper the analytical theory of slow magnetic reconnection at a neutral X-point of a two-dimensional magnetic field has been developed for the case of steady incompressible fluid flows when both resistivity and viscosity are important. Previous authors have only considered vanishing resistive and viscous effects (Syrovatskii 1979) or vanishing viscous effects (Graig & Rickard 1994). In the first case of ideal magnetohydrodynamics the stream function of any non-trivial flow contains a logarithmic singularity at the separatrices.

It is more non-trivial, however, that in the second case the singularity is resolved by resistivity only by flows with no advection across the separatrices. If trans-separatrix flows are desired, they lead again to the same singularities (Priest *et al.* 1994). The simplest ways to try and avoid them is to include either nonlinear inertial effects or fluid viscosity. In many cases (such as fast reconnection (Priest & Forbes 1986)) it is expected that the nonlinear effects become important mainly in the central region, but for slow enough reconnection a linear approach (see § 1.2) may be appropriate.

We have proceeded here to try and avoid the anti-reconnection theorem by including viscous effects, which Priest *et al.* (1994) had earlier started to explore. That first attempt was only partially successful, since it enabled us to find an approximate solution having a mathematically weak singularity, namely one with a discontinuity in the third derivative of the stream function across the separatrix. It lead, however, to a discontinuity in the corresponding pressure distribution, which was, of course, not physically acceptable.

In this connection we have found in § 2.1 an exact solution that uses the same function as our previous approximate one but in a separable rather than a self-similar form. Both of them have the same third-order discontinuity, but the new solution leads to an explanation of the origin of such a discontinuity and thereby suggests a technique for smoothing it away. Indeed, the solution represents simply a superposition of two unidirectional flows parallel to the separatrices, and so effectively it is the implicit one-dimensionality that causes the above discontinuity, since such an Ansatz turns out to be much too restrictive to satisfy both the proper asymptotic condition at infinity and the smoothness condition at the separatrix.

This point of view is confirmed and developed in § 2.2 where we have proved that, if the fluid is allowed to move more 'freely', assuming a quasi-quadratic form for the stream function, it is possible to obtain a physically acceptable solution with the desired degree of smoothness and asymptotic behaviour. An explicit example of such a solution demonstrates that it differs significantly from the previous exact solution most of all in the current density distribution. Although it is concentrated as before near separatrices, its central spike now decays more rapidly along the separatrices and turns gradually into infinite regions of reverse current growing in value quadratically with distance, instead of stabilizing at some constant value as in the first solution.

It is shown then in § 2.3 that a solution of quasi-quadratic form is just a very particular representative of a wide class of exact analytical solutions having a quasi-polynomial form. The characteristic feature of all of them is that the corresponding current density monotonically grows in value sufficiently far from the centre of the configuration along the separatrices, which means that the distributions of current density for such solutions are unbounded in the whole plane.

However, their more complex polynomial behaviour at small and intermediate distances from the centre indicate that one can obtain also under certain conditions a current distribution that is localized in a finite region, where the current density vanishes near the boundary. It would be interesting in future to explore how far such a region may extend. Unfortunately, it is difficult to study this question by using quasi-polynomial solutions of orders higher than one because of its complexity, so we shall use in the following paper (Part 2) another, simpler, semi-analytical approach to the problem.

We are very grateful to R. E. Grundy for useful advice on solving equations of the form discussed in § 2.1. V. S. Titov thanks very much E. R. Priest for warm hospitality during his stay at the University of St Andrews and Y. R. Romanovsky for helpful discussions. We are delighted to acknowledge financial support from the UK Particle Physics and

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Appendix A. Determination of the non-unidirectional part of the flow: the piecewise analytical function $f_1(x)$

