Functional Analysis and Its Applications, Vol. 36, No. 2, pp. 83–92, 2002
Translated from Funktsional′nyi Analiz i Ego Prilozheniya, Vol. 36, No. 2, pp. 1–11, 2002

Optimization in Mean and Phase Transitions in Controlled Dynamical Systems*

V. I. Arnold

Received November 14, 2001

ABSTRACT. The time mean of a smooth objective function along a phase trajectory of a controlled dynamical system is maximized. The simplest singularities of the dependence of the optimal mean value on the parameter in generic one-parameter families of controlled systems of this kind are listed. It turns out that the most common generic stable singularity is the discontinuity of the first or second derivative of the optimal mean value with respect to the parameter.

KEY WORDS: time mean, generic singularities, variational problems, mixed strategies, Harnack theorem, Sturm theorem.

By *optimization in mean* I mean the maximization of the time mean of an objective function given on the phase space of the controlled dynamical system. For this kind of optimization, it is sometimes useful to bring the phase point to the best equilibrium position of the dynamical system, but sometimes it proves more useful to follow a different strategy, for instance, by taking the phase point from time to time from one equilibrium position to another and then returning back. If the controlled system depends on parameters, then the most useful strategy can vary depending on the parameter values. Accordingly, the optimal mean value treated as a function of the parameters can have singularities. In the present paper, we explicitly indicate the simplest singularities (which turn out to be stable) occurring in these "phase transitions" (where the type of the optimal strategy changes as the parameter varies) in the simplest systems; for example, in generic families of controlled systems with a single phase variable, a single control variable, and a single parameter, these are discontinuities of the first or second derivative of the optimal mean value with respect to the parameter.

For larger dimensions of the phase space, of the space of control variables, and of the parameter space, one can also manage to study similar "phase transitions" in generic families, even though the computations become increasingly complicated. The starting point of the present paper was an example pertaining to control of cement mills [1].

1. One-Dimensional System without Equilibria

For simplicity, we assume that the phase space is the circle S^1 with coordinate $x \bmod 2\pi$. The *equation of motion of a controlled dynamical system* is an equation of the form

$$\frac{dx}{dt} = v(x, u), \tag{1}$$

where u is the *control parameter*. For simplicity, we also assume that this parameter is a point of (another) circle and hence the smooth vector field v is 2π-periodic in either of the variables**.

Let f be a smooth *objective function* on the phase circle. We intend to maximize the *time mean*

$$f_* = \lim_{T\to\infty} \frac{1}{T} \int_0^T f(x(t))\, dt, \tag{2}$$

*Partially supported by RFBR-02-01-00655.

**Instead of the direct product $\{(x,u)\}$, one could consider a bundle over the phase space or a singular submanifold of the total space of the tangent bundle of the phase space, but I preserve the notation (1).

of this function by choosing an appropriate *control* $u = U(t)$. (Here $x(t)$ stands for the solution of (1) corresponding to this control and to a given initial condition $x(0)$). This optimization problem can be studied for a given initial condition (then both the optimal strategy and the optimal value f_* depend on the initial condition); another option is to maximize the mean value over all possible choices of the initial condition $x(0)$ as well.

In this section, we assume that v is *nowhere zero* and study the maximization of the mean value without prescribing the initial condition; i.e., $x(0)$ is also varied in the optimization.

Since there are no equilibria, it follows that the phase point makes many revolutions along the phase circle in large time. The optimal motion strategy is to move as slowly as possible at the points where the objective function f takes larger values and as quickly as possible at the points where this function takes smaller values. The time mean can be rewritten as the space mean along the entire phase circle with a time-dependent weight:

$$\int_0^T f(x(t))\,dt = \int_{X_0}^{X_1} f(x)w(x,u)\,dx,$$

where $w = 1/v$ (since $dt = dx/v$ by (1)). The time of motion along the orbit can be expressed in a similar way:

$$T = \int_{X_0}^{X_1} w(x,u)\,dx \quad \text{(for the part of the orbit from } X_0 \text{ to } X_1).$$

The function w is everywhere positive. We seek an optimal strategy by choosing the value of the control parameter u in dependence on the position of the phase point x. Thus, some mass on the phase circle must be distributed with density $\rho(x)\,dx = w(x,u(x))\,dx$ in such a way that the mean value f_* of the objective function f with respect to the distribution ρ is maximal. The only condition on the choice of the distribution is that the value $\rho(x)$ must lie between the maximum and the minimum of $1/v$ with respect to the control parameter u (x is given):

$$\min_u w(x,u) = r(x) \leqslant \rho(x) \leqslant R(x) = \max_u w(x,u).$$

If we fix the total mass, then it is clear that the optimal distribution is as follows: for some constant c, the density ρ must be chosen to be minimal at the points at which the objective function f is less than c and maximal at the points at which the objective function f is greater than c. Indeed, for any other distribution, one can transfer a part of the mass to the domain with greater values of the objective function without violating the conditions.

The optimal mean does not depend on the value of the total mass $\oint \rho\,dx$, because the mean is preserved under the multiplication of the density by a constant. Therefore, it suffices to consider the distribution (depending on c) with density

$$\rho_c(x) = r(x) \quad \text{for } f(x) < c, \qquad \rho_c(x) = R(x) \quad \text{for } f(x) > c,$$

to evaluate the *c-mean value* of the objective function f with respect to this distribution,

$$f_c = \oint f(x)\rho(x)\,dx \Big/ \oint \rho(x)\,dx,$$

and then choose a constant c maximizing this c-mean and set $f_* = \max_c f_c$.

We assume for simplicity that the objective function has only two critical points, namely, a nondegenerate maximum and a nondegenerate minimum.

Theorem 1. *The optimal mean value $f_*(p)$ of the averaged objective function in a family of smooth controlled dynamical systems* (1) (*with the above properties*) *generically depending on a parameter p is a smooth function of the parameter p outside a discrete set of points such that in some neighborhood of each of these points the function f_* has a discontinuity of the first or the second derivative, so that the graph of f^* is diffeomorphic to that of either $|p|$ or $p\,|p|$.*

The sources of these singularities are the singularities of the functions $R(x) = \max_u w(x,u)$ and $r(x) = \min_u w(x,u)$ bounding the density. For generic controlled systems v, these singularities of the maximum and minimum functions graphs are diffeomorphic to the graphs singularities of the functions $\pm|x|$. They originate from the coincidence of critical values at two maximum (minimum) points.

In generic one-parameter families of functions v, there occure three more complicated singularities of the maximum and minimum functions (for certain exceptional parameter values): a) the coincidence of critical values attained at three distinct critical points u_i of the same type; b) the tangency of the graphs of two competing functions of local maximum or minimum that are attained at two distinct critical points; c) an instantaneous appearance of a "double maximum" attained at a triple critical point, as is the case for the function $-u^4$, or a similar phenomenon for the minima.

Under these occurrences, the following perestroikas take place on the graph of the maximum or minimum function. In case a), a small segment between two break points is created or annihilated. In case b), a smooth arc between two close points of the same smooth branch of the graph is created or annihilated. In case c), an instantaneous singularity occurs on the graph such that in the vicinity of this singularity the graph is diffeomorphic to the curve $x^3 = y^4$, and on one side of this parameter value there is no singularity (at this place) on the graph, whereas on the other side the graph has an ordinary break (as the graph of the function $y = |x|$) that is small near the critical value of the parameter.

Other sources of singularities of the optimal mean value are perestroikas of the critical points and values of the objective function f. A generic objective function has nondegenerate ("Morse") maxima and minima at some isolated points of the phase circle, and the critical values are pairwise distinct. If a system depends on a parameter, then, for generic one-parameter families of systems and for isolated parameter values, the critical point of the objective function can degenerate (as is the case for the function $f = 1 + x^3$), or the critical values of the objective function at two distinct critical points can coincide (as is the case for the function $f = x^4 - x^2$).

Finally, for isolated parameter values, there can be degenerations such that both the objective function and the functions r and R defining the constraints have only nondegenerate singularities, but one of the critical points of one type (say, for r) turns out to be critical in another sense as well (say, for f).