As was shown in § 2.2 the expression for $f_1(x)$ may be written as

$$f_1(x) = \frac{1}{2}\int_0^x (x-\xi)^2\,\tilde{h}(\xi)\,\mathrm{d}\xi + c_4\tfrac{1}{2}x^2 + \tilde{h}''(0)\,x + \tfrac{1}{4}\tilde{h}'(0), \tag{A 1}$$

where $\tilde{h}(0)$ is a piecewise analytical function (2.14). Some of the arbitrary constants entering into this function are determined from the required continuity and oddness of $\tilde{h}(x)$, which in turn provide the necessary smoothness and evenness of $f_1(x)$. These requirements are applied at the point $x=0$ and yield $f_1'(0)=0$ and $f_1'''(0)=0$ or, after using (2.17) and the third derivative of (A 1), they give

$$\tilde{h}''(0) = 0 \quad \text{and} \quad \tilde{h}(0) = 0. \tag{A 2}$$

So the former equation determines a linear relation between the constants c_i^1, $i=1..3$, while the latter simply yields

$$c_3^1 = 0.$$

The continuity of $\tilde{h}(x)$ and hence of $f_1'''(x)$ at the points x_1 and x_2 requires

$$\begin{aligned} &c_1^1\,u_1(x_1)^2 + c_2^1\,u_2(x_1)^2 \\ &\quad = c_1^2\,u_1(x_1)^2 + c_2^2\,u_2(x_1)^2 + c_3^2\,u_1(x_1)\,u_2(x_1), \end{aligned} \tag{A 3}$$

$$c_1^2\,u_1(x_2)^2 + c_2^2\,u_2(x_2)^2 + c_3^2\,u_1(x_2)\,u_2(x_2) = c_2^3\,u_2(x_2)^2. \tag{A 4}$$

In (A 4) we have also used the requirement (ii) about $f_1(x)$ vanishing at infinity (mentioned in § 2.2), which means automatically that $\tilde{h}(x)$ also vanishes there and therefore one needs to put

$$c_1^3 = 0.$$

Also, to have exponential rather than power-law decay of $\tilde{h}(x)$ at infinity, we put

$$c_3^3 = 0.$$

The same asymptotic requirement (ii) leads after expanding $(x-\xi)^2$ in the integral term of (A 1) to three additional conditions, which may be written briefly as

$$\int_0^\infty \tilde{h}(\xi)\,\mathrm{d}\xi + c_4 = 0, \tag{A 5}$$

$$-\int_0^\infty \xi\,\tilde{h}(\xi)\,\mathrm{d}\xi + \tilde{h}''(0) = 0, \tag{A 6}$$

$$\int_0^\infty \xi^2\,\tilde{h}(\xi)\,\mathrm{d}\xi + \tfrac{1}{2}\tilde{h}'(0) = 0. \tag{A 7}$$

One can consider the above relationships (A 2)–(A 7) as a linear system of equations for the constants c_j^i, c_4, $i,j=1,2,3$, when $\tilde{h}$ is replaced by (2.14). These equations are generally independent, so they determine nine of the constants, while the tenth remains free. However, it will be determined later by a normalization condition requiring an appropriate coefficient in the logarithmic asymptotics of the function $f_0(x)$ (see Appendix

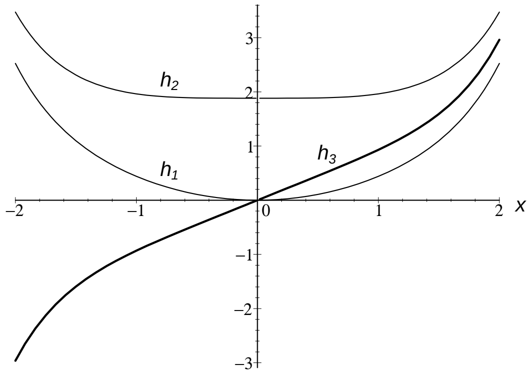

FIGURE 6. Smooth even, $h_1(x)$ and $h_2(x)$, and odd, $h_3(x)$, functions used in the expression for the Green's function (B 14).

B). Thus one can indeed construct a two-parameter family of solutions for equation (2.10) with the necessary asymptotic and smoothness properties (ii) and (iii). The corresponding parameters in this family are the coordinates x_1 and x_2 of the points where smoothness of the stream function breaks down in its fourth derivatives.

Note also that one can construct in a similar way even smoother solutions by an appropriate increase of the number of such points. This is because introducing an extra such point brings three additional arbitrary constants, so that the resulting freedom in the determination of the constants is enough to make both $\tilde{h}(x)$ and $\tilde{h}'(x)$ continuous not only at this new point but also at one of the previous points.