The consideration of each of the above-listed cases is easy (with regard for results known in singularity theory; e.g., see [2–5]). Theorem 1 becomes complicated only because the number of cases is large.

Suppose that the parameter p has a generic value p_0 at which none of cases a), b), c) takes place. Let c be a noncritical value of the objective function f. Then the functions r, R, and $D = R - r$ have the simplest "modulus" singularities (with a jump discontinuity of the first derivative of the function at some special points $x = \xi_i$ that smoothly depend on the parameter p near the point p_0).

Lemma 1. *The function given by the "total mass" of the distribution ρ_c,*

$$M(c) = \oint r(x)\,dx + \int_{f(x)>c} D(x)\,dx,$$

is smooth except for isolated points c, where it has a discontinuity of the second derivative; *at such a point we have $c = f(\xi_i)$ (for one of the points $x = \xi_i$ special for D).*

Indeed, $dM/dc = -\sum_j D(x_j)/|f'(x_j)|$, where the x_j are the roots of the equation $f(x_j) = c$.

If these points x_j are not special for D, then the derivative, and as well as the points x_j, smoothly depends on c. If one of the points x_j coincides with a special point ξ_i, then $f(\xi_i) = c$, and the first derivative of M with respect to c has the same "modulus" singularity as the function D at the point c, because x_j smoothly depends on the (noncritical) value c of the objective function. Hence, the function M itself has the simplest discontinuity of the second derivative at this point c, because

$$M(c+t) = M(c) + at + b_{\pm}t^2 + O(|t|^3) \tag{3}$$

for small $|t|$, where $b_+ > b_-$, b_+ is taken in the domain $t > 0$, and b_- in the domain $t < 0$.

Lemma 2. *For parameter values $p = p_0 + s$ close to p_0, the behavior of M is described by a formula similar to* (3) *in which $c = f(\xi_i)$, together with ξ_i, depends on the small parameter s smoothly and the values of all coefficients $M(c)$, a, b_+, and b_- smoothly depend on the small parameter s.*

The proof of this parametric version of Lemma 1 is similar to that of the lemma itself and goes by using the above formula for dM/dc and taking account of the smooth dependence of the singular points ξ_i of the function D (as well as the values of this function and of its one-sided derivatives with respect to x at these singular points) on the parameter p in the vicinity of p_0.

Now consider a somewhat more general function than M. This function expresses the integral of f with respect to the distribution ρ_c,

$$N(c) = \oint f(x)r(x)\,dx + \int_{f(x)>c} f(x)R(x)\,dx.$$

For this function, the derivative with respect to c is expressed by the formula

$$\frac{dN}{dc} = -\sum_j \frac{f(x_j)D(x_j)}{|f'(x_j)|}, \qquad f(x_j) = c.$$

Therefore, the behavior of this function near the special value $c = f(\xi_i)$ is described by a formula similar to (3) (which depends on the parameter p smoothly near p_0, as is the case in formula (3)) with other coefficients describing a simple discontinuity of the second derivative of N with respect to c, just as in the case of M.

Let us now study the *c-mean value of the objective function*, i.e., the ratio

$$f_c = N(c)/M(c),$$

whose maximum with respect to c is just the optimal mean value f_*. Our task is to study the dependence of this maximum on the value of the parameter p. We assume that the family in question is generic.

Lemma 3. *If for a generic value p_0 of the parameter p a nondegenerate maximum f_* of the function f_c is attained at a nonspecial point ($c \neq f(\xi_i)$), then this maximum f_* is a smooth function of the parameter p (in a neighborhood of the point p_0). If the maximum is attained at a special point ($c = f(\xi_i)$), then the function f_* has the simplest discontinuity of the second derivative at the point $p = p_0$:*

$$f_*(p_0 + s) = f_*(p_0) + As + B_{\pm}s^2 + O(|s|^3),$$

where the numbers B_+ and B_- (which are not equal) are used for $s > 0$ and $s < 0$, respectively.

We first note that c is a noncritical value of the objective function f. Indeed, let c be the maximum of f. Then $M(c) = \oint r(x)\,dx$ and $N(c) = \oint f(x)r(x)\,dx$. When slightly reducing the value c (by t), a segment whose length is of the order of $\sqrt{t}$ appears in the distribution ρ_c (near the maximum point), on which the density ρ increases by D. Hence, the distribution of masses is shifted in such a way that the fraction of points with greater values of the averaged function f increases, and hence the mean f_c itself increases. Hence, the *value c that is maximal for f is not a maximum point of the function f_c of the variable c.*

If c is the minimum of the objective function, then

$$M(c) = \oint R(x)\,dx, \qquad N(c) = \oint f(x)R(x)\,dx.$$

When increasing the value c by a small increment t, the density ρ_c is reduced on a segment whose length is of the order of $\sqrt{t}$ (near the minimum point of the objective function). Therefore, the mean value increases, because the relative contribution of the points with smaller values of the objective function is reduced. Thus, the minimum value c of the function f is not a maximum point of the c-mean either.

Remark. We use the assumption on the uniqueness of the maximum and minimum points of the objective function only in the above argument. The case in which the number of local critical points is greater than two requires additional investigation. However, the other parts of our argument do not depend on the number of critical points (and the results remain valid for the noncritical values c for the case in which the number of critical points exceeds two).

If in Lemma 3 the *maximum is attained at a nonspecial point*, then, according to Lemma 2, one speaks of the maximum of a smooth function that depends on a parameter as well. If the maximum is nondegenerate, then the maximal value smoothly depends on the parameter. In this situation, the case of competition of several local maxima can be investigated just as in the study of the above cases a), b), c). A discontinuity of the first derivative of the global maximum with respect to the parameter occurs; namely, $f_*(p_0+t) = f_*(p_0) + At + B_\pm t + O(t^2)$ (B_+ for $t \geqslant 0$ and B_- for $t \leqslant 0$).

If the maximum in Lemma 3 is attained at a special point, then we obtain the following formulas: the expansions

$$f_{c(s)+t} = f_0(s) + A(s)t + B_\pm(s)t^2 + O(|t|^3)$$

hold for $p = p_0 + s$, where $c(s) = f(\xi_i(s))$ is the value of the objective function at a singular point of the function D for $p = p_0 + s$; the coefficients f_0, A, and $B_\pm$ are smooth functions for small values of $|s|$, and $B_+ \neq B_-$ (B_+ is used for $t > 0$ and B_- for $t < 0$). Both quantities $B_\pm(0)$ are negative, because the critical point is a maximum at $s = 0$.

For a critical value of the parameter p (for which $s = 0$), the maximum is attained at $t = 0$, i.e., $A(0) = 0$. For small s, the summand that is linear in t has the derivative $A(s)$ of order s. Therefore, the critical point $t_*(s)$ of the function $f_{c(s)+t}$ with respect to t is defined (in the first approximation with respect to s) by the condition $2B_\pm t_* + A = 0$, which gives the asymptotic behavior $t_* = q_\pm s + O(s^2)$, where $q_\pm = -A'(0)/B_\pm(0)$ (the indices of the coefficients q and B are determined by the sign of t_*, which alternates as the sign of s alternates, because the quantities q_+ and q_-, as well as B_+ and B_-, are of the same sign).

Substituting the above value $t_*(s)$, we arrive at the following expression for the maximal value of the c-mean (attained at the point $t_*(s)$):

$$f_{c(s)+t_*(s)} = f_{00} + f_{01}s + f_{02}s^2 + C_\pm s^2 + O(|s|^3)$$

(for small s), which proves Lemma 3.

Remark. Strictly speaking, to extract Theorem 1 from these lemmas, one should consider the remaining exceptional cases of the values of the parameter p for which the singularities of the function D are subjected to surgeries (in accordance with the perestroikas a), b), and c) discussed above).

For these values of the parameter p, more complicated singularities than those studied above can occur at some points c_* on the curve $\{M(c), N(c)\}$.

However, these unconventional singularities of the curve can affect the behavior of the optimal mean value f_* of the objective function in dependence on the parameter p only if the above-mentioned point c_* is just the point at which the ratio N/M is maximal, i.e., the point of tangency of the curve $\{M, N\}$ with the radius vector.