Appendix B. Green's function for unidirectional visco-resistive inertialess flows

It was shown in § 2.2 that the required Green's function $G(\xi, x)$ must be a solution of the equation

$$x^2 \frac{\mathrm{d}^2 g}{\mathrm{d}x^2} + x \frac{\mathrm{d}g}{\mathrm{d}x} = \frac{\mathrm{d}^4 g}{\mathrm{d}x^4} + \delta(x - \xi) \tag{B 1}$$

with logarithmic asymptotics at large $|x|$. It is promising to seek such a solution with the function $F(x)$ (2.4) satisfying equation (2.7). Indeed, if our equation had constant coefficients, we could then simply take $\frac{1}{2}\pi\, F(x-\xi)/\Gamma\left(\frac{3}{4}\right)^2$ as a Green's function, using the translational symmetry and the proper asymptotic behaviour of $F(x)$. Since the coefficients are not constant, however, one can only claim that there is a solution $G(\xi, x)$ of (B 1) such that

$$G(0, x) = \tfrac{1}{2}\pi\, F(x)/\Gamma\left(\tfrac{3}{4}\right)^2 . \tag{B 2}$$

Nevertheless, this fact gives us the hope that for any other ξ one can indeed construct $G(\xi, x)$ with the necessary asymptotic behaviour.

Let us consider now this problem systematically, noticing first that any solution of

(B 1) may be sought in the form

$$g(\xi, x) = \sum_{i=1}^{4} C_i(\xi, x)\, g_i(x), \tag{B 3}$$

where the four functions $g_i(x)$ are functionally independent solutions of the homogeneous equation (2.3) and the coefficients $C_i(\xi, x)$ are new unknowns to be determined by the method of variation of parameters. According to this method, we substitute (B 3) into (B 1) and impose the following constraints on $C_i(\xi, x)$:

$$\sum_{i=1}^{4} \frac{\mathrm{d}C_i}{\mathrm{d}x} \frac{\mathrm{d}^k g_i}{\mathrm{d}x^k} = 0, \quad k = 0, 1, 2, \tag{B 4}$$

$$\sum_{i=1}^{4} \frac{\mathrm{d}C_i}{\mathrm{d}x} \frac{\mathrm{d}^3 g_i}{\mathrm{d}x^3} = -\delta(x - \xi). \tag{B 5}$$

These form a linear system of four equations for the four derivatives $\mathrm{d}C_i/\mathrm{d}x$ which can be found in a standard way. Then, integrating the corresponding expressions over x, one can find the functions $C_i(\xi, x)$ themselves. How far this general procedure is successful depends on the particular choice of solutions $g_i(x)$. It is useful to choose them as smooth as possible, since the system (B 4)–(B 5) contains derivatives of third order. So our previous choice of particular solutions for equation (2.3) based on the modified Bessel functions $I_{1/4}(x^2/4)$ and $K_{1/4}(x^2/4)$ (see (2.11)–(2.13)) is not good enough for our present purposes, since they are not smooth enough at $x = 0$. The analysis shows, however, that one can obtain the desired result, using the function $I_{-1/4}(x^2/4)$ instead of $K_{1/4}(x^2/4)$, so that

$$g_1 = \int_0^x h_1\, \mathrm{d}x, \quad g_2 = \int_0^x h_2\, \mathrm{d}x, \quad g_3 = \int_0^x h_3\, \mathrm{d}x, \quad g_4 = 1, \tag{B 6}$$

$$h_1 = u_1^2, \quad h_2 = u_2^2, \quad h_3 = u_1 u_2 \,\mathrm{sign}\, x, \tag{B 7}$$

$$u_1 = |x|^{1/2} I_{1/4}\left(x^2/4\right), \quad u_2 = |x|^{1/2} I_{-1/4}\left(x^2/4\right). \tag{B 8}$$

All of the functions $g_i(x)$ here are smooth everywhere due to the smoothness of the functions $h_i(x)$ which behave as shown in figure 6.