On the other hand, this tangency (at a conventional singular point of the curve), which was studied in Lemma 3, can happen only at some exceptional values of the parameter p. Unconventional singularities of the curve can also occur at some values of p (which are exceptional in another sense). For generic families, the exceptional values of these two kinds are distinct. For this reason, for generic families, the perestroikas of singularities of the functions R and r under the modification of the parameter p give no new singularities of the dependence of the optimal mean value on the parameter.

Thus, only "modulus" singularities remain in question (with the simplest discontinuity of the first derivative), as is the case of the dependence on the parameter of the maximum of a generic smooth function of one variable that depends on a single parameter, and the singularities "of type $s\,|s|$" with the simplest discontinuity of the second derivative that are provided by Lemma 3.

Remark. A passage through a singularity of one of the types a), b), c) of the maximum and minimum functions in the course of the modification of the parameter leads to certain singularities of the dependence of the functions $M(c)$ and $N(c)$ on the parameter s; these singularities can be of the following three forms.

Case a): smooth dependence on the parameter is violated (for some sign of the parameter increment s) by a correction of the order of s^2.

Case b): smooth dependence on the parameter is violated (for some sign of s) by a correction of the order of $|s|^{3/2}$.

Case c): smooth dependence on the parameter is violated (for some sign of s) by a correction of the order of $|s|^{7/2}$.

The last result is obtained from the normal form of the "swallowtail" singularity (describing the collision of two maximum points),

$$w = g + hxu + asu^2 - bu^4.$$

The critical points u are described by the equation

$$hx = 4bu^3 - 2asu, \tag{4}$$

and the critical values, by the formula

$$w_* = g + 3bu^4 - asu^2. \tag{5}$$

The collision occurs at the zero value of the parameter s at $x = 0$. Formulas (4) and (5) (for a given value of the parameter s) define a curve (on the plane $\{(x, w_*)\}$) parametrized by the critical point u. This curve is the graph of the triple critical value (depending on x) of the function w of u.

The local maxima correspond to the values u placed to the left of the first passage of the curve through the double point (at which $x = 0$ and $u^2 = as/(2b)$) and to the right of the second passage. We can now readily estimate the modification of the function

$$M_+ = \int w_*\,dx = \int w_*(dx/du)\,du$$

under the modification of s by formulas (4) and (5). Let us substitute the expressions (4) and (5) for x and w_* into this integral. The increment $M_+(s > 0) - M_+(s < 0)$ turns out to be of the order of $|s|u_*^5$, i.e., of the order of $|s|^{7/2}$ (for small $|s|$). The area of the curvilinear triangle bounded by our curve (between the cusp points and the point of self-intersection of the graph) is of the same order for an appropriate sign of the parameter s.

In the same way, along with Theorem 1, we describe other singularities of the dependence of the curves $\{M, N\}$ on p.

2. Mixed Strategies

Now consider a situation that is in a sense opposite to the previous one: let a dynamical controlled system have *many equilibria*; the problem is to make a choice among them. However, it often occurs that it is optimal to make a *dynamical walk* among these equilibria, which gives a greater advantage in the mean than any steady equilibrium regime, rather than simply *choose* one of the competing stationary regimes.

The equation of motion of the controlled system has the same form (1), but now the vector field v has an entire curve K of fixed points, $v(x, u) = 0$ (in the product of the one-dimensional phase space with coordinate x by the one-dimensional space of values of the control parameter with coordinate u).

If the entire phase trajectory reduces or converges to a fixed point of this kind, then the time mean of the objective function f along this trajectory is the value of the objective function f at this very point. Therefore, to optimize the time mean, it is natural first to consider the restriction of the objective function f to the K, to find (on K) the point of maximum of the restriction of the objective function to K, and finally to try to approach this point by choosing the control.

As we shall see now, one can approach the time mean provided by this equilibrium point of conditional maximum under an appropriate strategy of control. Therefore, we list (in Theorem 2 below) the generic singularities of this nature of the "optimal" time means.

However, the strategies and the possibility of their implementation are by no means obvious, and the absence of better strategies is not obvious either.

For example, we can consider the system $\dot{x} = u + x^2$ with parabolic curve K of fixed points ($u = -x^2$) and with the objective function $f = x$. In this example, neither the phase space nor the space of values of the control parameter $\{u\}$ is compact, but in the general considerations below I take no precautions necessary to include noncompact cases of the above type into the general theory (it is usually sufficient to impose boundary conditions on v at infinity under which the phase point cannot go there in finite time).

Definition. Oriented curve C on the plane with coordinates (x, u) is said to be *admissible* if along this curve we have $dx > 0$ if $v(x, u) > 0$ and $dx < 0$ if $v(x, u) < 0$.

In the above example, we must move to the left below the parabola $u = -x^2$ and to the right above the parabola (this rule is realized by the dynamical system).

We also assume for simplicity that an admissible curve is strictly "vertical" ($dx \equiv 0$) not only on K but also in a neighborhood of the curve K of fixed points.

Proposition. *For any admissible closed curve, there is a control $u = U(t)$ such that the point $(x(U(t)), u = U(t))$ periodically moves along this admissible curve.*

Example. For the equation $\dot{x} = u + x^2$, for an admissible curve we take a rectangle (whose angles could be rounded) with sides $x = A$ and $x = B$ and take $U = a$ and $U = -b$ (we assume, for instance, that $0 < A < B$, $a > 0$, and $b > B^2$).

Here we can take the control $U(t)$ in the form of a piecewise constant discontinuous function with values a and $-b$. For the initial point of the admissible curve, we take the point $(x = A, U = a)$. The value $U = a$ is preserved until x grows up to B, after which we choose $U = -b$ and keep it equal to $-b$ until x reduces up to A. After this, everything is repeated periodically.

The construction of control for any admissible curve C is based on the same idea. If, say, a point (x, u) of the curve C belongs to the domain $v > 0$ and if $du > 0$ at this point, then we go from the curve C by slightly increasing the value u and wait a bit until the motion with velocity v returns the phase point to C, after which we repeat the process to obtain a step-like broken line approaching C. The passage to the limit of infinitesimal steps makes no problem.

If A and B are close to some common value X of the phase coordinate x, then the time mean of the objective function f along the admissible curve belonging to the strip $A < x < B$ is close to $f(X)$.

Thus, we have constructed special controls for which the time means of the objective function turn out to be almost equal to the value of the function itself at an arbitrarily chosen point on the curve K of equilibria.

Remark. Some points of the curve of equilibria are attracting for the corresponding constant value of the control parameter u (for instance, these are the points of K in the left half-plane ($x < 0$) in the example $\dot{x} = u + x^2$). One can readily construct a control U that leads the phase point to an attracting point of this kind. In this case, the time mean is obviously equal to the value of the objective function at this attracting point.

However, admissible curves can also define much more complicated "admissible strategies," which enable us to approach the maximum of the restriction of the objective function to the curve of fixed points of the controlled dynamical system, which is by no means simple if the maximum point is not attracting.

We thus obtain a family of admissible strategies that realizes the maximum of the restriction of the objective function to the curve of fixed points of the controlled dynamics as the time mean which is optimal for this class of strategies. We can readily prove the following assertion.

Theorem 2. *The maximum $F(p)$ of the restriction of a smooth function f to the curve of fixed points ($K_p = \{x, u : v(x, u; p) = 0\}$) of a smooth dynamical controlled system $\dot{x} = v(x, u; p)$ smoothly depending on the parameter p is a smooth function of p outside a discrete set of singular points, and in a neighborhood of each of these points the graph of the function F is diffeomorphic to the graph of one of the two functions $F_1 = |p|$ and $F_2 = \{0$ for $p < 0$, $1 + \sqrt{p}$ for $p > 0\}$ (for generic f and v).*

Let the curve of equilibria of the controlled system be nonsingular for some value of the parameter p (so that 0 is not a critical value of the function v). Then our special admissible strategies provide the mean value (of the objective function) that is equal to the maximum of a generic smooth function of two variables with respect to one of the two variables:

$$F(p) = \max_{K_p} f(x) = \max_{s \in S^1} G(s, t), \qquad t = p - p_0.$$

As in the case of maximum of the function w with respect to u, which was treated in §1, the graph of the function F has the same singularities as the skyline of a generic landscape G; namely, for some $a > b$, the relations

$$F(p_0 + t) = c + at + O(t^2) \quad \text{for } t \geqslant 0,$$
$$F(p_0 + t) = c + bt + O(t^2) \quad \text{for } t \leqslant 0,$$

hold for small t. This is the first case of Theorem 2 ("modulus singularity").