Solving equations (B 4)–(B 5) is significantly simplified if one transforms $\mathrm{d}^3 g_i/\mathrm{d}x^3$ so as to decrease the order of the derivatives of u_1 and u_2: this can be done by using (B 6)–(B 8) and taking into account the fact that u_1 and u_2 satisfy the parabolic cylinder equation

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} - \frac{x^2}{4} u = 0. \tag{B 9}$$

We thereby derive from (B 4)–(B 5)

$$\frac{\mathrm{d}C_1}{\mathrm{d}x} = -\frac{h_2(x)}{2W^2} \delta(x - \xi), \tag{B 10}$$

$$\frac{\mathrm{d}C_2}{\mathrm{d}x} = -\frac{h_1(x)}{2W^2} \delta(x - \xi), \tag{B 11}$$

$$\frac{\mathrm{d}C_3}{\mathrm{d}x} = \frac{h_3(x)}{W^2} \delta(x - \xi), \tag{B 12}$$

$$\frac{\mathrm{d}C_4}{\mathrm{d}x} = \left(h_2(x)\, g_1(x) + h_1(x)\, g_2(x) - 2h_3(x)\, g_3(x)\right) \frac{\delta(x - \xi)}{2W^2}, \tag{B 13}$$

where $W = u_2\,\mathrm{d}u_1/\mathrm{d}x - u_1\,\mathrm{d}u_2/\mathrm{d}x$ is the Wronskian. For the functions u_1 and u_2 given by (B 8) and satisfying (B 9), W is constant and as $x \to 0$ they yield the value

$$W = 2\frac{\sqrt{2}}{\pi}.$$

Now the coefficients $C_i(\xi, x)$ can easily be found by integrating these equations, and substituting them into (B 3), to obtain

$$\begin{aligned} g(\xi,x) = {} & \tfrac{1}{32}\pi^2\,\mathrm{sign}(\xi - x)\,\left(h_2(\xi)\,g_1(x) + h_1(\xi)\,g_2(x) - 2h_3(\xi)\,g_3(x)\right. \\ & \left. - h_2(\xi)\,g_1(\xi) - h_1(\xi)\,g_2(\xi) + 2h_3(\xi)\,g_3(\xi)\right) \\ & + a_1(\xi)\,g_1(x) + a_2(\xi)\,g_2(x) + a_3(\xi)\,g_3(x) + a_4(\xi). \end{aligned}$$

Here $a_i(\xi)$, $i = 1, \ldots, 4$, are arbitrary constants of integration of (B 10)–(B 13). They depend parametrically on ξ and may be chosen to make $g(\xi, x)$ the Green's function. One can verify that the functions

$$a_1(\xi) = a_2(\xi) = \tfrac{1}{32}\pi^2, \quad a_3(\xi) = \tfrac{1}{32}\pi^2\,\left(h_1(\xi) + h_2(\xi)\right)$$

enable us to obtain the logarithmic asymptotics for $g(\xi, x)$ as $|x| \to \infty$, while

$$a_4(\xi) = \tfrac{1}{32}\pi^2\left(h_3(\xi)\,g_1(\xi) + h_3(\xi)\,g_2(\xi) - \left(h_1(\xi) + h_2(\xi)\right)\,g_3(\xi)\right)$$

makes $g(\xi, x)$ decay like $1/\xi$ as $\xi \to \infty$. Thus the resulting $g(\xi, x) \equiv G(\xi, x)$ may be written after simple transformations as

$$\begin{aligned} G(\xi,x) = {} & \tfrac{1}{32}\pi^2\,\left[\left(h_2(\xi)\,\mathrm{sign}(\xi - x) - h_3(\xi)\right)\left(g_1(x) - g_1(\xi)\right)\right. \\ & + \left(h_1(\xi)\,\mathrm{sign}(\xi - x) - h_3(\xi)\right)\left(g_2(x) - g_2(\xi)\right) \\ & \left. + \left(h_1(\xi) + h_2(\xi) - 2h_3(\xi)\,\mathrm{sign}(\xi - x)\right)\left(g_3(x) - g_3(\xi)\right)\right]. \end{aligned} \tag{B 14}$$

Asymptotically this expression becomes as $|x| \to \infty$

$$G(\xi, x) \to \tfrac{1}{8}|\xi|\left[I_{1/4}\left(\tfrac{1}{4}\xi^2\right) + I_{-1/4}\left(\tfrac{1}{4}\xi^2\right)\right]\,K_{1/4}\left(\tfrac{1}{4}\xi^2\right)\,\log|x|,$$

where we have used the relationship

$$I_{-1/4}(X) - I_{1/4}(X) = \frac{\sqrt{2}}{\pi}\,K_{1/4}(X).$$

So, to obtain the asymptotics of the ideal solution, i.e. $\frac{1}{2}\log|x|$, for the unidirectional part of the above solution (2.18), one needs to impose on $f_1(x)$ the following normalization condition:

$$\int_0^{\infty} \xi\left[I_{1/4}\left(\tfrac{1}{4}\xi^2\right) + I_{-1/4}\left(\tfrac{1}{4}\xi^2\right)\right]\,K_{1/4}\left(\tfrac{1}{4}\xi^2\right)\,f_1''(\xi)\,\mathrm{d}\xi = 1, \tag{B 15}$$

where the evenness of $f_1''(\xi)$ and the asymptotic expression for $G(\xi, x)$ have been used to reduce the integral (2.18). This condition determines an additional linear equation for the free constants entering into the expression for $f_1(x)$ (see (2.14)–(2.17)). Note also that, as expected, $G(0, x)$ may be transformed with the help of (B 15) to the form (B 2).

For the unidirectional part of the current density (2.20) it is useful to determine an explicit expression, which is obtained by substituting (2.18) into (2.20). After using (B 14) it becomes

$$j_0(x) = \frac{1}{2} - \int_{-\infty}^{+\infty} f_1''(\xi)\,L(\xi, x)\,\mathrm{d}x,$$

where

$$L(\xi,x)= \tfrac{1}{16}\pi^2\,x\,[h_3(x)\,(h_1(\xi)+h_2(\xi))-h_3(\xi)\,(h_1(x)+h_2(x)) \\ +\mathrm{sign}(\xi-x)\,(h_1(x)\,h_2(\xi)+h_1(\xi)\,h_2(x)-2h_3(x)\,h_3(\xi))]\,.$$

Appendix C. Unidirectional flows with a uniform forced vorticity

As mentioned in § 2.1 the most general unidirectional flows are described by

$$x^2\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}+x\frac{\mathrm{d}f}{\mathrm{d}x}=\frac{\mathrm{d}^4 f}{\mathrm{d}x^4}+c, \tag{C 1}$$

where the arbitrary constant c determines a spatially uniform distribution of forced vorticity. We show here that its presence does not change the conclusions of § 2.1.

Applying again, as in Appendix B, the method of variation of parameters, we seek a particular solution of (C 1) as

$$f(x)=\sum_{i=1}^{4} C_i(x)\,g_i(x), \tag{C 2}$$

where $g_i(x)$ are given by (B 6)–(B 8). After substituting (C 2) into (C 1) one can derive instead of (B 10)–(B 13) the following expressions for the parameters:

$$C_1=-\frac{c\,g_2(x)}{2W^2},\quad C_2=-\frac{c\,g_1(x)}{2W^2},\quad C_3=\frac{c\,g_3(x)}{2W^2},$$
$$C_4=\frac{c}{2W^2}\int_0^x\left(h_2(\xi)\,g_1(\xi)+h_1(\xi)\,g_2(\xi)-2h_3(\xi)\,g_3(\xi)\right)\,\mathrm{d}\xi$$
$$\equiv\frac{c}{2W^2}\left(g_1(x)\,g_2(x)-g_3(x)^2\right).$$

So we obtain now from (C 2) an explicit form of the particular solution for (C 1), namely

$$f(x)=-\frac{c}{2W^2}g_1(x)\,g_2(x). \tag{C 3}$$

Since $|g_i(x)|\sim e^{x^2/2}/x^2$ as $|x|\to\infty$, $i=1,\,2,\,3$, the asymptotics of (C 3) dominates any solution of (C 1) with $c=0$, so that its absolute value exponentially grows at infinity. Thus, if $c\neq 0$, the unidirectional flows do not possess the desired logarithmic asymptotics – it may be achieved only in the opposite case described in § 2.1.

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