However, if this curve of equilibria K_{p_0} has a singularity, then, for a generic family, this curve is subjected to a Morse surgery as the parameter p passes through the value p_0 (typical examples are $v = x^2 \pm u^2 + p$ and $p_0 = 0$).

In this case, as the parameter p passes through the critical value, on one side of this value there appears a new segment of values x (and hence $f(x)$) on the curve K_p, namely, a segment whose length is of the order of $\sqrt{|t|}$. If the value F_0 of the function $f(x)$ at the critical point is greater than the values on the other part of the curve of fixed points, then the maximal values $F(t) = F_0 + F_1\sqrt{|t|} + O(t)$ occur (on one side of the value $t = 0$).

This is the other case of Theorem 2 (on the other side of $t = 0$ the function F is smooth and less than F_0; a diffeomorphism of the plane that reduces the graph to the normal form indicated in the theorem is obtained by a variable shift along the axis of values on the plane of the graph, which transforms the smooth part of the graph to a horizontal line, together with a subsequent dilation of the vertical coordinate that normalizes this coordinate and smoothly depends on the horizontal coordinate).

Here we assumed as above for simplicity that $x \in S^1$ and $u \in S^1$. However, more general cases (for instance, with an arbitrary compact phase manifold) present no new complications, especially if the dimensions of the space of control variables $\{u\}$ and the parameter space ($\{p\}$) are small [9].

However, even for our simplest case in which both spaces are one-dimensional, Theorem 2 describes the singularities only for time means that are obtained by using strategies that are optimal in the set of our special admissible strategies. I do not know if the singularities of the dependence on the parameter for true optimal time means are the same.

It is of interest to compare these "true" singularities with the singularities of the maximum functions of generic families of functions depending on arbitrarily many (and even on infinitely many) variables (for generic families of functions depending on the same number of parameters as the controlled dynamical system considered in the family).

For instance, the maximum functions for generic families of functions on a given manifold are topologically equivalent to Morse smooth functions. It is not known if this is the case for the maximal values of the time means over all controls in generic families of controlled systems.

The problems of generic phase transitions, i.e., problems concerning the singularities in the dependence of the maximal values of the means on the parameters of the problem are also of interest for systems of partial differential equations in which the role of time t over which the mean is taken is played by a point of an integral manifold whose dimension is greater than one. In this case, it is natural to assume that the objective function depends not only on the desired "values of fields" (which play the role of the values x for the points of the "orbit" $\{(x(t))\}$ in the case of one-dimensional time treated above) but also on their partial derivatives $\partial x_i/\partial t_j$ and on the values of the control parameters u_k. Already in our one-dimensional problem (1) one could consider a more general objective function $f(x,u)$ instead of $f(x)$ with almost no changes in the results and proofs.

For an example of a problem with "many-dimensional" time, one can consider cases of "quasiperiodic" functions or pseudoperiodic sums of linear functions with periodic functions arising in pseudoperiodic topology. Consider a periodic function (for instance, a trigonometric polynomial) in n variables and restrict it to an "irrational" plane of dimension k in n-dimensional affine space (which is the universal covering of the torus on which the original periodic function is in fact defined); this plane plays the role of integral manifold. The means of diverse analytic and topological characteristics of such "quasiperiodic" restrictions were recently studied by Gusein-Zade (see [6, 7]). The phase transitions should appear here in the form of singularities of the dependence of these means on the parameters (for instance, on the coefficients of the original trigonometric polynomial and on the choice of the affine "integral plane"), but they are less studied, although the problem of their investigation is long-standing and was discussed, for instance, in the paper [8] continued in [7,6].

The quasiperiodic analog of the Harnack theorem gives upper bounds for both topological and metric characteristics of pseudoperiodic functions and manifolds (via the degree of trigonometric polynomial or the Newton diagram of the periodic function whose restriction to the irrational plane defines the quasiperiodic object under consideration). The length of the curve (on the torus of standard size) on which a trigonometric polynomial of degree n vanishes is bounded above by a constant that depends on n alone rather than on the coefficients of this polynomial. The lengths of all algebraic curves of a fixed degree on the sphere of unit radius in the three-dimensional space are also bounded above. The Sturm–Hurwitz theorem on the zeros of Fourier series has similar analogs: the length of the curve on the standard torus on which a trigonometric polynomial vanishes is bounded *below* by some constant $c(n)$ (independent of the coefficients of the polynomial) if this polynomial is orthogonal to all trigonometric polynomials of degree n (see [10]).

References

1. V. I. Arnold, "Convex hulls and the increase of efficiency of systems under impulse loading," Sib. Mat. Zh., **28**, No. 4, 27–31 (1987).
2. V. I. Arnold, A. N. Varchenko, S. M. Gusein-Zade, Singularities of Differentiable Maps, I [in Russian], Nauka, Moscow, 1982; English transl.: V. I. Arnold, S. M. Gusein-Zade, A. N. Varchenko, Singularities of Differentiable Maps, Vol. I, Birkhäuser Boston, Inc., Boston, Mass., 1985.
3. V. I. Arnold, V. A. Vassiliev, V. V. Goryunov, and O. V. Lyashko, Singularities. I. Local and Global Theory, Dynamical Systems-6, Current problems in Mathematics. Fundamental Directions [in Russian], Vol. 6, VINITI, Moscow, 1988; English transl.: V. I. Arnold, V. V. Goryunov, O. V. Lyashko, and V. A. Vasilev, Singularity Theory. I, Dynamical Systems VI, Encyclopaedia Math. Sci., Vol. 6, Springer-Verlag, Berlin, 1993.

4. V. I. Arnold, V. A. Vassiliev, V. V. Goryunov, and O. V. Lyashko, Singularity. II. Classification and Applications, Dynamical Systems-8, Current problems in Mathematics. Fundamental Directions [in Russian], Vol. 39, VINITI, Moscow, 1989; English transl.: V. I. Arnold, V. V. Goryunov, O. V. Lyashko, and V. A. Vasilev, Singularity Theory. II, Dynamical Systems VIII, Encyclopaedia Math. Sci., Vol. 39, Springer-Verlag, Berlin, 1993.
5. V. I. Arnold, Catastrophe Theory [in Russian], Nauka, Moscow, 1990; English transl.: V. I. Arnold, Catastrophe Theory, Springer-Verlag, Berlin, 1992.
6. S. M. Gusein-Zade, "On the topology of quasiperiodic functions," In: Pseudoperiodic Topology (V. Arnold, M. Kontsevich, and A. Zorich, eds.), Amer. Math. Soc. Transl., Ser. 2, Vol. 197, 1999, pp. 1–8.
7. S. M. Gusein-Zade, "The number of critical points of a quasiperiodic potential," Funkts. Anal. Prilozhen., **23**, No. 2, 55–56 (1989).
8. V. I. Arnold, "Remarks on quasicrystallic symmetries," Phys. D, **33**, 21–25 (1988).
9. V. I. Arnold, "On a variational problem, connected with phase transitions of the means in controlled dynamical systems," In: Nonlinear Problems in Mathematical Physics and Related Topics, in honor of O. A. Ladyzhenskaia, International Mathematical Series, Vol. 1, Kluwer/Plenum, 2002.
10. V. I. Arnold, "The longest curves of given degree and a quasicrystal Harnack theorem in pseudoperiodic topology," Funkts. Anal. Prilozhen., **36** (2002).

V. A. Steklov Institute of Mathematics
CEREMADE, Universitè Paris-Dauphine

Translated by A. I. Shtern